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INTERFACIAL CRACKS AT THE SIDES OF A HOLE IN A BIMATERIAL UNDER ANTI-PLANE SHEAR

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Abstract

An infinite crack terminates at the side of a central crack-breaker hole and another one of finite length originates at the other side of the hole thereby forming interfacial cracks in an infinite elastic bimaterial body. Mode III fields near the crack tip, which may be of theoretical and engineering importance, are obtained in terms of the elliptic integral of the first kind and shown to depend on known fields for a tunnel crack of the same radius as the hole with similar load sites.

1. Introduction

The central circular hole containing finite cracks in a homogeneous material has been studied by many authors; see for example [4-6]. The non-homogeneous case, in the form studied here, appears new. The bimaterial investigated is an infinite body containing a central crack-breaker hole of radius $r = a$ into which an infinite crack located along $\theta = \pm\pi,\ r > a,$ terminated. A crack of length $b - a$ originates on the other side of the hole along $\theta = 0.$ The matrix firmly bounded along its interface anti-plane tractions are prescribed so that Q_1 acts on $r = a,\ 0 < \theta < \pi$ and Q_2 on

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$r = a, \ -\pi < \theta < 0$ (Figure 1). The subscript 1 refers to material 1 while 2 refers to material 2.

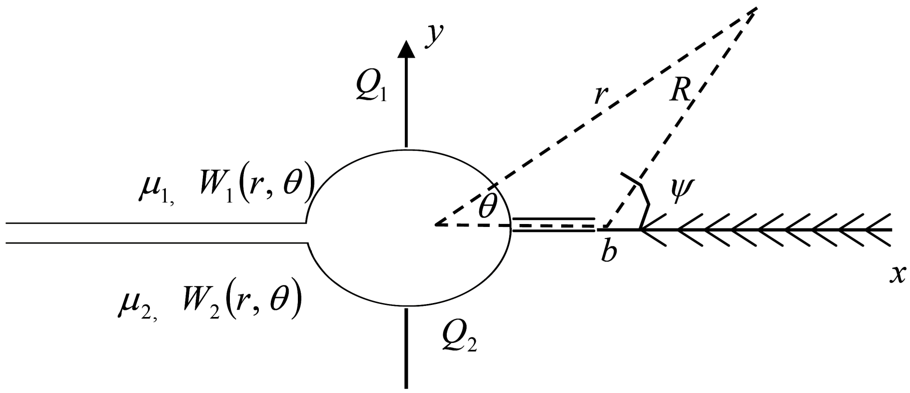

Figure 1. A sketch of the problem.

2. Basic Equations

The only non-vanishing components of displacement, $W_j(r, \theta), \ j = 1, 2$ satisfy the boundary value problem:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) W_j(r, \theta) = 0, \quad r > a, \ -\pi \le \theta \le \pi, \quad j = 1, 2, \tag{1}$$

$$\frac{\partial}{\partial r} W_j(a, \theta) = \frac{Q_j}{\mu_j}, \quad 0 < \theta < \pi (j = 1), \ -\pi < \theta < 0 \quad (j = 2), \tag{2}$$

$$\frac{\partial}{\partial r} W_j(r, \pm\pi) = 0, \ \ r > a; \quad \frac{\partial W_j}{\partial \theta}(r, 0) = 0, \ \ a < r < b. \tag{3}$$

Polar stresses are related to displacements through

$$\sigma_{j\theta z}(r, \theta) = \frac{\mu_j}{r}\frac{\partial W_j}{\partial \theta}(r, \theta); \quad \sigma_{jrz}(r, \theta) = \mu_j \frac{\partial W_j}{\partial r}(r, \theta). \tag{4}$$

Continuity conditions satisfied by the fields are:

$$W_1(r, 0) = W_2(r, 0), \ \ r \ge b; \quad \sigma_{1\theta z}(r, 0) = \sigma_{2\theta z}(r, 0), \ \ r \ge b. \tag{5}$$

3. Transformation of the Problem

The problem is transformed by using the holomorphic mapping function

$$f(z) = \frac{1}{2}\left(\frac{z}{a} + \frac{a}{z}\right) - \delta, \qquad z = re^{i\theta}, \tag{6}$$

where

$$\delta = \frac{1}{2}\left(\frac{b}{a} + \frac{a}{b}\right). \tag{7}$$

Let

$$f(z) = \operatorname{Re} f + i \operatorname{Im} f = \rho e^{i\phi},$$

where Re and Im denote the real and imaginary parts, respectively.

Along the boundary

$$\frac{\partial\phi}{\partial\theta}(r, \pm\theta) \neq 0; \qquad \frac{\partial\phi}{\partial r}(a, \theta) = \frac{-\sin\theta}{a(\delta - \cos\theta)}. \tag{8}$$

Since $b \geq a$ for $\delta \geq 1$, we see that $\rho(a, \theta) = \delta - \cos\theta$ and so, for $j = 1, 2$,

$$\frac{\partial\phi}{\partial r}(a, \phi) = (-1)^j \frac{\sqrt{1 - (\rho - \delta)^2}}{a\rho}, \quad \delta - 1 < \rho < \delta + 1, \quad -\pi < \theta < \pi.$$

The conformality property $W_j(r, \theta) \equiv W_j(\rho, \phi)$ yields

$$\frac{\partial W_j}{\partial r}(a, \theta) = \frac{\partial W_j}{\partial\phi}(\rho, \pm\pi)\frac{\partial\phi}{\partial r}(a, \theta), \qquad -\pi \leq \theta \leq \pi.$$

The mapping therefore reformulates the task to a search for $W_j(\rho, \phi)$ in the problem:

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)W_j(\rho, \phi) = 0, \quad \rho \geq 0, \quad -\pi \leq \phi \leq \pi, \quad j = 1, 2, \tag{9}$$

$$W_1(\rho, 0) = W_2(\rho, 0), \quad \rho \geq 0; \quad \mu_1 \frac{\partial W_1}{\partial\phi}(\rho, 0) = \mu_2 \frac{\partial W_2}{\partial\phi}(\rho, 0), \quad \rho \geq 0, \tag{10}$$

$$\frac{\partial W_j}{\partial\phi}(\rho, (-1)^{j-1}\pi) = (-1)^j \frac{aQ_j}{\mu_j}\frac{\rho}{\sqrt{1 - (\rho - \delta)^2}}, \quad \delta - 1 < \rho < \delta + 1 \tag{11a}$$

$$= 0 \qquad \text{otherwise.} \tag{11b}$$

The behaviours of the stresses are

$$\sigma_{j\rho z}(\rho, \phi) = \sigma_{j\phi z}(\rho, \phi) = 0(\rho^{-3/2}) \quad \text{as} \quad \rho \to 0$$

$$= 0(\rho^{-1/2}) \quad \text{as} \quad \rho \to \infty.$$

The Mellin transformation applied to (9)-(11) yields

$$\left(\frac{d^2}{ds^2} + s^2\right)\overline{W_j}(s, \phi) = 0, \quad -\frac{1}{2} < \operatorname{Re} s < \frac{1}{2}, \quad j = 1, 2, \tag{12}$$

$$\overline{W_1}(s, 0) = \overline{W_2}(s, 0); \quad \mu_1 \frac{dW_1}{d\phi}(s, 0) = \mu_2 \frac{dW_2}{d\phi}(s, 0), \tag{13}$$

$$\frac{d^2\overline{W_j}}{d\phi^2}(s, (-1)^{j-1}\pi) = (-1)^j \frac{aQ}{\mu_j} h(a, b; s), \tag{14}$$

where

$$h(a, b; s) = \int_{\partial-1}^{\partial+1} \frac{\rho^s d\rho}{\sqrt{1-(\rho-\delta)^2}}, \tag{15}$$

and the Mellin transformation is defined by

$$\overline{W_j}(s, \phi) = \int_0^\infty W_j(\rho, \phi)\rho^{s-1}d\rho, \quad -\frac{1}{2} < \operatorname{Re} s < \frac{1}{2}.$$

The displacement is then given by the inversion formula defined by

$$W_j(\rho, \phi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i} \overline{W_j}(s, \phi)\rho^{-s}ds, \quad -\frac{1}{2} < c < \frac{1}{2}. \tag{16}$$

4. Solution of the Reformulated Problem

Consider the solution of (12) given as

$$\overline{W_j}(s, \phi) = A_j(s)\sin s\phi + B_j(s)\cos s\phi, \quad j = 1, 2. \tag{17}$$

Application of (13) and (14) to (17) gives

$$B_1(s) = B_2(s) \quad \text{and} \quad A_2(s) = \frac{\mu_1}{\mu_2} A_1(s), \tag{18}$$

$$A_j = \frac{a}{2\mu_j}\{(1+\gamma)Q_2 - (1-\gamma)Q_1\}\frac{h(a,\ b;\ s)}{s\cos\pi s}, \tag{19}$$

$$B_j = \frac{a}{2}\left\{(1+\gamma)\frac{Q_1}{\mu_1} + (1-\gamma)\frac{Q_2}{\mu_2}\right\}\frac{h(a,\ b;\ s)}{s\cos\pi s}. \tag{20}$$

In view of (19), (20) and (16), we have

$$W_j(\rho,\ \phi) = \frac{a}{2\mu_j}\{(1+\gamma)Q_2 - (1-\gamma)Q_1\}\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} h(a,\ b;\ s)\rho^{-s}\,\frac{\sin s\phi}{s\cos\pi s}\,ds\right)$$

$$+ \frac{a}{2}\left\{(1+\gamma)\frac{Q_1}{\mu_1} + (1-\gamma)\frac{Q_2}{\mu_2}\right\}\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} h(a,\ b;\ s)\rho^{-s}\,\frac{\cos s\phi}{s\sin\pi s}\,ds\right). \tag{21}$$

Series technique and the method of residues are applied to evaluate (21). The order of singularities of $h(a,\ b;\ s)$ is incorporated through term-by-term integration of a series of the form

$$(1-\tau)^{-1/2} = \sum_{k=0}^{\infty} b_k\tau^k, \quad |\tau| < 1, \quad b_k = \frac{(2k)!}{2^{2k}(k!)^2}.$$

The appropriate form for $\delta - 1 < \rho < \delta + 1$ is

$$\rho^s\{(1-\rho+\delta)(1+\rho-\delta)\}^{-1/2} = \rho^{s-1/2}\lambda^{1/2}(1-\lambda\rho)^{-1/2}(1-\varepsilon\rho^{-1})^{-1/2},$$

where $\lambda = (\delta+1)^{-1}$ and $\varepsilon = \delta - 1$.

Therefore

$$h(a,\ b;\ s) = (\delta+1)^{-1/2}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} b_k b_m \lambda^k\varepsilon^m\,\frac{\rho^{s+k-m+1/2}}{s+k-m+1/2}\Bigg|_{\delta-1}^{\delta+1}.$$

To apply Jordan's Lemma [1] the poles of $h(a,\ b;\ s)$ are separated into those in the right half plane $\operatorname{Re} s > 0$ for which $k < m$ and those in the left half plane $\operatorname{Re} s < 0$ for which $k \geq m$:

The separation is attained by writing

$$M(t;\ s) = \sum_{m=0}^{\infty}\sum_{k=m+1}^{\infty} b_k b_m\lambda^k\varepsilon^m\,\frac{t^{k-m+1/2}}{s+k-m+1/2} + \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\varepsilon^m b_k b_m\lambda^k\,\frac{t^{1/2}}{s+1/2}$$

$$+ \sum_{k=0}^{\infty}\sum_{m=k+1}^{\infty} b_k b_m\lambda^k\varepsilon^m\,\frac{t^{k-m+1/2}}{s+k-m+1/2}, \tag{22}$$

so that

$$h(a,\, b;\, s) = (\delta + 1)^{-1/2}\{M(\delta + 1;\, s)(\delta + 1)^{s} - M(\delta - 1;\, s)(\delta - 1)^{s}\}. \qquad (23)$$

The integrals to be evaluated are as follows:

$$I^{(1)}_{\delta+1}(\rho,\, \phi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M(\delta + 1;\, s)\left(\frac{\rho}{\delta + 1}\right)^{-s} \frac{\sin s\phi}{s\cos \pi s}\, ds, \quad \rho < \delta + 1, \quad k \geq m,$$

$$I^{(1)}_{\delta-1}(\rho,\, \phi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M(\delta - 1;\, s)\left(\frac{\rho}{\delta - 1}\right)^{-s} \frac{\sin s\phi}{s\cos \pi s}\, ds, \quad \rho > \delta - 1, \quad k < m,$$

$$I^{(2)}_{\delta+1}(\rho,\, \phi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M(\delta + 1;\, s)\left(\frac{\rho}{\delta + 1}\right)^{-s} \frac{\cos s\phi}{s\sin \pi s}\, ds, \quad \rho < \delta + 1, \quad k \geq m,$$

$$I^{(2)}_{\delta-1}(\rho,\, \phi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M(\delta - 1;\, s)\left(\frac{\rho}{\delta - 1}\right)^{-s} \frac{\cos s\phi}{s\sin \pi s}\, ds, \quad \rho > \delta - 1, \quad k < m.$$

The solution for $\delta - 1 < \rho < \delta + 1$ is therefore

$$W_j(\rho,\, \phi) = (\delta + 1)^{-1/2}\frac{a}{2}\left[\frac{1}{\mu_j}\{(1 + \gamma)Q_2 - (1 - \gamma)Q_1\}\{I^{(1)}_{\delta+1}(\rho,\, \phi) + I^{(1)}_{\delta-1}(\rho,\, \phi)\}\right.$$
$$\left. + \left\{(1 + \gamma)\frac{Q_1}{\mu_1} + (1 - \gamma)\frac{Q_2}{\mu_2}\right\}\{I^{(2)}_{\delta+1}(\rho,\, \phi) + I^{(2)}_{\delta-1}(\rho,\, \phi)\}\right].$$

For $j = 1,\, 2,$ let

$$q^{(1)}_j(\rho,\, \phi) = \ln\left(\frac{\rho}{\delta + (-1)^{j-1}}\right)\sin\left(n - \frac{1}{2}\right)\phi$$
$$+ (-1)^{j-1}\phi\cos\left(n - \frac{1}{2}\right)\phi + \frac{2(-1)^{j}\sin\left(n - \frac{1}{2}\right)}{2n - 1}\phi,$$

$$q^{(2)}_j(\rho,\, \phi) = (-1)^{j}\ln\left(\frac{\rho}{\delta + (-1)^{j-1}}\right)\cos\left(n - \frac{1}{2}\right)\phi$$
$$+ \phi\sin\left(n - \frac{1}{2}\right)\phi + \frac{2}{2n - 1}\cos\left(n - \frac{1}{2}\right)\phi.$$

We have

$$I_{\delta+1}^{(1)}(\rho,\ \phi) = \frac{2}{\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}b_{n-1+m}b_m\lambda^{n-1+m}\varepsilon^m q_1^{(1)}(\rho,\ \phi)\rho^{n-1/2},$$

$$I_{\delta-1}^{(1)}(\rho,\ \phi) = \frac{2}{\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}b_m b_{m+n}\lambda^n\varepsilon^{m+n}q_2^{(1)}(\rho,\ \phi)\rho^{1/2-n},$$

$$I_{\delta+1}^{(2)}(\rho,\ \phi) = \frac{2}{\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}b_{n-1+m}b_m\lambda^{n-1+m}\varepsilon^m q_1^{(2)}(\rho,\ \phi)\rho^{n-1/2},$$

$$I_{\delta-1}^{(2)}(\rho,\ \phi) = \frac{2}{\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}b_m b_{m+n}\lambda^m\varepsilon^{m+n}q_2^{(2)}(\rho,\ \phi)\rho^{1/2-n}.$$

5. Crack Tip Equations

The characteristics fracture parameters at the crack tip are obtained asymptotically when $0 < \rho < 1$ from (21) as $\rho \to 0.$ That is

$$W_j(\rho,\ \phi) = \frac{a}{2\mu_j}\{(1+\gamma)Q_2 - (1-\gamma)Q_1\}h\left(a,\ b;\ -\frac{1}{2}\right)\rho^{1/2}\sin\frac{\phi}{2}\ \text{ as } \rho \to 0. \quad (24)$$

From (15) and entry 3. 131 6 [2],

$$h\left(a,\ b;\ -\frac{1}{2}\right) = \frac{2}{(1+\delta)^{1/2}}F\left(\frac{\pi}{2},\ \sqrt{\frac{2}{1+\delta}}\right),$$

where $F\left(\frac{\pi}{2},\ p\right)$ is the elliptic integral of the first kind; since

$$\sqrt{\frac{2}{1+\delta}} = \frac{\text{geometric mean of } a \text{ and } b}{\text{arithmetic mean of } a \text{ and } b} < 1.$$

Let $(R,\ \psi)$ be polar coordinates at the crack tip (Figure 1). In view of

$$r\cos\theta = b + R\cos\psi \quad \text{and} \quad r\sin\theta = R\sin\psi,$$

(6) leads to

$$\rho e^{i\phi} = \frac{1}{2a}\left(1 - \frac{a^2}{b^2}\right)\mathrm{Re}^{i\psi} + 0\left[\left(\frac{R}{b}\right)^2\right], \quad b > a.$$

Hence

$$\rho^{1/2} e^{i(\phi/2)} = \left[\frac{1}{2a}\left(1 - \frac{a^2}{b^2}\right)\right]^{1/2} R^{1/2} e^{i(\psi/2)} \quad \text{as} \quad R \to 0.$$

The implication of these on (24) is

$$W_j(R, \psi) = \{(1+\gamma)Q_2 - (1-\gamma)Q_1\} \frac{1}{\mu_j}\left(\frac{a}{\pi}\right)^{1/2}\left(1 - \frac{a^2}{b^2}\right)^{1/2}$$

$$\times h\left(a, b; -\frac{1}{2}\right)\left(\frac{R}{\pi}\right)^{1/2} \sin\frac{\psi}{2} \quad \text{as} \quad R \to 0. \tag{25}$$

The corresponding stress intensity factor is

$$K_{III}(b-a; Q_1, Q_2) = (b-a)^{1/2}\{(1+\gamma)Q^2 - (1-\gamma)Q_1\}$$

$$\times \left\{\frac{1}{\pi}\frac{a}{b}\left(1 + \frac{a}{b}\right)\right\}^{1/2} h\left(a, b; -\frac{1}{2}\right). \tag{26}$$

The energy release rate is

$$G = \frac{1}{4}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right) K_{III}^2(b-a; Q_1, Q_2; \gamma).$$

6. Conclusion

The crack tip displacement fields are derived in terms of the elliptic integral of the first kind and the stress intensity factor is obtained in the standard form [3]

$$K_{III}(b-a; Q_1, Q_2; \gamma) = K\left(\frac{a}{b}\right)(\pi a)^{1/2}\{(1+\gamma)Q_2 - (1-\gamma)Q_1\},$$

where

$$K\left(\frac{a}{b}\right) = \frac{2\sqrt{2}}{\pi}\left(\frac{b-a}{b+a}\right)^{1/2}\left(\frac{a}{b}\right)^{1/2} F\left(\frac{\pi}{2}, \frac{2\sqrt[2]{\frac{a}{b}}}{1 + \frac{a}{b}}\right).$$

Let $Q_2 = \beta Q_1$. Then

$$K_{III}(b-a; Q_1, Q_2; \gamma) = \{\beta(1+\gamma) + \gamma - 1\} K\left(\frac{a}{b}\right) K_m^0(a, Q_1), \tag{27}$$

where $K_{III}^{0}(a, Q_1) = (\pi a)^{1/2} Q_1$ is the known stress intensity factor for a tunnel crack of width a in an infinite homogeneous material under anti-plane shear [5]. The case $\beta = -1$ leads to self equilibrated tractions.

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