Far East Journal of Applied Mathematics
Volume 41, Number 2, 2010, Pages 145-152
Published Online: June 18, 2010
This paper is available online at http://pphmj.com/journals/fjam.htm

A NEW HIERARCHY OF EVOLUTION EQUATIONS AND SOLITON SOLUTIONS

GUO XIURONG

Department of Public Course
Shandong University of Science and Technology
Taian 271019, P. R. China

Abstract

With the help of the Zakharov-Shabat eigenvalue problem, a new soliton hierarchy of evolution equations is obtained. A reduced case of the hierarchy is presented, which are the well-known AKNS equations, whose soliton solutions are produced. The approach has extensive applications.

1. Introduction

Researching for new integrable Hamiltonian systems has been a topic for us, for which some efficient methods have been proposed in [1, 2]. Tu [9] introduced a straightforward method for generating integrable Hamiltonian systems called the Tu scheme by Ma [7]. After this, some interesting soliton hierarchies of evolution equations were obtained [3, 4, 6, 8]. The Tu scheme is presented briefly in the following.

Let G be a matrix Lie algebra over a field C and $\tilde{G}G \otimes C(\lambda, \lambda^{-1})$ be a resulting loop algebra, where $C(\lambda, \lambda^{-1})$ stands for a set of Laurent polynomials in the parameter λ.

2010 Mathematics Subject Classification: 17B, 35C08.

Keywords and phrases: Lie algebra, loop algebra, soliton solution.

This work is supported by Shandong University of Science and Technology, Grant No. 2009AZZ090.

Communicated by Yufeng Zhang

Received March 7, 2010

First, solving the stationary zero curvature equation for $V = V(u, \lambda)$:

$$V_x = [U, V] \tag{1}$$

so that for $V^{(n)} = (\lambda^n V)_+ + \Delta_n$, we have

$$V_x^{(n)} - [U, V^{(n)}] = Ce_1 + \cdots + Ce_p, \tag{2}$$

where $[U, V] = UV - VU$, and $\{e_1, \ldots, e_p\}$ is a set of basis of the Lie algebra G, $(\lambda^n V)_+ = \sum_{m=0}^{n} V_m \lambda^{n-m}$, $V_m = \sum_{i=1}^{p} a_{im} e_i(0)$, a_{im} $(i = 1, 2, \ldots, p)$ represent the smooth functions in x and t.

Second, use of the zero curvature equation

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0, \tag{3}$$

which is the compatibility condition of the Lax pair

$$\varphi_x = U\varphi, \quad \varphi_t = V^{(n)}\varphi \tag{4}$$

deduces an integrable hierarchy

$$u_t = K(u). \tag{5}$$

Third, search for a Hamiltonian operator J and a recurrence operator L from equation (1) so that the hierarchy (5) can be written as the Hamiltonian form

$$u_t = J \frac{\delta H_n}{\delta u} = JL \frac{\delta H_{n-1}}{\delta u}, \tag{6}$$

where $H_n (z \in Z)$ are common conserved densities of equation (5), which can be deduced by the trace identity [9]

$$\frac{\delta}{\delta u}\left\langle V, \frac{\partial U}{\partial \lambda}\right\rangle = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \left\langle V, \frac{\partial U}{\partial u_i}\right\rangle, \quad i = 1, 2, \ldots, p, \tag{7}$$

where $\langle A, B\rangle$ represents matrix traces, $A, B \in \tilde{G}$. If we could prove $JL = L^* J$, then the hierarchy (5) is Liouville integrable.

In this paper, we want to introduce the famous Zakharov-Shabat eigenvalue problem and another isospectral problem to constitute an isospectral Lax pair from their compatibility condition that a generalized zero curvature equation is obtained. By employing the Tu scheme, we work out a new soliton hierarchy of evolution equations. A reduced case of the hierarchy is just right coupled AKNS equations, whose exact solitary solutions are obtained by using logarithm expansion.

2. A New Integrable Hamiltonian Hierarchy

Let [9]

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Then the Lie algebra A_1 can be expressed by

$$A_1 = \operatorname{span}\{h, e, f\}.$$

The resulting loop algebra reads

$$\tilde{A}_1 = \operatorname{span}\{h(n), e(n), f(n)\}, \quad h(n) = h\lambda^n, \quad e(n) = e\lambda^n, \quad f(n) = f\lambda^n. \tag{8}$$

Introducing a linear operator

$$L(X) = [U, X] + X, \quad U, X \in \tilde{A}_1,$$

we consider the Lax pair

$$\varphi_x = U\varphi, \quad \varphi_t = L(N)\varphi, \tag{9}$$

where

$$U = h(1) + qe(0) + rf(0), \quad N = \sum_{m \geq 0} (a_m h(-m) + b_m e(-m) + c_m f(-m)).$$

The compatibility condition of equation (9) presents that

$$U_t - L_x(N) + [U, L(N)] = 0. \tag{10}$$

The stationary form of equation (10) admits the following recurrence relations for N:

$$\begin{cases} (a_m + qc_m - rb_m)_x = 2qc_{m+1} - 2rb_{m+1} - r(b_m - 2qa_m) - q(2ra_m + c_m), \\ 2b_{m+1x} + (b_m - 2qa_m)_x = 2(b_{m+1} - 2qa_{m+1}) - 2q(a_m + qc_m - rb_m) + 4b_{m+2}, \\ (2ra_m + c_m - 2c_{m+1})_x = -2(2ra_{m+1} + c_{m+1}) + 4c_{m+2} + 2r(a_m + qc_m - rb_m). \end{cases} \tag{11}$$

Set $a_0 = b_0 = c_0 = b_1 = c_1 = 0$, $a_1 = \alpha = \text{const.}$, then from (11), we infer that $a_2 = 0$, $b_2 = \alpha q$, $c_2 = \alpha r$, $a_3 = -\frac{\alpha}{2} qr$, $b_3 = c_3 = a_4 = 0$, $b_4 = -\frac{\alpha}{2} q^2 r + \frac{\alpha}{4} q_x$, $c_4 = -\frac{\alpha}{2} qr^2 + \frac{\alpha}{4} r_x$, $b_5 = \frac{\alpha}{8}(q_{xx} - q_x)$, $c_5 = \frac{\alpha}{8}(-r_{xx} + r_x)$, ...

Denote by $L_+^{(n)} = [U, N_+^{(n)}] + N_+^{(n)}$, $N_+^{(n)} = \sum_{m=0}^{n} (a_m h(-m) + b_m e(-m) + c_m f(-m))\lambda^n$, a direct calculation reads that

$$\begin{aligned} -L_{+x}^{(n)} + [U, L_+^{(n)}] = {} & (4qa_{n+1} - 2b_{n+1} + 2b_{n+1x} - 4b_{n+2})e(0) \\ & + (4ra_{n+1} + 2c_{n+1} - 2c_{n+1x} - 4c_{n+2}) f(0) \\ & + (2qc_{n+1} - 2rb_{n+1})h(0) - 4b_{n+1}e(1) - 3c_{n+1} f(1). \end{aligned}$$

Note $\Delta_n = 2b_{n+1}e(0) - 2c_{n+1} f(0)$, $L^{(n)}(N) = L^{(n)}(N)_+ + \Delta_n$, we infer that

$$\begin{aligned} & -L_x^{(n)}(N) + [U, L^{(n)}(N)] \\ & = (4qa_{n+1} - 2b_{n+1} - 4b_{n+2})e(0) + (4ra_{n+1} + 2c_{n+1} - 4c_{n+2}) f(0). \end{aligned}$$

Hence, the generalized zero curvature equation

$$U_t - L_x^{(n)}(N) + [U, L^{(n)}(N)] = 0 \tag{12}$$

gives rise to the integrable hierarchy

$$\begin{aligned} u_t &= \begin{pmatrix} q \\ r \end{pmatrix}_t \\ &= \begin{pmatrix} -4qa_{n+1} + 2b_{n+1} + 4b_{n+2} \\ -4ra_{n+1} - 2c_{n+1} + 4c_{n+2} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} -2c_{n+2} + c_{n+1} + 2qa_{n+1} \\ 2b_{n+2} + b_{n+1} - 2qa_{n+1} \end{pmatrix}$$

$$= J \begin{pmatrix} -2c_{n+2} + c_{n+1} + 2qa_{n+1} \\ 2b_{n+2} + b_{n+1} - 2qa_{n+1} \end{pmatrix}, \tag{13}$$

where $J = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$ is a Hamiltonian operator obviously.

However, the trace identity fails to deduce the Hamiltonian structure of (13). As for how to solve it, it is an open problem.

Taking $\alpha = 2,\ n = 3$ in (13), we obtain

$$\begin{cases} q_t = -2q^2 r + q_{xx}, \\ r_t = 2qr^2 - r_{xx}. \end{cases} \tag{14}$$

Taking $\alpha = 2,\ n = 4$ in (13), we obtain

$$\begin{cases} q_t = 6qq_x r - q_{xxx}, \\ r_t = 6qrr_x - r_{xxx}. \end{cases} \tag{15}$$

Equations (14) and (15) are exactly the coupled equations for the case $n = 1$ and $n = 2$ in AKNS hierarchy.

3. Soliton Solutions

In what follows, we shall deduce the exact soliton solutions of equations (14) by using a logarithm expansion.

Set $q = q(x, t) = q(\xi)$, $r = r(x, t) = r(\xi)$, $\xi = x - ct$ and substituting into equations (14), we have the ordinary differential equations in the variable ξ

$$\begin{cases} -cq' = -2q^2 r + q'', \\ -cr' = 2qr^2 - r''. \end{cases} \tag{16}$$

Set

$$q = \sum_{m=0}^{n_1} a_m (\ln f)_\xi^m, \quad r = \sum_{m=0}^{n_2} b_m (\ln f)_\xi^m.$$

By balancing the nonlinear terms and the highest derivative terms, we may take $n_1 = n_2 = 1.$ Hence, we have

$$q = a_0 + a_1(\ln f)_\xi, \quad r = b_0 + b_1(\ln f)_\xi, \tag{17}$$

where

$$(\ln f)_\xi = \frac{f'(\xi)}{f(\xi)} = \frac{f'}{f}.$$

Assume that $f(\xi)$ satisfies the equation [5]

$$f'' = \alpha f' + \beta f, \tag{18}$$

where α, β are all constants.

Substituting (17), (18) into (16) and comparing the coefficients of $(\ln f)_\xi^i$ $(i = 0, 1, 2, 3)$, we infer that

$$\begin{cases} -ca_1\beta = -2a_0^2b_0 + a_1\alpha\beta, \\ -ca_1\alpha = -2(a_0^2b_1 + 2a_0a_1b_0) + a_1(\alpha^2 - 2\beta), \\ ca_1 = -2(2a_0a_1b_1 + a_1^2b_0) - 3a_1\alpha, \\ -a_1^2b_1 + a_1 = 0, \\ -cb_1\beta = 2a_0b_0^2 - \alpha\beta b_1, \\ -\alpha b_1c = 4a_0b_0b_1 + 2a_1b_0^2 - \alpha^2b_1 + 2\beta b_1, \\ cb_1 = 2a_0b_1^2 + 4a_1b_0b_1 + 3\alpha b_1, \\ a_1b_1^2 - b_1 = 0. \end{cases} \tag{19}$$

A solution of (19) is obtained by using Maple

$$\begin{cases} a_1 = \dfrac{a_0(\sqrt{4\beta + \alpha^2} - 3\alpha)}{\sqrt{4\beta + \alpha^2}\alpha + \alpha^2 - 2\beta}, \quad b_0 = -\dfrac{\beta}{a_0}, \quad c = \sqrt{4\beta + \alpha^2}, \\ b_1 = -\dfrac{\sqrt{4\beta + \alpha^2}\alpha^4 - \sqrt{4\beta + \alpha^2}\beta\alpha^2 + \sqrt{4\beta + \alpha^2}\beta^2 + \alpha^5 + \beta\alpha^3 - 3\alpha\beta^2}{a_0(\alpha^3\sqrt{4\beta + \alpha^2} - 2\alpha\beta\sqrt{4\beta + \alpha^2} + \alpha^4 + 2\beta^2)}, \end{cases} \tag{20}$$

where a_0 is a nonzero constant.

Inserting (20) into (17), the soliton solutions of equations (14) are given by

$$\begin{cases} q = \dfrac{a_0 c_2 e^{\frac{1}{2}(\alpha - \sqrt{4\beta+\alpha^2})\xi}(3\sqrt{4\beta+\alpha^2}\alpha - \alpha^2 - 4\beta)}{(\sqrt{4\beta+\alpha^2}\alpha + \alpha^2 - 2\beta)(c_1 e^{\frac{1}{2}(\alpha+\sqrt{4\beta+\alpha^2})\xi} + c_2 e^{\frac{1}{2}(\alpha-\sqrt{4\beta+\alpha^2})\xi})}, \\ r = -\dfrac{c_1 e^{\frac{1}{2}(\alpha+\sqrt{4\beta+\alpha^2})\xi}(\alpha\sqrt{4\beta+\alpha^2}(\beta\alpha^2 - 3\beta^2 + \alpha^4) + 3\alpha^4\beta + 4\beta^3 + \alpha^6 - 3\beta^2\alpha^2)}{a_0(\alpha\sqrt{4\beta+\alpha^2}(\alpha^2 - 2\beta) + \alpha^4 + 2\beta^2)(c_1 e^{\frac{1}{2}(\alpha+\sqrt{4\beta+\alpha^2})\xi} + c_2 e^{\frac{1}{2}(\alpha-\sqrt{4\beta+\alpha^2})\xi})}, \end{cases} \tag{21}$$

where c_1, c_2 are arbitrary constants, $\xi = x - ct = x - \sqrt{4\beta + \alpha^2}t$.

Then let us consider the property of the exact solution (21) of equations (14) with the case of $\alpha = \beta = a_0 = 1 = C_1 = 1$, $C_2 = -1$, respectively.

Similarly, we can find the soliton solutions of the second AKNS coupled equations (15) as follows:

$$\begin{cases} q = \dfrac{c_1(2a_0 + a_1(\alpha + \sqrt{4\beta+\alpha^2}))e^{\frac{1}{2}(\alpha+\sqrt{4\beta+\alpha^2})\xi} + c_2(2a_0 + a_1(\alpha - \sqrt{4\beta+\alpha^2}))e^{\frac{1}{2}(\alpha-\sqrt{4\beta+\alpha^2})\xi}}{2(c_1 e^{\frac{1}{2}(\alpha+\sqrt{4\beta+\alpha^2})\xi} + c_2 e^{\frac{1}{2}(\alpha-\sqrt{4\beta+\alpha^2})\xi})}, \\ r = \dfrac{c_1(2a_0 + a_1(\alpha - \sqrt{4\beta+\alpha^2}))e^{\frac{1}{2}(\alpha+\sqrt{4\beta+\alpha^2})\xi} + c_2(2a_0 + a_1(\alpha + \sqrt{4\beta+\alpha^2}))e^{\frac{1}{2}(\alpha-\sqrt{4\beta+\alpha^2})\xi}}{-2a_1^2(c_1 e^{\frac{1}{2}(\alpha+\sqrt{4\beta+\alpha^2})\xi} + c_2 e^{\frac{1}{2}(\alpha-\sqrt{4\beta+\alpha^2})\xi})}, \end{cases} \tag{22}$$

where a_0, $a_1 \neq 0$, c_1, c_2 are arbitrary constants, $\xi = x + \frac{t}{a_1^2}(\alpha^2 a_1^2 + 6a_0 a_1 \alpha + 6a_0^2 - 2a_1^2\beta)$.

The property of the exact solution (22) of equations (15) with the case of $\alpha = \beta = a_0 = 1 = C_1 = a_1 = 1$, $C_2 = -3$ can be seen from the following graph.

4. Conclusion

We have derived a new soliton hierarchy of evolution equations by using the Zakharov-Shabat eigenvalue problem. A reduced case of the hierarchy is exactly the well-known AKNS equations. Also some soliton solutions are obtained. The approach can be applied into other extensive equations.

References

[1] M. J. Ablowitz and H. Segur, Solitons and Inverse Scattering Transformation, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Penn, 1981.

[2] V. I. Arnold, Mathematical Methods of Classical Mechanics, Springer, Berlin, 1978.

[3] Engui Fan, Phys. Lett. A 274 (2000), 135.

[4] Engui Fan, J. Math. Phys. 42 (2001), 4327.

[5] D. D. Ganji and M. Abdollahzadeh, J. Math. Phys. 50 (2009), 013519.

[6] Fukui Guo and Yufeng Zhang, J. Math. Phys. 44(2) (2003), 5793.

[7] Wen-Xiu Ma, Chinese J. Contemp. Math. 13 (1992), 79.

[8] Wen-Xiu Ma, J. Phys. A 25 (1992), 5329.

[9] G. Z. Tu, J. Math. Phys. 30(2) (1989), 330.