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ON SOME GEOMETRICAL PROPERTIES OF CERTAIN VECTOR-VALUED SEQUENCE SPACES

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Abstract

In this paper, we introduce the generalized Cesáro vector-valued sequence space $ces(X, p_n, q_n)$ equipped with the Luxemburg norm. Further, we show some topological properties of this space with respect to the Luxemburg norm. Finally, it is proved that the space $ces(X, p_n, q_n)$ has Kadec-Klee (H) property and we also give a counterexample concerning not rotundity of the vector-valued space $ces(X, p_n, q_n)$.

1. Introduction

Recently there has been a lot of interest in investigating geometric properties of sequence spaces besides topological and some other usual properties. In the

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literature, there are many papers concerning the geometric properties of different sequence spaces. For example, in [11], Mursaleen et al. studied some geometric properties of normed Euler sequence space. Karakaya [5] defined a new sequence space involving lacunary sequence space equipped with Luxemburg norm and studied Kadec-Klee (*H*), rotund (*R*) properties of this space. In addition, some related papers on this topic can be found in [1], [4], [13], [15] and [19].

Especially, some geometric properties of the Cesáro sequence spaces have been studied by many authors including Cui and Hudzik [2], Liu et al. [10], Cui et al. [3].

Shiue [16] first defined Cesáro sequence space with norm. In a recent paper, Suantai [17] generalized the normed Cesáro sequence spaces to the paranormed sequence spaces by making use of Köthe sequence spaces. He showed that the Cesáro sequence space $ces(p)$ equipped with Luxemburg norm has rotund (R) and Kadec-Klee (H) properties. Also, in [14], Sanhan and Suantai showed that the Cesáro sequence space $ces(p)$, where the sum runs over $2^r \leq k < 2^{r+1}$, equipped with Luxemburg norm has property (H) but it is not rotund.

In [6], Khan and Rahman introduced sequence space $ces[(p_n), (q_n)]$. Afterwards Mursaleen and Khan [12] generalized this space to the vector-valued sequence space and studied dual of this space. In the space $ces[(p_n), (q_n)]$, if we specialize $q_n = 1$ for all $n \in N$, then we get $ces[(p_n), (q_n)] = ces(p)$ defined in [14].

In this work, our purpose is to generalize paranormed sequence space $ces[(p_n), (q_n)]$ to vector-valued space $ces(X, p_n, q_n)$ and to investigate some topological properties and geometrical properties as Kadec-Klee (*H*) and rotund (*R*) according to Luxemburg norm of this space.

2. Preliminaries and Notations

Let $(X, \|\cdot\|)$ be a Banach space with a scalar field K and the space of all sequences in X be denoted by $w(X)$. When $X = \mathbb{R}$ or $\mathbb{C}$, the corresponding spaces are written as w.

Let $(X, \|\cdot\|)$ be a real Banach space and $B(X)$ (resp. $S(X)$) be the closed unit ball (resp. unit sphere) of X.

For any sequence (x_n) in X such that $\| x_n \| \to 1$ as $n \to \infty$, if the weak convergence of (x_n) to x (write $x_n \xrightarrow{w} x$) implies that $\| x_n - x \| \to 0$ as $n \to \infty$, then a point $x \in S(X)$ is said to be an H-point of $B(X)$. If every point of $S(X)$ is an H-point of $B(X)$, then it is said to have H-property. Briefly, X is said to have the H-property, if every weakly convergent sequence on the unit sphere is convergent in norm. A point $z \in S(X)$ is an extreme point of $B(X)$, if for any $x, y \in S(X)$, $z = \frac{x+y}{2}$ implies $x = y$. A Banach space X is rotund (R) if every point of $S(X)$ is an extreme point of $B(X)$.

Let (q_n) and (p_n) with $\inf p_r > 0$ be sequences of the positive real numbers. Now, we shall define generalized vector-valued sequence spaces $ces(X, p_n, q_n)$ which is equivalent to the space $ces(X, p, q)$ defined in [12] as follows:

$$ces(X, p_n, q_n) = \left\{ x = (x_k) : \sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_k \| x(k) \| \right)^{p_r} < \infty \right\},$$

where $Q_{2^r} = q_{2^r} + q_{2^r+1} + \cdots + q_{2^{r+1}-1}$ and $\sum_r$ denotes summation over the range $2^r \le k < 2^{r+1}$.

It is trivial that the sequence space $ces(X, p_n, q_n)$ may be reduced to some new sequence spaces in the special cases of X, (p_n) and (q_n), for all $n \in \mathbb{N}$. For instance, the sequence space $ces(X, p_n, q_n)$ corresponds to the sequence space $ces[(p_n), (q_n)]$ introduced by Khan and Rahman [6] in the case $X = \mathbb{C}$ or $\mathbb{R}$ in (2.1). Also if $q_n = 1$, for all $n \in \mathbb{N}$, then the space $ces(X, p_n, q_n)$ reduces to $ces(X, p_n)$ which is equivalent to the space $ces(X, p)$ defined by [18]. Besides this, if $q_n = 1$ and $p_n = p$, for all $n \in \mathbb{N}$, then we can write the space $ces_p(X)$ in place of the space $ces(X, p_n, q_n)$, where

$$ces_p(X) = \left\{ x = (x_k) : \sum_{r=0}^{\infty} \left(2^{-r} \sum_r \| x(k) \| \right)^{p} < \infty \right\}.$$

The sequence space $ces(X, p_n, q_n)$ has paranorm defined by

$$h(x) = \left[\sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k \| x(k) \|\right)^{p_r}\right]^{1/M}, \tag{2.1}$$

where $H = \sup_r p_r < \infty$ and $M = \max(1, H)$. By using standard techniques, it is easily verified that $ces(X, p_n, q_n)$ is paranormed space with (2.1).

If the functional σ on $w(X)$ has the following properties, it is called *modular* on $w(X)$:

(i) $\sigma(x) = 0 \Leftrightarrow x = 0$;

(ii) $\sigma(\alpha x) = \sigma(x), \forall \alpha \in \mathbb{F}$ with $|\alpha| = 1$, for all $x \in w(X)$;

(iii) $\sigma(\alpha x + \beta y) \leq \sigma(x) + \sigma(y)$ if $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, for all $x, y \in w(X)$, if the property (iii) is replaced by

(iv) $\sigma(\alpha x + \beta y) \leq \alpha\sigma(x) + \beta\sigma(y)$, for all $\alpha, \beta \in R^+$ with $\alpha + \beta = 1$; then we say that σ is a convex modular.

We can introduce the modular σ on the vector-valued sequence space $ces(X, p_n, q_n)$ as follows:

$$\sigma : ces(X, p_n, q_n) \to [0, \infty],$$

where $\sigma(x) = \sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k \| x(k) \|\right)^{p_r}$. For all $x \in ces(X, p_n, q_n)$, we define a norm as follows:

$$\| x \|_L = \inf\left\{\tau > 0 : \sigma\left(\frac{x}{\tau}\right) \leq 1\right\}.$$

The $\| \cdot \|_L$ is called the *Luxemburg norm* on the sequence space $ces(X, p_n, q_n)$.

Note that Luxemburg norm on the sequence space $ces(X, p_n, q_n)$ is defined as follows:

$$\| x \|_L = \inf\left\{\tau > 0 : \sigma\left(\frac{x}{\tau}\right) = \sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k \left\| \frac{x(k)}{\tau} \right\|\right)^{p_r} \leq 1\right\}.$$

Let us give an inequality that we will require throughout this paper: Let $p = (p_r)$ be a bounded sequence of positive real numbers. Then we have

$$|a_r + b_r|^{p_r} \leq C[|a_r|^{p_r} + |b_r|^{p_r}],$$

where $C = \max(1, 2^{H-1}),\ H = \sup_r p_r$.

3. Main Results

We shall give some propositions which we need in the sequel of this paper.

Proposition 3.1. *The functional* σ *is a convex modular on* $ces(X, p_n, q_n)$.

Proof. Let $x, y \in ces(X, p_n, q_n)$. It is obvious that;

(i) $\sigma(x) = 0 \Leftrightarrow x = 0$ and;

(ii) $\sigma(\lambda x) = \sigma(x)$, for all scalar λ with $|\lambda| = 1$.

$$\begin{aligned}
\sigma(\lambda x) &= \sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k\|\lambda x(k)\|\right)^{p_r} \\
&= \sum_{r=0}^{\infty}|\lambda|^{p_r}\left(\frac{1}{Q_{2^r}}\sum_r q_k\|x(k)\|\right)^{p_r} \\
&= \sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k\|x(k)\|\right)^{p_r} \\
&= \sigma(x).
\end{aligned}$$

(iii) For $\lambda, \beta \geq 0$ with $\lambda + \beta = 1$, by the convexity $t \to |t|^{p_r}$, for every $r \in \mathbb{N}$, we have

$$\begin{aligned}
\sigma(\lambda x + \beta y) &= \sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k\|\lambda x(k) + \beta y(x)\|\right)^{p_r} \\
&\leq \lambda^{p_r}\sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k\|x(k)\|\right)^{p_r} + \beta^{p_r}\sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k\|y(k)\|\right)^{p_r} \\
&\leq \lambda\sigma(x) + \beta\sigma(y).
\end{aligned}$$

□

Proposition 3.2. (i) *If* $\| x \|_L < 1,$ *then* $\sigma(x) \leq \| x \|_L;$

(ii) $\| x \|_L = 1$ *if and only if* $\sigma(x) = 1.$

Proof. It is provided with standard techniques as [14]. □

Proposition 3.3. *For* $x \in ces(X, p_n, q_n),$ *we have*

(i) *if* $0 < a < 1$ *and* $\| x \|_L > a,$ *then* $\sigma(x) > a^H,$ *where* $H = \sup_r p_r;$

(ii) *if* $a \geq 1$ *and* $\| x \|_L < a,$ *then* $\sigma(x) < a^H.$

Proof. It is provided with standard techniques as [14]. □

Proposition 3.4. *Let* (x_n) *be a sequence in* $ces(X, p_n, q_n).$

(i) *If* $\lim_{n \to \infty} \| x_n \|_L = 1,$ *then* $\lim_{n \to \infty} \sigma(x_n) = 1;$

(ii) *If* $\lim_{n \to \infty} \sigma(x_n) = 0,$ *then* $\lim_{n \to \infty} \| x_n \|_L = 0.$

Proof. (i) Suppose that $\lim_{n \to \infty} \| x_n \|_L = 1.$ Let $\varepsilon \in (0, 1).$ Then there exists n_0 such that $1 - \varepsilon < \| x_n \|_L < 1 + \varepsilon,$ for all $n \geq n_0.$ Since $(1-\varepsilon)^H < \| x_n \|_L < (1+\varepsilon)^H$ for all $n \geq n_0,$ by Proposition 3.3(i) and (ii), we have $\sigma(x_n) \geq (1 - \varepsilon)^H$ and $\sigma(x_n) \leq (1 - \varepsilon)^H.$ Therefore, $\lim_{n \to \infty} \sigma(x_n) = 1.$

(ii) Suppose that $\| x_n \|_L \nrightarrow 0.$ Then there are an $\varepsilon \in (0, 1)$ and a subsequence (x_{n_k}) of (x_n) such that $\| x_{n_k} \|_L > \varepsilon,$ for all $k \in \mathbb{N}.$ By Proposition 3.3(i), we obtain that $\sigma(x_{n_k}) > \varepsilon^H,$ for all $k \in \mathbb{N}.$ This implies that $\sigma(x_{n_k}) \nrightarrow 0$ as $n \to \infty.$ Hence $\sigma(x_n) \nrightarrow 0.$ □

We now show that the $ces(X, p_n, q_n)$ is a Banach space with respect to Luxemburg norm.

Theorem 3.5. *The space* $ces(X, p_n, q_n)$ *is a Banach space with respect to Luxemburg norm defined by*

$$\| x \|_L = \inf\left\{\tau > 0 : \sigma\left(\frac{x}{\tau}\right) \leq 1\right\}.$$

Proof. We show that every Cauchy sequence in $ces(X, p_n, q_n)$ is convergent according to the Luxemburg norm. Let $(x^n(k))$ be a Cauchy sequence in $ces(X, p_n, q_n)$ and $\varepsilon \in (0, 1)$. Thus there exists n_0 such that $\| x^n - x^m \|_L < \varepsilon^M$, for all $m, n \geq n_0$. By Proposition 3.2(i), we obtain

$$\sigma(x^n - x^m) < \| x^n - x^m \|_L < \varepsilon^M, \tag{3.1}$$

for all $n, m \geq n_0$, that is,

$$\sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_{r} q_k \| x^n(k) - x^m(k) \| \right)^{p_r} < \varepsilon^M,$$

for $m, n \geq n_0$. For fixed k, we get that

$$\| x^n(k) - x^m(k) \| < \varepsilon.$$

Hence, we obtain that the sequence $(x^n(k))$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $x^m(k) \to x(k)$ as $m \to \infty$. Therefore, for fixed k,

$$\| x^n(k) - x(k) \| < \varepsilon,$$

for all $n \geq n_0$. Now, we will show that the sequence $(x(k))$ is the element of $ces(X, p_n, q_n)$. From inequality (3.1), we can write

$$\sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_{r} q_k \| x^n(k) - x^m(k) \| \right)^{p_r} < \varepsilon,$$

for all $m, n \geq n_0$. For every $k \in \mathbb{N}$, we have $x^m(k) \to x(k)$, so we obtain that

$$\sigma(x^n - x^m) \to \sigma(x^n - x)$$

as $m \to \infty$. Since for all $n \geq n_0$,

$$\sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_{r} q_k \| x^n(k) - x^m(k) \| \right)^{p_r} \to \sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_{r} q_k \| x^n(k) - x(k) \| \right)^{p_r}$$

as $m \to \infty$, by (3.1), we have $\sigma(x^n - x) < \| x^n - x \|_L < \varepsilon$, for all $n \geq n_0$. This means that $x_n \to x$ as $n \to \infty$. So we have $(x_{n_0} - x) \in ces(X, p_n, q_n)$. Since $ces(X, p_n, q_n)$ is a linear space, we have $x = x_{n_0} - (x_{n_0} - x) \in ces(X, p_n, q_n)$. Therefore, the vector-valued sequence space $ces(X, p_n, q_n)$ is a Banach space with respect to Luxemburg norm. This completes the proof. □

Proposition 3.6. *Let $x \in ces(X, p_n, q_n)$ and $(x_n) \subseteq ces(X, p_n, q_n)$. If $\sigma(x_n) \to \sigma(x)$ as $n \to \infty$ and $x_n(k) \to x(k)$ as $n \to \infty$, for all $k \in \mathbb{N}$, then $x_n \to x$ as $n \to \infty$.*

Proof. Let $\varepsilon > 0$. Since $\sigma(x) = \sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k \| x(k) \|\right)^{p_r} < \infty$, there exists $k \in \mathbb{N}$ such that

$$\sum_{r=r_0+1}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k \| x(k) \|\right)^{p_r} < \frac{\varepsilon}{3}\frac{1}{2^M}, \tag{3.2}$$

where $M = \max\{1, 2^{H-1}\}$, $H = \sup_r p_r$.

Since $\sigma(x_n) - \sum_{r=0}^{r_0}\left(\frac{1}{Q_{2^r}}\sum_r q_k \| x_n(k) \|\right)^{p_r} \to \sigma(x) - \sum_{r=0}^{r_0}\left(\frac{1}{Q_{2^r}}\sum_r q_k \| x(k) \|\right)^{p_r}$ as $n \to \infty$ and $x_n(k) \to x(k)$ as $n \to \infty$, for all $k \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that

$$\left| \sum_{r=r_0+1}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k \| x_n(k) \|\right)^{p_r} - \sum_{r=r_0+1}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k \| x(k) \|\right)^{p_r} \right| < \frac{\varepsilon}{3}\frac{1}{2^M}, \tag{3.3}$$

for all $n \geq n_0$. Also, since $x_n(k) \to x(k)$ as $n \to \infty$, for all $k \in \mathbb{N}$, we have $\sigma(x_n) \to \sigma(x)$ as $n \to \infty$. Hence, for all $n \geq n_0$, we have $\| x_n(k) - x(k) \| < \varepsilon$. As a result, for all $n \geq n_0$, we have

$$\sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k \| x_n(k) - x(k) \|\right)^{p_r} < \frac{\varepsilon}{3}. \tag{3.4}$$

Then, from (3.2), (3.3) and (3.4) it follows that for $n \geq n_0$,

$$\sigma(x_n - x) = \sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k \| x_n(k) - x(k) \|\right)^{p_r}$$

$$= \sum_{r=0}^{r_0}\left(\frac{1}{Q_{2^r}}\sum_r q_k \| x_n(k) - x(k) \|\right)^{p_r}$$

$$+ \sum_{r=r_0+1}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k \| x_n(k) - x(k) \|\right)^{p_r}$$

$$< \frac{\varepsilon}{3} + 2^M\left[\sum_{r=r_0+1}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k \| x_n(k) \|\right)^{p_r}\right.$$

$$\left. + \sum_{r=r_0+1}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k \| x(k) \|\right)^{p_r}\right]$$

$$= \frac{\varepsilon}{3} + 2^M\left[\sigma(x_n) - \sum_{r=0}^{r_0}\left(\frac{1}{Q_{2^r}}\sum_r q_k \| x_n(k) \|\right)^{p_r}\right.$$

$$\left. + \sum_{r=r_0+1}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k \| x(k) \|\right)^{p_r}\right]$$

$$< \frac{\varepsilon}{3} + 2^M\left[\sigma(x) - \sum_{r=0}^{r_0}\left(\frac{1}{Q_{2^r}}\sum_r q_k \| x(k) \|\right)^{p_r}\right.$$

$$\left. + \frac{\varepsilon}{3}\frac{1}{2^M} + \sum_{r=r_0+1}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k \| x(k) \|\right)^{p_r}\right]$$

$$= \frac{\varepsilon}{3} + 2^M\left[\sum_{r=r_0+1}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k \| x(k) \|\right)^{p_r}\right.$$

$$\left. + \frac{\varepsilon}{3}\frac{1}{2^M} + \sum_{r=r_0+1}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_k \| x(k) \|\right)^{p_r}\right]$$

$$= \frac{\varepsilon}{3} + 2^M \left[2 \sum_{r=r_0+1}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_k \| x(k) \| \right)^{p_r} + \frac{\varepsilon}{3} \frac{1}{2^M} \right]$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This shows that $\sigma(x_n - x) \to 0$ as $n \to \infty$. Hence, by Proposition 3.4(ii), we have $\| x_n - x \|_L \to 0$ as $n \to \infty$. □

Now, we shall give main results of this paper involving geometric properties of the space $ces(X, p_n, q_n)$.

Theorem 3.7. *The space* $ces(X, p_n, q_n)$ *has the property Kadec-Klee* (*H-property*).

Proof. Let $x \in S(ces(X, p_n, q_n))$ and $(x_n) \subseteq B(ces(X, p_n, q_n))$ such that $\| x_n \|_L \to \| x \|_L = 1$ and $x_n \xrightarrow{w} x$ as $n \to \infty$. From Proposition 3.2(ii), we have $\sigma(x) = 1$, so it follows from Proposition 3.4(i) that $\sigma(x_n) \to \sigma(x)$ as $n \to \infty$. Since $x_n \xrightarrow{w} x$ and the ith-coordinate mapping $\pi_i : ces(X, p_n, q_n) \to \mathbb{R}$ defined by $\pi_k(k) \to x(k)$ is continuous linear function on $ces(X, p_n, q_n)$, it follows that $x_n(k) \to x(k)$ as $n \to \infty$, for all $k \in \mathbb{N}$. Thus we obtain by Proposition 3.6 that $x_n \to x$ as $n \to \infty$. □

Theorem 3.8. *Let* $p = (p_k)$ *be a bounded sequence of real numbers such that* $p_k > 1$, *for all* $k \in \mathbb{N}$. *Then the space* $ces(X, p_n, q_n)$ *is not rotund* (R).

Proof. For the proof, we will give a counterexample.

Let $z \in X$ be such that $\| z \| = 1$. Take $q_k = \frac{3}{4}$, for all $k \in \mathbb{N}$, $x = (0, 2z, 0, 0, \ldots)$ and $y = (0, z, z, 0, \ldots)$. Then $\sigma(x) = \sigma(y) = 1$ and $\sigma\left(\frac{x+y}{2}\right) = 1$. This shows that $ces(X, p_n, q_n)$ is not rotund. □

Remark 3.9. If we take $q_k = 1$ in the space $ces(X, p_n, q_n)$, then we obtain the vector-valued sequence space $ces(X, p)$. Let $p_k > 1$ and $q_k = 1$. We choose

$x_k = (1, 0, 0, 0, \ldots)$ and $y_k = (0, 1, 1, 0, \ldots)$. Then it can be seen that $\sigma(x) = \sigma\left(\frac{x+y}{2}\right) = \sigma(y) = 1,$ where

$$\sigma(x) = \sum_{r=0}^{\infty}\left(\frac{1}{2^r}\sum_{r} \| x_k \|\right)^{p_r}.$$

Therefore, the space $ces(X, p)$ is not rotund for $q_k = 1$. Also, if $X = \mathbb{C}$ or $\mathbb{R}$ and $q_k = 1,$ for all $k \in \mathbb{N},$ then we get the sequence space $ces(p)$ defined by [14]. They showed that this space is not rotund.

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