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THREE DIMENSIONAL QUARTIC FUNCTIONAL EQUATION IN FUZZY NORMED SPACES

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Abstract

In this paper, the author investigates the generalized Hyers-Ulam-Rassias stability of a three dimensional quartic functional equation

$$\begin{aligned} & g(2x+y+z)+g(2x+y-z)+g(2x-y+z) \\ & +g(-2x+y+z)+16g(y)+16g(z) \\ = & 8[g(x+y)+g(x-y)+g(x+z)+g(x-z)] \\ & +2[g(y+z)+g(y-z)]+32g(x) \end{aligned} \tag{0.1}$$

in fuzzy normed space.

1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [28] concerning the stability of group homomorphisms. Hyers [9] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [2] for additive mappings and by Rassias [23] for

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linear mappings by considering an unbounded Cauchy difference. The paper of Rassias [23] has provided a lot of influences in the development of what we call generalized Hyers-Ulam stability of functional equations. A generalization of the Rassias theorem was obtained by Găvruța [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach. In 1982, Rassias [22] followed the innovative approach of the Rassias theorem [23] in which he replaced the factor $\| x \|^p + \| y \|^p$ by $\| x \|^p \| y \|^p$ for $p, q \in R$ with $p + q = 1$.

In 2008, a special case of Găvruța's theorem for the unbounded Cauchy difference was obtained by Ravi et al. [26] by considering the summation of both the sum and the product of two p-norms in the spirit of Rassias approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 6, 10, 12, 13, 24, 25]).

Katsaras [14] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [7, 16, 29]. In particular, Bag and Samanta [3], following Cheng and Mordeson [5], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michálek type [15]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [4].

We use the definition of fuzzy normed spaces given in [3] and [18-21].

Definition 1.1. Let X be a real linear space. Then a function $N : X \times \mathbb{R} \to [0, 1]$ (the so-called *fuzzy subset*) is said to be a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(F1) $N(x, c) = 0$ for $c \leq 0$;

(F2) $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$;

(F3) $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$ if $c \neq 0$;

(F4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;

(F5) $N(x, \cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;

(F6) for $x \neq 0$, $N(x, \cdot)$ is (upper semi) continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed linear space*. We may regard $N(X, t)$ as the truth-value of the statement 'the norm of x is less than or equal to the real number t'.

Example 1.2. Let $(X, \|\cdot\|)$ be a normed linear space. Then

$$N(x, t) = \begin{cases} \dfrac{t}{t + \|x\|}, & t > 0, \quad x \in X, \\ 0, & t \leq 0, \quad x \in X \end{cases}$$

is a fuzzy norm on X.

Example 1.3. Let $(X, \|\cdot\|)$ be a normed linear space. Then

$$N(x, t) = \begin{cases} 0, & t \leq 0, \\ \dfrac{t}{\|x\|}, & 0 < t \leq \|x\|, \\ 1, & t > \|x\| \end{cases}$$

is a fuzzy norm on X.

Definition 1.4. Let (X, N) be a fuzzy normed linear space and x_n be a sequence in X. Then x_n is said to be *convergent* if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the *limit* of the sequence x_n and we denote it by $N - \lim_{n\to\infty} x_n = x$.

Definition 1.5. A sequence x_n in X is called *Cauchy* if for each $\varepsilon > 0$ and $t > 0$ there exists n_0 such that for all $n \geq n_0$ and $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

Definition 1.6. Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed space is called a *fuzzy Banach space*.

Definition 1.7. A mapping $f : X \to Y$ between fuzzy normed spaces X and Y is continuous at a point x_0 if for each sequence $\{x_n\}$ covering to x_0 in X, the sequence $f\{x_n\}$ converges to $f(x_0)$. If f is continuous at each point of $x_0 \in X$, then f is said to be *continuous* on X.

The stability of various functional equations in fuzzy normed spaces was investigated in [11, 17-21, 27].

In this paper, the author investigates a fuzzy version of the generalized Hyers-Ulam-Rassias stability of a three dimensional quartic functional equation

$$\begin{aligned} & g(2x+y+z)+g(2x+y-z)+g(2x-y+z) \\ & + g(-2x+y+z)+16g(y)+16g(z) \\ = \; & 8[g(x+y)+g(x-y)+g(x+z)+g(x-z)] \\ & +2[g(y+z)+g(y-z)]+32g(x) \end{aligned} \tag{1.1}$$

in the fuzzy normed vector space setting.

2. Fuzzy Stability of the Quartic Functional Equation (1.1)

Throughout this section, assume that X, (Z, N') and (Y, N') are linear space, fuzzy normed space and fuzzy Banach space, respectively. Now we use the following notation for a given mapping $f : X \to Y$:

$$\begin{aligned} Dg(x, y, z) = \; & g(2x+y+z)+g(2x+y-z) \\ & + g(2x-y+z)+g(-2x+y+z)+16g(y)+16g(z) \\ & -8[g(x+y)+g(x-y)+g(x+z)+g(x-z)] \\ & -2[g(y+z)+g(y-z)]-32g(x) \end{aligned}$$

for all $x, y, z \in X$.

Now the author investigates the generalized Hyers-Ulam-Rassias stability of the functional equation (1.1).

Theorem 2.1. *Let* $\beta \in \{-1, 1\}$ *be fixed and* $\alpha : X^3 \to Z$ *be a mapping such that for some a with* $0 < \left(\frac{a}{16}\right)^{\beta} < 1$,

$$N'(\alpha(2^{\beta}x, 0, 0), r) \geq N'(a^{\beta}\alpha(x, 0, 0), r) \tag{2.1}$$

for all $x \in X$ *and* $a > 0$, *and*

$$\lim_{n\to\infty} N'(\alpha(2^{\beta n}x, 2^{\beta n}y, 2^{\beta n}z), 16^{\beta n}r) = 1 \tag{2.2}$$

for all $x \in X$ *and* $r > 0$. *Suppose that a function* $g : X \to Y$ *satisfies the inequality*

$$N(Dg(x, y, z), r) \geq N'(\alpha(x, y, z), r) \tag{2.3}$$

for all $r > 0$ *and* $x, y \in X$. *Then the limit*

$$Q(x) = N - \lim_{n \to \infty} \frac{g(2^{\beta n} x)}{16^{\beta n}} \tag{2.4}$$

exists for all $x \in X$ *and the mapping* $Q : X \to Y$ *is a unique quartic mapping such that*

$$N(g(x) - Q(x), r) \geq N'(\alpha(x, 0, 0), r|16 - a|) \tag{2.5}$$

for all $x \in X$ *and* $r > 0$.

Proof. First assume $\beta = 1$. Replacing (x, y, z) by $(x, 0, 0)$ in (2.3), we get

$$N(g(2x) - 16g(x), r) \geq N'(\alpha(x, 0, 0), r) \tag{2.6}$$

for all $x \in X$ and $r > 0$. Replacing x by $2^n x$ in (2.6), we obtain

$$N\left(\frac{g(2^{n+1}x)}{16} - g(2^n x), \frac{r}{16}\right) \geq N'(\alpha(2^n x, 0, 0), r) \tag{2.7}$$

for all $x \in X$ and $r > 0$. Using (2.1), (F3) in (2.7), we arrive

$$N\left(\frac{g(2^{n+1}x)}{16} - g(2^n x), \frac{r}{16}\right) \geq N'\left(\alpha(x, 0, 0), \frac{r}{a^n}\right) \tag{2.8}$$

for all $x \in X$ and $r > 0$. It is easy to verify from (2.8) that

$$N\left(\frac{g(2^{n+1}x)}{16^{n+1}} - \frac{g(2^n x)}{16^n}, \frac{r}{16 \cdot 16^n}\right) \geq N'\left(\alpha(x, 0, 0), \frac{r}{a^n}\right) \tag{2.9}$$

holds for all $x \in X$ and $r > 0$. Replacing r by $a^n r$ in (2.9), we get

$$N\left(\frac{g(2^{n+1}x)}{16^{n+1}} - \frac{g(2^n x)}{16^n}, \frac{a^n r}{16 \cdot 16^n}\right) \geq N'(\alpha(x, 0, 0), r) \tag{2.10}$$

for all $x \in X$ and $r > 0$. It is easy to see that

$$\frac{g(2^n x)}{16^n} - g(x) = \sum_{i=0}^{n-1} \frac{g(2^{i+1} x)}{16^{i+1}} - \frac{g(2^i x)}{16^i} \tag{2.11}$$

for all $x \in X$. From equations (2.10) and (2.11), we have

$$\begin{aligned} N\left(\frac{g(2^n x)}{16^n} - g(x), \sum_{i=0}^{n-1} \frac{a^i r}{16 \cdot 16^i}\right) &\geq \min \bigcup_{i=0}^{n-1} \left\{\frac{g(2^{i+1} x)}{16^{i+1}} - \frac{g(2^i x)}{16^i}, \frac{a^i r}{16 \cdot 16^i}\right\} \\ &\geq \min \bigcup_{i=0}^{n-1} \{N'(\alpha(x, 0, 0), r)\} \\ &\geq N'(\alpha(x, 0, 0), r) \end{aligned} \tag{2.12}$$

for all $x \in X$ and $r > 0$. Replacing x by $2^m x$ in (2.12) and using (2.1), (F3), we obtain

$$N\left(\frac{g(2^{n+m} x)}{16^{n+m}} - \frac{g(2^m x)}{16^m}, \sum_{i=0}^{n-1} \frac{a^i r}{16 \cdot 16^i}\right) \geq N'\left(\alpha(x, 0, 0), \frac{r}{a^m}\right) \tag{2.13}$$

for all $x \in X$, $r > 0$ and $m, n \geq 0$. Replacing r by $a^m r$ in (2.13), we get

$$N\left(\frac{g(2^{n+m} x)}{16^{n+m}} - \frac{g(2^m x)}{16^m}, \sum_{i=m}^{m+n-1} \frac{a^i r}{16 \cdot 16^i}\right) \geq N'(\alpha(x, 0, 0), r) \tag{2.14}$$

for all $x \in X$, $r > 0$ and $m, n \geq 0$. Using (F3) in (2.14), we obtain

$$N\left(\frac{g(2^{n+m} x)}{16^{n+m}} - \frac{g(2^m x)}{16^m}, r\right) \geq N'\left(\alpha(x, 0, 0), \frac{r}{\sum_{i=m}^{m+n-1} \frac{a^i}{16 \cdot 16^i}}\right) \tag{2.15}$$

for all $x \in X$, $r > 0$ and $m, n \geq 0$. Since $0 < a < 16$ and $\sum_{i=0}^{n} \left(\frac{a}{16}\right)^i < \infty$, the

Cauchy criterion for convergence and (F5) imply that $\left\{\frac{g(2^n x)}{16^n}\right\}$ is a Cauchy sequence in (Y, N). Since (Y, N) is a fuzzy Banach space, this sequence converges to some point $Q(x) \in Y$. So we can define the mapping $Q : X \to Y$ by

$$Q(x) = N - \lim_{n\to\infty} \frac{g(2^n x)}{16^n}$$

for all $x \in X$. Letting $m = 0$ in (2.15), we get

$$N\left(\frac{g(2^n x)}{16^n} - g(x), r\right) \geq N'\left(\alpha(x, 0, 0), \frac{r}{\sum_{i=0}^{n-1} \frac{a^i}{16 \cdot 16^i}}\right) \tag{2.16}$$

for all $x \in X$ and $r > 0$. Letting $n \to \infty$ in (2.16) and using (F6), we arrive

$$N(g(x) - Q(x), r) \geq N'(\alpha(x, 0, 0), r(16 - a))$$

for all $x \in X$ and $r > 0$. To prove that Q satisfies (1.1), replacing (x, y, z) by $(2^n x, 2^n y, 2^n z)$ in (2.3), respectively, we obtain

$$N\left(\frac{1}{16^n} Dg(2^n x, 2^n y, 2^n z), r\right) \geq N'(\alpha(2^n x, 2^n y, 2^n z), 16^n r) \tag{2.17}$$

for all $r > 0$ and $x, y, z \in X$. Now

$$N(Q(2x + y + z) + Q(2x + y - z) + Q(2x - y + z) + Q(-2x + y + z)$$

$$+16Q(y) + 16Q(z) - 8[Q(x + y) + Q(x - y) + Q(x + z) + Q(x - z)]$$

$$-2[Q(y + z) + Q(y - z)] - 32Q(x))$$

$$\geq \min\left\{N\left(Q(2x + y + z) - \frac{g(2^n(2x + y + z))}{16^n}, \frac{r}{14}\right),\right.$$

$$N\left(Q(2x + y - z) - \frac{g(2^n(2x + y - z))}{16^n}, \frac{r}{14}\right),$$

$$N\left(Q(2x-y+z)-\frac{g(2^n(2x-y+z))}{16^n},\frac{r}{14}\right),$$

$$N\left(Q(-2x+y+z)-\frac{g(2^n(-2x+y+z))}{16^n},\frac{r}{14}\right),$$

$$N\left(16Q(y)-\frac{16g(2^n(y))}{16^n},\frac{r}{14}\right),\ N\left(16Q(z)-\frac{16g(2^n(z))}{16^n},\frac{r}{14}\right),$$

$$N\left(-8Q(x+y)+\frac{8g(2^n(x+y))}{16^n},\frac{r}{14}\right),\ N\left(-8Q(x-y)+\frac{8g(2^n(x-y))}{16^n},\frac{r}{14}\right),$$

$$N\left(-8Q(x+z)+\frac{8g(2^n(x+z))}{16^n},\frac{r}{14}\right),\ N\left(-8Q(x-z)+\frac{8g(2^n(x-z))}{16^n},\frac{r}{14}\right),$$

$$N\left(-2Q(y+z)+\frac{2g(2^n(y+z))}{16^n},\frac{r}{14}\right),\ N\left(-2Q(y-z)+\frac{2g(2^n(y-z))}{16^n},\frac{r}{14}\right),$$

$$N\left(-32Q(x)+\frac{32g(2^n(x))}{16^n},\frac{r}{14}\right),$$

$$N\left(\frac{g(2^n(2x+y+z))}{16^n}+\frac{g(2^n(2x+y-z))}{16^n}+\frac{g(2^n(2x-y+z))}{16^n}\right.$$

$$+\frac{g(2^n(-2x+y+z))}{16^n}+\frac{16g(2^n(y))}{16^n}+\frac{16g(2^n(z))}{16^n}$$

$$-\frac{8g(2^n(x+y))}{16^n}-\frac{8g(2^n(x-y))}{16^n}-\frac{8g(2^n(x+z))}{16^n}$$

$$-\frac{8g(2^n(x-z))}{16^n}-\frac{2g(2^n(y+z))}{16^n}-\frac{2g(2^n(y-z))}{16^n}$$

$$\left.\left.-\frac{32g(2^n(x))}{16^n},\frac{r}{14}\right)\right\} \tag{2.18}$$

for all $x, y, z \in X$ and $r > 0.$ Using (F5), (2.17) in (2.18), we arrive

$$N(Q(2x + y + z) + Q(2x + y - z) + Q(2x - y + z) + Q(-2x + y + z)$$

$$+16Q(y) + 16Q(z) - 8[Q(x + y) + Q(x - y) + Q(x + z) + Q(x - z)]$$

$$-2[Q(y + z) + Q(y - z)] - 32Q(x)) \tag{2.19}$$

$$\geq \min\{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, N'(\alpha(2^n x, 2^n y, 2^n z), 16^n r)\} \tag{2.20}$$

for all $x, y, z \in X$ and $r > 0.$ Letting $n \to \infty$ in (2.19) and using (2.2), we see that

$$N(Q(2x + y + z) + Q(2x + y - z) + Q(2x - y + z) + Q(-2x + y + z)$$

$$+16Q(y) + 16Q(z) - 8[Q(x + y) + Q(x - y) + Q(x + z) + Q(x - z)]$$

$$-2[Q(y + z) + Q(y - z)] - 32Q(x)) = 1$$

for all $x, y, z \in X$ and $r > 0.$ Using (F2) in the above inequality gives

$$Q(2x + y + z) + Q(2x + y - z) + Q(2x - y + z) + Q(-2x + y + z) + 16Q(y) + 16Q(z)$$

$$= 8[Q(x + y) + Q(x - y) + Q(x + z) + Q(x - z)]$$

$$+2[Q(y + z) + Q(y - z)] + 32Q(x)$$

for all $x, y, z \in X$. Hence Q satisfies the quartic functional equation (1.1). In order to prove $Q(x)$ is unique, let $Q'(x)$ be another quartic functional equation satisfying (1.1) and (2.5). Hence

$$N(Q(x) - Q'(x), r) = N\left(\frac{Q(2^n x)}{16^n} - \frac{Q'(2^n x)}{16^n}, r\right)$$

$$\geq \min\left\{N\left(\frac{Q(2^n x)}{16^n} - \frac{f(2^n x)}{16^n}, \frac{r}{2}\right), N\left(\frac{f(2^n x)}{16^n} - \frac{Q'(2^n x)}{16^n}, \frac{r}{2}\right)\right\}$$

$$\geq N'\left(\alpha(2^n x, 0, 0), \frac{r16^n(16 - a)}{2}\right)$$

$$\geq N'\left(\alpha(x, 0, 0), \frac{r16^n(16 - a)}{2a^n}\right)$$

for all $x \in X$ and $r > 0.$ Since

$$\lim_{n\to\infty} \frac{r16^n(16-a)}{2a^n} = \infty,$$

we obtain

$$\lim_{n\to\infty} N'\left(\alpha(x, 0, 0), \frac{r16^n(16-a)}{2a^n}\right) = 1.$$

Thus

$$N(Q(x) - Q'(x), r) = 1$$

for all $x \in X$ and $r > 0,$ hence $Q(x) = Q'(x).$ Therefore $Q(x)$ is unique.

For $\beta = -1,$ we can prove the result by a similar method. This completes the proof of the theorem. □

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