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RUNNING WITH THE TWIN PRIMES, THE GOLDBACH CONJECTURE, THE FERMAT PRIMES NUMBERS, THE FERMAT COMPOSITE NUMBERS, AND THE MERSENNE PRIMES

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Abstract

An integer t is a twin prime (see [7] or [8] or [11] or [12] or [13]), if t is a prime number ≥ 3 and if $t - 2$ or $t + 2$ is also a prime number ≥ 3. Example: 41 and 43 are twin primes. It is conjectured that there are infinitely many twin primes. A Fermat prime is a prime of the form $F_s = 2^{2^s} + 1$, where s is an integer ≥ 0, and a Fermat composite number [or a Fermat composite] is a non-prime number of the form $F_s = 2^{2^s} + 1$, where s is an integer ≥ 1; it is conjectured that there are infinitely many Fermat composite numbers, and it is very hard to decide whether or not there are infinitely many Fermat primes. A Mersenne prime is a prime of the form $M_m = 2^m - 1$, where m is a prime [it is conjectured that there are infinitely many Mersenne primes], and the Goldbach conjecture states that every even integer $e \geq 4$ is of the form $e = p + q$, where (p, q) is a couple of prime(s). Here, we state a simple conjecture (Q.), we generalize the Fermat induction, and we use it to give a simple and detailed proof that (Q.) is stronger than the Goldbach conjecture, the twin

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primes conjecture, the Mersenne primes conjecture, the Fermat composite numbers conjecture and the Fermat primes conjecture [in the sense that there are infinitely many Fermat primes]; this helps us to explain why it is natural and not surprising to conjecture that the twin primes conjecture, the Mersenne primes conjecture, the Fermat composite numbers conjecture and the Fermat primes conjecture [in the sense that there are infinitely many Fermat primes] are all special cases of the Goldbach conjecture.

0. Prologue

Briefly, the immediate part of the generalized Fermat induction is based around the following simple definitions. Let n be an integer $\geq 2.$ Then we say that $c(n)$ is a *cache* of n, if $c(n)$ is an integer of the form $0 \leq c(n) < n$ [Example 0. If $n = 4,$ then $c(n)$ is a cache of n if and only if $c(n) \in \{0, 1, 2, 3\}$]. Now, for every couple of integers $(n, c(n))$ such that $n \geq 2$ and $0 \leq c(n) < n$ [observe that $c(n)$ is a cache of n], we define $c(n, 2)$ as follows: $c(n, 2) = 1$ if $c(n) \equiv 1 \bmod[2];$ and $c(n, 2) = 0$ if $c(n) \not\equiv 1 \bmod[2].$ It is immediate that $c(n, 2)$ exists and is well defined, since $n \geq 2$ [Example 1. If $n = 9,$ then $c(n, 2) = 0$ if $c(n) \in \{0, 2, 4, 6, 8\}$ and $c(n, 2) = 1$ if $c(n) \in \{1, 3, 5, 7\}$]. In this paper, induction will be made on n and $c(n, 2)$ [where n is an integer ≥ 2 and $c(n)$ is a cache of n].

1. Introduction and Non-standard Definitions

The prime numbers are well known. We say that e is *Goldbach*, if e is an even integer ≥ 4 and is of the form $e = p + q,$ where (p, q) is a couple of prime(s). The Goldbach conjecture (see [18] or [17] or [7] or [8] or [9] or [10] or [11] or [12] or [14] or [1] or [15]) states that every even integer $e \geq 4$ is Goldbach. We say that e is *Goldbachian*, if e is an even integer $\geq 4,$ and if every even integer v with $4 \leq v \leq e$ is Goldbach [there is no confusion between *Goldbach* and *Goldbachian*, since *Goldbachian* clearly implies *Goldbach*]. A *Fermat prime* is a prime of the form $F_s = 2^{2^s} + 1,$ where s is an integer $\geq 0,$ and a *Fermat composite number* [or a *Fermat composite*] is a non-prime number of the form $F_s = 2^{2^s} + 1,$ where s is an integer $\geq 1.$ It is known (see [4]) that for every $j \in \{0, 1, 2, 3, 4\},$ F_j is a Fermat prime, and it is also known (see [4]) that F_5 and F_6 are Fermat composites [[indeed, $F_5 = 641 \times 6700417$ and $F_6 = 274177 \times 67280421310721$ (see [4])]]. It is conjectured that there are infinitely many Fermat composite numbers, and it is very hard to decide whether or not there are infinitely many Fermat primes. A *Mersenne prime* is a prime of the form $M_m = 2^m - 1$, where m is a prime (see [2] or [3] or [4] or [6] or [5] or [16]). The Mersenne primes are well known and it is conjectured that there are infinitely many Mersenne primes (see [5]). The twin primes are defined in Abstract. Now, for every integer $n \geq 2$, we define $\mathcal{G}'(n)$, g'_n, $\mathcal{T}(n)$, t_n, $t_{n,1}$, $t_{n,2}$, $\mathcal{F}(n)$, f_n, $f_{n,1}$, $f_{n,2}$, $\mathcal{FCO}(n)$, o_n, $o_{n,1}$, $o_{n,2}$, $\mathcal{M}(n)$, m_n, $m_{n,1}$ and $m_{n,2}$ as follows: $\mathcal{G}'(n) = \{g'; 1 < g' \leq 2n, \text{ and } g' \text{ is Goldbachian}\}$, $g'_n = \max_{g' \in \mathcal{G}'(n)} g'$,

$$\mathcal{T}(n) = \{t;\ t \text{ is a } \textit{twin prime} \text{ and } 1 < t < 2n\} \ [\text{note } 3 \in \mathcal{T}(n)],$$

$t_n = \max_{t \in \mathcal{T}(n)} t$, $t_{n,1} = t_n^{t_n}$, $t_{n,2} = t_{n,1}^{t_{n,1}}$,

$$\mathcal{F}(n) = \{F_4\} \cup \{f;\ 1 < f < 2n, \text{ and } f \text{ is a } \textit{Fermat prime}\}$$

[we recall that $F_4 = 2^{2^4} + 1$, and F_4 is prime], $f_n = \max_{f \in \mathcal{F}(n)} f$, $f_{n,1} = f_n^{f_n}$, $f_{n,2} = f_{n,1}^{f_{n,1}}$;

$$\mathcal{FCO}(n) = \{F_5\} \cup \{o;\ 1 < o < 2n, \text{ and } o \text{ is a } \textit{Fermat composite}\}$$

[we recall that $F_5 = 2^{2^5} + 1$ and F_5 is not prime], $o_n = \max_{o \in \mathcal{FCO}(n)} o$, $o_{n,1} = o_n^{o_n}$, $o_{n,2} = o_{n,1}^{o_{n,1}}$,

$$\mathcal{M}(n) = \{m;\ 1 < m < 2n, \text{ and } m \text{ is a } \textit{Mersenne prime}\} \ [\text{note } 3 \in \mathcal{M}(n)],$$

$m_n = \max_{m \in \mathcal{M}(n)} m$, $m_{n,1} = m_n^{m_n}$ and $m_{n,2} = m_{n,1}^{m_{n,1}}$. Using the previous denotations, let us define

Definition 1.0 (Fundamental 1)**.** For every integer $n \geq 2$, we put

$$\mathcal{D}(n,\ 2) = \{t_{n,2}\} \cup \{f_{n,2}\} \cup \{o_{n,2}\} \cup \{m_{n,2}\}.$$

From Definition 1.0 and definition of g'_n, it becomes immediate to see.

Assertion 1.1. Let n be an integer ≥ 2. Then

(1.1.0) $g'_{n+1} \leq 2n + 2$.

(1.1.1) $g'_{n+1} < 2n + 2$ if and only if $g'_{n+1} = g'_n$.

(1.1.2) $g'_{n+1} = 2n + 2$ if and only if $2n + 2$ is Goldbachian.

(1.1.3) $2n + 2$ is Goldbachian if and only if $2n$ is Goldbachian and $2n + 2$ is Goldbach.

Assertion 1.2. Let n be an integer ≥ 3; consider $d_{n,2} \in \mathcal{D}(n, 2)$, and look at the couple $(d_n, d_{n,1})$ [Example 0. If $d_{n,2} = f_{n,2}$, then $d_n = f_n$ and $d_{n,1} = f_{n,1}$. Example 1. If $d_{n,2} = o_{n,2}$, then $d_n = o_n$ and $d_{n,1} = o_{n,1}$. Example 2. If $d_{n,2} = m_{n,2}$, then $d_n = m_n$ and $d_{n,1} = m_{n,1}$. Example 3. If $d_{n,2} = t_{n,2}$, then $d_n = t_n$ and $d_{n,1} = t_{n,1}$]. Then $0 < d_n < d_{n,1} < d_{n,2}$ and $d_{n-1,2} \leq d_{n,2}$.

Now, using the previous definitions, let (Q.) be the following statement:

(Q.). For every integer $r \geq 3$, *one and only one* of the following two properties w(Q.r) and x(Q.r) are satisfied.

w(Q.r). $2r + 2$ is not Goldbach.

x(Q.r). For every $d_{r,2} \in \mathcal{D}(r, 2)$, we have $d_{r,2} > g'_{r+1}$.

Let us remark that if for every integer $r \geq 3$, property x(Q.r) of statement (Q.) is satisfied, then the *twin primes conjecture, the Fermat primes conjecture* [in the sense that there are infinitely many Fermat primes], the Fermat composite numbers conjecture, and the Mersenne primes conjecture are simultaneously special cases of the Goldbach conjecture. It is easy to see that property x(Q.r) of statement (Q.) is satisfied for large values of r. In this paper, using only the immediate part of the generalized Fermat induction, we prove a theorem which immediately implies the following result (E.):

(E.). Suppose that statement (Q.) holds. Then the twin primes conjecture, the Fermat primes conjecture [in the sense that there are infinitely many Fermat primes], the Fermat composite numbers conjecture, the Mersenne primes conjecture and the Goldbach conjecture simultaneously hold.

Result (E.) *helps us to explain why to conjecture that the twin primes conjecture, the Fermat primes conjecture* [in the sense that there are infinitely many Fermat primes], the Fermat composite numbers conjecture, and the Mersenne primes conjecture are simultaneously special cases of the Goldbach conjecture is not surprising.

2. The Proof of Theorem which Implies Result (E.)

The following theorem immediately implies result (E.) mentioned above.

Theorem 2.1. *Let* $(n, c(n))$ *be a couple of integers such that* $n \geq 3$ *and* $c(n)$ *be a cache of* n. *Now*, *suppose that statement* (Q.) *holds*. *We have the following*:

(0.) *If* $c(n) \equiv 0 \bmod[2]$, *then* $2n + 2 - c(n)$ *is Goldbachian*.

(1.) *If* $c(n) \equiv 1 \bmod[2]$, *then for every* $d_{n,2} \in \mathcal{D}(n, 2)$, *we have* $d_{n,2} > 1 + g'_{n+1} - c(n)$.

To prove Theorem 2.1, we use

Lemma 2.2. *Suppose that* $n = 3$. *Then Theorem* 2.1 *is contented*.

Proof. Clearly $c(n) \in \{0, 1, 2\}$, and it suffices to show that Theorem 2.1 is satisfied for all $c(n) \in \{0, 1, 2\}$. So, we have to distinguish two cases [namely, case where $c(n) \in \{0, 2\}$ and case where $c(n) = 1$].

Case 0. $c(n) \in \{0, 2\}$. Clearly $c(n) \equiv 0 \bmod[2]$ and we have to show that property (0.) of Theorem 2.1 is satisfied by the couple $(n, c(n))$. Recall $n = 3$, so $2n + 2 = 8$ [note that 8 is Goldbachian], and clearly $2n + 2$ is Goldbachian; in particular, $2n + 2 - c(n)$ is Goldbachian [use the definition of Goldbachian and note (by the previous) that $2n + 2$ is Goldbachian, $n = 3$ and $c(n) \in \{0, 2\}$]. So, property (0.) of Theorem 2.1 is satisfied by the couple $(n, c(n))$, and Theorem 2.1 is contented. Case 0 follows.

Case 1. $c(n) = 1$. Clearly $c(n) \equiv 1 \bmod[2]$ and we have to show that property (1.) of Theorem 2.1 is satisfied by the couple $(n, c(n))$. Since $n = 3$, clearly $g'_{n+1} = g'_4 = 8$, $\mathcal{T}(n) = \{2, 3, 5\}$, $t_n = 5$, $t_{n,1} = 5^5$, $t_{n,2} = t_{n,1}^{t_{n,1}}$, $\mathcal{F}(n) = \{5, f_4\}$

[where $f_4 = 2^{2^4} + 1$], $f_n = f_4$, $f_{n,1} = f_4^{f_4}$, $f_{n,2} = f_{n,1}^{f_{n,1}}$ [where $f_{n,1} = f_4^{f_4}$], $\mathcal{FCO}(n) = \{f_5\}$ [where $f_5 = 2^{2^5} + 1$], $o_n = f_5$, $o_{n,1} = f_5^{f_5}$, $o_{n,2} = o_{n,1}^{o_{n,1}}$ [where $o_{n,1} = f_5^{f_5}$], $\mathcal{M}(n) = \{3\}$, $m_n = 3$, $m_{n,1} = 3^3 = 27$ and $m_{n,2} = 27^{27}$; clearly $\mathcal{D}(n, 2) = \{t_{n,2}, f_{n,2}, o_{n,2}, m_{n,2}\}$, and using the previous equalities, it becomes immediate to see that for every $d_{n,2} \in \mathcal{D}(n, 2)$, we have $d_{n,2} > g'_{n+1}$; in particular, for every $d_{n,2} \in \mathcal{D}(n, 2)$, we have $d_{n,2} > 1 + g'_{n+1} - c(n)$. So, property (1.) of Theorem 2.1 is satisfied by the couple $(n, c(n))$, and Theorem 2.1 is contented. Case 1 follows, and Lemma 2.2 immediately follows. □

Using Lemma 2.2 and the meaning of Theorem 2.1, it becomes easy to see:

Remark 0. If Theorem 2.1 is false, then there exists $(n, c(n))$ such that $(n, c(n))$ is a counter-example with *n minimum* and $c(n, 2)$ *maximum*.

Consequence 0 (Application of Remark 0 and Lemma 2.2)**.** Suppose that Theorem 2.1 is false, and let $(n, c(n))$ be a counter-example with *n minimum* and $c(n, 2)$ *maximum*. Then $n \geq 4$.

Proof. Clearly $n \geq 4$ [use Lemma 2.2]. □

Remark 1. Suppose that Theorem 2.1 is false, and let $(n, c(n))$ be a counter-example with *n minimum* and $c(n, 2)$ *maximum*. Then we have the following two simple properties (R.1.0) and (R.1.1):

(R.1.0) [The use of the *minimality* of *n*]. Put $u = n - 1$, then for every $d_{u,2} \in \mathcal{D}(u)$, we have $d_{u,2} > g'_{u+1}$.

Indeed, let $u = n - 1$ and let $c(u) = j$, where $j \in \{0, 1\}$; now consider the couple $(u, c(u))$ [note that $u < n$, $u \geq 3$ (use Consequence 0), $c(u)$ is a cache of *u*, and the couple $(u, c(u))$ clearly exists]. Then by the *minimality* of *n*, the couple $(u, c(u))$ is not a counter-example of Theorem 2.1. Clearly, $c(u) \equiv j \bmod[2]$ [because $c(u) = j$, where $j \in \{0, 1\}$], and therefore, property (j.) of Theorem 2.1 is satisfied by the couple $(u, c(u))$ [[Example 1.0. If $j = 0$ (i.e., if $c(u) = j = 0$), then property (0.) of Theorem 2.1 is satisfied by the couple $(u, c(u))$; so $2u + 2$ is Goldbachian. Example 1.1. If $j = 1$ (i.e., if $c(u) = j = 1$), then property (1.) of Theorem 2.1 is satisfied by the couple $(u, c(u))$; so, for every $d_{u,2} \in \mathcal{D}(u)$, we have $d_{u,2} > g'_{u+1}$]].

(R.1.1) [The use of the *maximality* of $c(n, 2)$: the immediate part of the generalized Fermat induction]. If $c(n) \equiv 0 \bmod[2]$, then for every $d_{n,2} \in \mathcal{D}(n, 2)$, we have $d_{n,2} > g'_{n+1}$.

Indeed, if $c(n) \equiv 0 \bmod[2]$, then clearly $c(n, 2) = 0$. Now, let the couple $(n, y(n))$ such that $y(n) = 1$. Clearly $y(n)$ is a cache of n such that $y(n, 2) = 1$ [note that $n \geq 4$ (use Consequence 0)]. Clearly $y(n, 2) > c(n, 2)$, where $y(n)$ and $c(n)$ are two caches of n [since $c(n, 2) = 0$ and $y(n, 2) = 1$, by the previous]; then by the maximality of $c(n, 2)$, the couple $(n, y(n))$ is not a counter-example of Theorem 2.1 [because $(n, c(n))$ is a counter-example of Theorem 2.1 such that n is minimum and $c(n, 2)$ is maximum, and the couple $(n, y(n))$ is of the form $y(n, 2) > c(n, 2)$, where $y(n)$ and $c(n)$ are two caches]. Note that $y(n) \equiv 1 \bmod[2]$ [since $y(n) = 1$, by the definition of $y(n)$], and therefore, property (1.) of Theorem 2.1 is satisfied by the couple $(n, y(n))$; so, for every $d_{n,2} \in \mathcal{D}(n, 2)$, we have $d_{n,2} > 1 + g'_{n+1} - y(n)$, and clearly, for every $d_{n,2} \in \mathcal{D}(n, 2)$, we have $d_{n,2} > g'_{n+1}$ [because $y(n) = 1$].

Consequence 1 (Application of Remark 1)**.** Suppose that Theorem 2.1 is false, and let $(n, c(n))$ be a counter-example with *n minimum* and $c(n, 2)$ *maximum*. Then we have the following four properties:

(c.1.0) $2n$ is Goldbachian [i.e., $g'_n = 2n$].

(c.1.1) For every $d_{n-1,2} \in \mathcal{D}(n-1, 2)$, we have $d_{n-1,2} > g'_n$.

(c.1.2) For every $d_{n,2} \in \mathcal{D}(n, 2)$, we have $d_{n,2} > g'_n$.

(c.1.3) If $c(n) \equiv 0 \bmod[2]$, then $2n + 2$ is Goldbach.

Proof. Property (c.1.0) is easy [indeed consider the couple $(u, c(u))$ such that

$u = n - 1$ and $c(u) = 0,$ and apply Example 1.0 of property (R.1.0) of Remark 1]; property (c.1.1) is also easy [consider the couple $(u, c(u))$ such that $u = n - 1$ and $c(u) = 1,$ and apply Example 1.1 of property (R.1.0) of Remark 1]; and property (c.1.2) is an immediate consequence of property (c.1.1) via Assertion 1.2 [indeed, note that $d_{n-1, 2} \leq d_{n, 2},$ by using Assertion 1.2]. Now, to prove Consequence 1, it suffices to show property (c.1.3). Fact: $2n + 2$ is Goldbach. Indeed, observing [by using property (R.1.1) of Remark 1] that for every $d_{n, 2} \in \mathcal{D}(n, 2),$ we have $d_{n, 2} > g'_{n+1},$ clearly property x(Q.n) of statement (Q.) is satisfied, and recalling that statement (Q.) holds, then we immediately deduce that property w(Q.n) of statement (Q.) is not satisfied; therefore, $2n + 2$ is Goldbach. □

Proof of Theorem 2.1. We reason by reduction to absurd. Suppose that Theorem 2.1 is false and let $(n, c(n))$ be a counter-example with n minimum and $c(n, 2)$ maximum [such a couple exists, by Remark 0]. Then we observe the following:

Observation 0. $c(n) \not\equiv 0 \bmod[2].$

Otherwise,

$$c(n) \equiv 0 \bmod[2] \tag{0.0}$$

and clearly

$$2n + 2 - c(n) \text{ is not Goldbachian} \tag{0.1}$$

[indeed note $c(n) \equiv 0 \bmod[2]$ [by congruence (0.0)], and in particular, property (0.) of Theorem 2.1 is not satisfied by the couple $(n, c(n))$; so $2n + 2 - c(n)$ is not Goldbachian]. (0.1) immediately implies that

$$2n + 2 \text{ is not Goldbachian} \tag{0.2}$$

[indeed, recalling that $c(n)$ is a cache of n such that $c(n) \equiv 0 \bmod[2]$ [by congruence (0.0)], clearly $c(n) \geq 0$ and $2n + 2 - c(n) \geq 4$ [note that $n \geq 4,$ by Consequence 0]; now, using the previous and the definition of Goldbachian via (0.1), we immediately deduce that $2n + 2$ is not Goldbachian]. Now, we have the following two simple Facts:

Fact 0.0.0. $g'_{n+1} = g'_n$. Indeed, observing [via (0.2)] that $2n + 2$ is not Goldbachian, clearly $g'_{n+1} < 2n + 2$ [use the definition of g'_{n+1} via the definition of g'_n, and observe (by the previous) that $2n + 2$ is not Goldbachian] and property (1.1.1) of Assertion 1.1 implies that $g'_{n+1} = g'_n$.

Fact 0.0.1. $2n + 2$ *is not Goldbach*. Otherwise, observing [via property (c.1.0) of Consequence 1] that $2n$ is Goldbachian, then using the previous, it immediately follows that $2n + 2$ is Goldbach and $2n$ is Goldbachian; consequently, $2n + 2$ is Goldbachian [using the fact that $2n + 2$ is Goldbach and $2n$ is Goldbachian and apply property (1.1.3) of Assertion 1.1], and this contradicts (0.2). The Fact 0.0.1 follows.

These two simple *Facts* made, observing [by Fact 0.0.1] that $2n + 2$ is not Goldbach, clearly property w(Q.*n*) of statement (Q.) is satisfied, and recalling that statement (Q.) holds, then we immediately deduce that property x(Q.*n*) of statement (Q.) is not satisfied; therefore,

$$\textit{there exists } d_{n,2} \in \mathcal{D}(n, 2) \textit{ such that } d_{n,2} \leq g'_{n+1}. \tag{0.0.2}$$

Now, using Fact 0.0.0, then (0.0.2) immediately implies that there exists $d_{n,2} \in \mathcal{D}(n, 2)$ such that $d_{n,2} \leq g'_n$, and this contradicts property (c.1.2) of Consequence 1. Observation 0 follows.

Observation 0 implies that

$$c(n) \equiv 1 \operatorname{mod}[2] \tag{1.0}$$

and clearly

$$\textit{there exists } d_{n,2} \in \mathcal{D}(n, 2) \textit{ such that } d_{n,2} \leq g'_{n+1} \tag{1.1}$$

[indeed note $c(n) \equiv 1 \operatorname{mod}[2]$ (by congruence (1.0)), and, in particular, property (1.) of Theorem 2.1 is not satisfied by the couple $(n, c(n))$; so there exists $d_{n,2} \in \mathcal{D}(n, 2)$ such that $d_{n,2} \leq 1 + g'_{n+1} - c(n)$, and consequently, there exists $d_{n,2} \in \mathcal{D}(n, 2)$ such that $d_{n,2} \leq g'_{n+1}$, because $c(n) \geq 1$ (since $c(n) \equiv 1 \operatorname{mod}[2]$ [by congruence (1.0)], and $c(n)$ is a cache of n)]. (1.1) clearly says that property x(Q.*n*) of statement (Q.) is not satisfied, and recalling that statement (Q.) holds, then we immediately deduce that property w(Q.n) of statement (Q.) is satisfied; therefore,

$$2n + 2 \text{ is not Goldbach}. \tag{1.2}$$

(1.2) immediately implies that $g'_{n+1} < 2n + 2$; now, using property (1.1.1) of Assertion 1.1 and the previous inequality, we immediately deduce that

$$g'_{n+1} = g'_n. \tag{1.3}$$

Now, using equality (1.3), then (1.1) clearly says that there exists $d_{n,2} \in \mathcal{D}(n, 2)$ such that $d_{n,2} \leq g'_n$, and this contradicts property (c.1.2) of Consequence 1. Theorem 2.1 follows. □

Remark 2. Note that to prove Theorem 2.1, we consider a couple $(n, c(n))$ such that $(n, c(n))$ is a counter-example with *n minimum* and $c(n, 2)$ *maximum*. In properties (c.1.0), (c.1.1) and (c.1.2) of Consequence 1 (via property (R.1.0) of Remark 1), the minimality of n is used; and in property (c.1.3) of Consequence 1 (via property (R.1.1) of Remark 1), the maximality of $c(n, 2)$ is used. Consequence 1 helps us to give a simple and detailed proof of Theorem 2.1.

Corollary 2.3. *Suppose that statement* (Q.) *holds*. *Then we have the following four properties*:

(2.3.0). *For every integer* $n \geq 1$, $2n + 2$ *is Goldbachian* [*i.e.*, $g'_{n+1} = 2n + 2$].

(2.3.1). *The Goldbach conjecture holds*.

(2.3.2). *For every integer* $n \geq 3$, *and for every* $d_{n,2} \in \mathcal{D}(n, 2)$, *we have* $d_{n,2} > 2n + 2$.

(2.3.3). *The twin primes*, *the Fermat primes*, *the Fermat composite numbers*, *and the Mersenne primes are all infinite*.

Proof. (2.3.0). It is immediate if $n \in \{1, 2\}$. If $n \geq 3$, then consider the couple $(n, c(n))$ with $c(n) = 0$. The couple $(n, c(n))$ is of the form $0 \leq c(n) < n$, where $n \geq 3$, $c(n) \equiv 0 \bmod[2]$, and $c(n)$ is a cache of n. Then property (0.) of Theorem 2.1 is satisfied by the couple $(n, c(n))$. So, $2n + 2$ is Goldbachian [because $c(n) = 0$], and consequently, $g'_{n+1} = 2n + 2$.

(2.3.1). Indeed, the Goldbach conjecture immediately follows, by using property (2.3.0).

(2.3.2). Let the couple $(n, c(n))$ be such that $c(n) = 1$. Then the couple $(n, c(n))$ is of the form $0 \leq c(n) < n$, where $n \geq 3$, $c(n) \equiv 1 \bmod[2]$, and $c(n)$ is a cache of *n*. Then property (1.) of Theorem 2.1 is satisfied by the couple $(n, c(n))$. So, for every $d_{n,2} \in \mathcal{D}(n, 2)$, we have $d_{n,2} > g'_{n+1}$ [because $c(n) = 1$]; now, observing [by property (2.3.0)] that $g'_{n+1} = 2n + 2$, then we immediately deduce that for every $d_{n,2} \in \mathcal{D}(n, 2)$, we have $d_{n,2} > 2n + 2$.

(2.3.3). Indeed, the twin primes, the Fermat primes, the Fermat composite numbers, and the Mersenne primes are all infinite, by using property (2.3.2) and the definition of $\mathcal{D}(n, 2)$. □

Using property (2.3.1) and property (2.3.3) of Corollary 2.3, then the following:

Result (E.). Suppose that statement (Q.) holds. Then the Goldbach conjecture holds, and moreover, the twin primes, the Fermat primes, the Fermat composite numbers, and the Mersenne primes are all infinite.

Conjecture 0. Statement (Q.) holds.

Epilogue. *To conjecture that the twin primes conjecture, the Fermat primes conjecture* [in the sense that there are infinitely many Fermat primes], the Fermat composite numbers conjecture, and the Mersenne primes conjecture are consequences of the Goldbach conjecture is not surprising. Indeed, let (Q'.) be the following statement:

(Q'.). For every integer $r \geq 3$, *at most one* of the following two properties w(Q'.*r*) and x(Q'.*r*) holds.

w(Q'.*r*). $2r + 2$ is not Goldbach.

x(P'.*r*). For every $d_{r,2} \in \mathcal{D}(n, 2)$, we have $d_{r,2} > g'_{r+1}$.

Note that statement (Q'.), somehow, resembles to statement (Q.). More precisely, statement (Q.) implies statement (Q'.) [Proof. In particular, the Goldbach conjecture holds [use property (2.3.1) of Corollary 2.3]; consequently, statement (Q'.) holds [use definition of statement (Q'.) and the previous].

Conjecture 1. Statement (Q.) and statement (Q'.) are equivalent.

Conjecture 1 implies that the twin primes, the Fermat primes conjecture [in the sense that there are infinitely many Fermat primes], the Fermat composite numbers conjecture, and the Mersenne primes conjecture are consequences of the Goldbach conjecture.

Proof. Suppose that Conjecture 1 holds. If the Goldbach conjecture holds, then clearly statement (Q'.) holds; observing that statement (Q'.) and statement (Q.) are equivalent, then (Q.) holds, and, result (E.) implies that the twin primes, the Fermat primes, the Fermat composite numbers and the Mersenne primes are all infinite. □

Conjecture 2. Suppose that statement (Q'.) holds. Then the Goldbach conjecture holds, and moreover, the twin primes, the Fermat primes, the Fermat composite numbers and the Mersenne primes are all infinite.

Conjecture 2 immediately implies that the twin primes conjecture, the Fermat primes conjecture [in the sense that there are infinitely many Fermat primes], the Fermat composite numbers conjecture, and the Mersenne primes conjecture are consequences of the Goldbach conjecture.

Proof. Suppose that Conjecture 2 holds. If the Goldbach conjecture holds, then clearly statement (Q'.) holds, and in particular, the twin primes, the Fermat primes, the Fermat composite numbers, and the Mersenne primes are all infinite. □

Conjecture 3. For every integer $r \geq 3,$ property x(Q'.*r*) of statement (Q'.) holds [note that property x(Q'.*r*) of statement (Q'.) is exactly property x(Q.*r*) of statement (Q.); moreover, it is immediate to see that property x(Q'.*r*) of statement (Q'.) is satisfied for large values of *r*].

Conjecture 3 also implies that the twin primes conjecture, the Fermat primes conjecture [in the sense that there are infinitely many Fermat primes], the Fermat composite numbers conjecture, and the Mersenne primes conjecture are consequences of the Goldbach conjecture.

Proof. Suppose that Conjecture 3 holds. If the Goldbach conjecture holds, then clearly, $g'_{n+1} = 2n + 2,$ and so for every $d_{n,2} \in \mathcal{D}(n, 2),$ we have

$$d_{n,2} > g'_{n+1} > 2n. \tag{3.0}$$

Observing that (3.0) holds for every integer $n \geq 3,$ then, in particular, it results that the twin primes, the Fermat primes, the Fermat composite numbers, and the Mersenne primes are all infinite. □

Now, using the previous three conjectures, it becomes natural and not surprising to the following conjecture:

Conjecture 4. The twin primes conjecture, the Fermat primes conjecture [in the sense that there are infinitely many Fermat primes], the Fermat composite numbers conjecture, and the Mersenne primes conjecture are consequences of the Goldbach conjecture.

From Conjecture 4, the following immediately comes:

Conjecture 5. There are infinitely many Fermat primes.

Conjecture 6. Let $(n, b(n))$ be a couple of integers such that $n \geq 4$ and $0 \leq b(n) < n.$ Then we have the following:

(0.) If $b(n) \equiv 0 \bmod[4];$ then $2n + 2 - b(n)$ is Goldbachian.

(1.) If $b(n) \equiv 1 \bmod[4];$ then $t_{n,2} > 1 + g'_{n+1} - b(n)$ and $f_{n,2} > 1 + g'_{n+1} - b(n).$

(2.) If $b(n) \equiv 2 \bmod[4];$ then $o_{n,2} > 2 + g'_{n+1} - b(n).$

(3.) If $b(n) \equiv 3 \bmod[4];$ then $m_{n,2} > 3 + g'_{n+1} - b(n).$

It is easy to see that Conjecture 6 simultaneously implies that: not only the Goldbach conjecture holds, but the twin primes, the Fermat primes, the Fermat composite numbers, and the Mersenne primes are all infinite, and to attack this conjecture, we must consider the generalized Fermat induction.

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