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# RIGHT WEAKLY REGULAR SEMIGROUPS

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## Abstract

A semigroup $S$ is right weakly regular (r.w.r.) if every right ideal of $S$ is idempotent. Various equivalent conditions to $S$ being r.w.r. and basic properties of r.w.r. semigroups are given. Constructions and examples of r.w.r. semigroups are given. Properties of the class of all r.w.r. semigroups are investigated and a radical associated with this class is developed. Right weak regularity is investigated for the multiplicative and adjoint semigroups of a ring and for simple and 0-simple semigroups.

## 1. Introduction

Here $S$ will always denote a semigroup. We say that $S$ is *right weakly regular* (*r.w.r.*) if every right ideal of $S$ is idempotent; i.e., if $B$ is a right ideal of $S$, then $B = B^2 = \{xy : x, y \in B\}$. Similarly, define *left weakly regular* (*l.w.r.*). We use $\mathbb{R}(S)$, $\mathbb{L}(S)$, and $\mathbb{I}(S)$ for the multiplicative semigroups of all right, left, and two-

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sided ideals, respectively, of $S$. So $S$ is r.w.r. (l.w.r.) if and only if $\mathbb{R}(S)$ (respectively, $\mathbb{L}(S)$) is a band. In this paper, we investigate the properties of r.w.r. semigroups and of the semigroups $\mathbb{R}(S)$. This work was motivated by the previous work on the semigroup of right ideals of a ring [8, 9], and by the substantial theory that has been developed on right weakly regular rings (for example, see [16, Section 20]). Right weakly regular rings are also called "*fully right idempotent*" [7], "*right fully idempotent*" [1] and "*weakly regular*" [3].

This paper is organized as follows: In Section 2, we present basic results on r.w.r. semigroups. In Section 3, we give other classes of examples and constructions for r.w.r. semigroups. Also in that section, properties of the class of all r.w.r. semigroups are investigated and a radical associated with that class is developed. Further examples are given to illustrate and delimit the theory. In Section 4, we investigate the relationship between a right weakly regular ring, its multiplicative semigroup, and its adjoint semigroup. In Section 5, simple and 0-simple monoids are considered and their relationships with $\mathbb{R}(S)$ and $\mathbb{L}(S)$ are developed.

## 2. Preliminaries and Basic Results

We use $\langle b\rangle_r$ and $\langle b\rangle$ for the principal right ideal (respectively, two-sided ideal) of $S$ generated by $b \in S$, and $E(S)$ for the set of all idempotent elements in $S$. If $B$ is a right (two-sided) ideal of $S$ and $B$ is a r.w.r. semigroup, then we call $B$ a *r.w.r.* (*two-sided*) *ideal*.

Observe that every right and every left ideal of a regular semigroup are idempotent. So if $S$ is regular, then $S$ is r.w.r. and l.w.r. Also note that if $S$ is a monoid, then $AS = A \subseteq SA$ for each right ideal $A$ of $S$, and $SL = L \subseteq LS$ for each left ideal $L$ of $S$.

**Lemma 2.1.** *If every principal right ideal of $S$ is idempotent, then $\langle x\rangle_r = xS$ for every $x \in S$, and $x \in xS$.*

**Proof.** For any $x \in S$, we have $\langle x\rangle_r = (\langle x\rangle_r)^2 = (xS \cup \{x\})^2 = (xS)^2 \cup (xSx) \cup (x^2S) \cup \{x^2\} \subseteq xS \subseteq \langle x\rangle_r$, so $xS = \langle x\rangle_r$ and $x \in xS$ for each $x \in S$. □

**Corollary 2.2.** *If $S$ is r.w.r., then $\langle x\rangle_r = xS$ and $x \in xS$ for each $x \in S$.*

**Proposition 2.3.** *The following are equivalent*:

(a) *S is r.w.r.*;

(b) *every principal right ideal of S is idempotent*;

(c) $x \in (xS)^2$ *for every* $x \in S$;

(d) $x \in (xS)^n$ *for every* $x \in S$ *and* $n > 1$;

(e) $x \in x\langle x\rangle$ *for each* $x \in S$;

(f) *if A*, *B are right ideals of S and* $A \subseteq B$, *then* $AB = A$;

(g) *every homomorphic image of S is r.w.r.*;

(h) *every ideal of S is a r.w.r. semigroup*.

**Proof.** In the following $x \in S$. Note that (a) implies (b) is trivial. Assume (b). Then $\langle x\rangle_r = (\langle x\rangle_r)^2 = (xS)^2$, and hence $x \in (xS)^2$. So (b) implies (c).

Assume (c). Let $B$ be a right ideal of $S$ and $b \in B$. Then from $(\langle b\rangle_r)^2 = (bS)^2 \cup bSb \cup b^2S \cup \{b^2\}$, and since $b \in (bS)^2$, we have $b \in (\langle b\rangle_r)^2$. Thus $B \subseteq B^2$. So (c) implies (a), and hence (a), (b) and (c) are equivalent.

Using $S$ is r.w.r., we have $xS = (xS)^2 = (xS)^n$ for $n > 1$, and hence $x \in (xS)^n$. Since $(xS)^n \subseteq (xS)^2$ for $n > 1$, the converse is immediate. Thus (a), (b), (c) and (d) are equivalent.

Using (a) through (c), we have $\langle x\rangle = xS \cup Sx$ and $x\langle x\rangle S = x^2S^2 \cup (xS)^2$. Since $x \in (xS)^2$, we have $x \in x\langle x\rangle S$. But $x \in x\langle x\rangle S \subseteq x\langle x\rangle$, yielding $x \in x\langle x\rangle$. Thus (a) through (c) imply (e).

Assume (e). Since $x\langle x\rangle \subseteq xS$, we have $x \in xS$, and hence $\langle x\rangle = xS \cup Sx \cup SxS$. So $x\langle x\rangle = x^2S \cup xSx \cup xSxS$. If $x \in xSx$, then $xS \subseteq xSxS$, and hence $x \in (xS)^2$. If $x \in x^2S$, then $x = x^2s$ for some $s \in S$, and hence $x = (x^2s)xs$, or $x \in (xS)^2$. So in any case, $x \in (xS)^2$ and thus (c) holds. So (a) through (e) are equivalent.

Assume (a). Then if $A$, $B$ are right ideals of $S$ with $A \subseteq B$, we have $A = A^2 \subseteq AB \subseteq A$ or $AB = A$. So (a) implies (f). The converse is immediate using $A = B$.

Observe that (a) implies (g) follows immediately from the basic properties of homomorphisms. The converse is trivial.

Let $I$ be an ideal of $S$ and $y \in I$. By (c), we have $y \in (yS)^2$, and this implies that $y \in (yS)^4$. Since $y \in (yS)^4 \subseteq (yI)^2$, we have $y \in (yI)^2$, and hence the semigroup $I$ satisfies (c). So $I$ is r.w.r. and (c) implies (h). The converse is immediate. This completes the logical circuit. □

**Proposition 2.4.** *Let $S$ have a right identity. Then $S$ is r.w.r. if and only if $xS = (xS)^2$ for each $x \in S$.*

**Proof.** If $xS = (xS)^2$ for each $x \in S$, then $x \in (xS)^2$ and hence $S$ is r.w.r. The converse is immediate. □

The condition that $S$ has a right identity cannot be eliminated in the previous proposition. To see this, observe that if $S$ is a semigroup with zero and at least one other element and that $ab = 0$ for each $a, b \in S$, then $xS = (xS)^2$ for each $x \in S$, but $S$ is not r.w.r.

**Proposition 2.5.** *The following are equivalent*:

(a) *$S$ is r.w.r.*;

(b) $A \cap B \subseteq BA$ *for each* $A, B \in \mathbb{R}(S)$;

(c) $A \cap B \subseteq (AB) \cap (BA)$ *for each* $A, B \in \mathbb{R}(S)$.

**Proof.** If $S$ is r.w.r, then $A \cap B = (A \cap B)^2 \subseteq AB$. Similarly, one gets $A \cap B \subseteq BA$. Assuming (b) and letting $A = B$ gives $A \subseteq A^2$ and hence $A = A^2$. So (a) and (b) are equivalent. Consequently (a) implies (c), and the converse is immediate.

□

**Corollary 2.6.** *Let $S$ be r.w.r.*

(a) *If $B$ is a right ideal of $S$ and $I$ is an ideal of $S$, then $B \cap I = BI$.*

(b) *For each* $x \in S$ *and each ideal I of S, we have* $(xS) \cap I = xI$.

(c) *If* $e \in E(S)$, *then eSe is r.w.r.*

**Proof.** (a) $B \cap I \subseteq BI \subseteq B \cap I$; so $B \cap I = BI$.

(b) $xI \subseteq (xI) \cap I \subseteq (xS) \cap I = (xS)I \subseteq x(SI) \subseteq xI$; so $xI = (xS) \cap I$.

(c) Let $T$ be a right ideal of $eSe$. Since $T = Te = eT$ and $TeS$ is a right ideal of $S$, we have $T = Te \subseteq TeS = (TeS)^2$ and $Te \subseteq (TeS)(TeSe)$. So $T = Te \subseteq (TeS)(TeSe)$ $= [T(eSe)][T(eSe)] \subseteq T^2$, and hence $T = T^2$. □

**Corollary 2.7.** *Let S be a monoid. Then the following are equivalent*:

(a) *S is r.w.r.*;

(b) *if B is a right ideal of S and I is an ideal of S, then* $B \cap I = BI$;

(c) *if* $x \in S$ *and I is an ideal of S, then* $(xS) \cap I = xI$;

(d) *if* $e \in E(S)$, *then eSe is r.w.r.*

**Proof.** Assume (b). Let $A$ and $B$ be right ideals of $S$. Then $B \cap A \subseteq B \cap (SA)$ $= BSA = BA$. Then by Proposition 2.5, we have that $S$ is r.w.r. This and Corollary 2.6 yield (a) and (b) are equivalent.

Assume (c). Let $x \in S$ and $I$ be an ideal of $S$. Then $xS \subseteq (xS) \cap (SxS) = x(SxS)$ $= (xS)^2$, and hence $xS = (xS)^2$. So by Proposition 2.4, (a) and (c) are equivalent.

Finally, the equivalence of (a) and (d) is immediate using Corollary 2.6 and the fact that $S$ has identity. □

For any ideal $I$ of a semigroup $S$, let $S/I$ be the Rees quotient semigroup determined by $S$ and $I$. Recall that this induces a natural homomorphism of $S$ onto $S/I$ and that $S/I$ is a semigroup with zero [12, p. 62].

**Proposition 2.8.** *Let S be a semigroup with zero.*

(a) *If S is r.w.r., then the only nilpotent ideal of S is* $(0)$.

(b) *If I is an ideal of S and* $S/I$ *is r.w.r., then every nilpotent ideal of S is contained in I.*

**Proof.** (a) Part (a) follows immediately from the definition of the terms. Part (b) then follows from part (a) and the fact that $S/I$ is r.w.r. with zero. □

**Lemma 2.9.** *Let S have a partial order relation* $\leq$ *such that* $ab \leq a$ *for each* $a, b \in S$. *If S is regular, then S is a band. If S is an inverse semigroup, then S is a semilattice.*

**Proof.** Let $b \in S$. Then $b = bb'b$ for some $b' \in S$, and hence $b = (bb')b \leq bb' \leq b$; so $b = bb'$. But $bb'$ is an idempotent. □

For results similar to Lemma 2.9 (see [11]).

**Proposition 2.10.** *If the semigroup* $\mathbb{R}(S)$ *is regular, then S is r.w.r. If* $\mathbb{R}(S)$ *is an inverse semigroup, then* $\mathbb{R}(S)$ *is a semilattice,* $R(S) = \mathbb{I}(S)$, *and* $AB = A \cap B$ *for each* $A, B \in \mathbb{R}(S)$.

**Proof.** Set inclusion serves as a partial ordering on $\mathbb{R}(S)$ with the properties required in Lemma 2.9. So if $\mathbb{R}(S)$ is regular, then $\mathbb{R}(S)$ is a band and $S$ is r.w.r. Consequently, if $\mathbb{R}(S)$ is an inverse semigroup, then the elements in $\mathbb{R}(S)$ commute pairwise and each is an idempotent; i.e., $\mathbb{R}(S)$ is a semilattice. Consequently $\mathbb{R}(S) = \mathbb{I}(S)$. Then Corollary 2.6 (a) gives $AB = A \cap B$, for each $A, B \in \mathbb{R}(S)$. □

**Corollary 2.11.** *The following are equivalent*:

(a) *S is r.w.r*;

(b) *if* $B \in \mathbb{R}(S)$, *then there exists* $n = n(B) > 1$ *such that* $B^n = B$;

(c) $\mathbb{R}(S)$ *is regular.*

## 3. Examples and Constructions

Let $\mathcal{W}$ be the class of all r.w.r. semigroups and $\mathcal{T}$ be the class of all semigroups that contain a r.w.r. ideal. Observe that both $\mathcal{W}$ and the class of all semigroups with zero are contained in $\mathcal{T}$.

**Proposition 3.1.** *Let* $S \in \mathcal{T}$.

(a) *There is a unique largest r.w.r. ideal of S, which we denote by* $\mathcal{W}(S)$.

(b) *If I is an ideal of S, then* $I \in \mathcal{T}$.

(c) *Every homomorphic image of S is in* $\mathcal{T}$.

**Proof.** (a) Let $I_\lambda$, $\lambda \in \Lambda$ be the set of all ideals of $S$ such that $I_\lambda \in \mathcal{W}$, and let $T = \bigcup I_\lambda$, $\lambda \in \Lambda$. Then for any $x \in T$, we have $x \in I_\gamma$ for some $\gamma \in \Lambda$, and hence $x \in (xI_\gamma)^2$. But $(xI_\gamma)^2 \subseteq (xT)^2$. Thus $x \in (xT)^2$ and hence the ideal $T$ is r.w.r. So $T$ is the desired ideal $\mathcal{W}(S)$.

(b) Let $I$ be an ideal of $S$. Then $I \cap \mathcal{W}(S)$ is an ideal of $\mathcal{W}(S)$, so it is r.w.r. But $I \cap \mathcal{W}(S)$ is also an ideal of the semigroup $I$. Thus $I \in \mathcal{T}$.

(c) Let $\phi : S \to \overline{S}$ be a surjective homomorphism. Observe that $\phi(\mathcal{W}(S))$ is a r.w.r. ideal of $\overline{S}$. So $\overline{S} \in \mathcal{T}$. □

**Proposition 3.2.** *Let* $S \in \mathcal{T}$.

(a) $\mathcal{W}(S/\mathcal{W}(S)) = 0$.

(b) *If* $S \in \mathcal{W}$, *then* $S/I$ *is in* $\mathcal{W}$ *for each ideal I of S.*

**Proof.** (a) Let $\overline{I}$ be a r.w.r. ideal of $S/\mathcal{W}(S)$, where the preimage $I$ under the natural homomorphism $\eta : \to S/\mathcal{W}(S)$ is an ideal of $S$. Using $\overline{I}$ is r.w.r., we have that for each $x \in I$, $\eta(x) = \eta(x)\eta(s_1)\eta(x)\eta(s_2)$, where $s_1, s_2 \in S$. This implies that $x = xs_1xs_2$ and hence $x \in (xI)^2$; so $I$ is r.w.r. Thus $I \subseteq \mathcal{W}(S)$ and hence $\overline{I}$ is zero.

(b) This follows from Proposition 3.1 (c). □

A general definition for radicals in semigroups was given by Hoehnke [10]. We use the equivalent definition of such a radical as discussed by Clifford [4] and by Roĭz and Schein [14].

**Corollary 3.3.** *Let* $\rho$ *be the function defined on* $\mathcal{T}$, *where if* $S \in \mathcal{T}$, *then* $\rho(S)$ *is the Rees congruence on S induced by the ideal* $\mathcal{W}(S)$. *Then* $\rho$ *is a Hoehnke type radical on the class* $\mathcal{T}$.

**Proof.** Let $S \in \mathcal{T}$ and $f : S \to \overline{S}$ be a surjective homomorphism. If $x, y \in S$

such that $x \neq y$ and $(x, y) \in \rho(S)$, then $x, y \in \mathcal{W}(S)$. So $(f(x), f(y)) \in \mathcal{W}(\overline{S})$, and hence $(f(x), f(y)) \in \rho(\overline{S})$. This, together with Proposition 3.2 (a) yields that ρ is a Hoehnke radical on the class $\mathcal{T}$. □

The concept of an Amitsur-Kurosh radical class for semigroups was defined in [17, Section 4]. From the previous remarks in this section and in Proposition 2.3 (h), we observe that $\mathcal{W}$ is an Amitsur-Kurosh radical class. We use $\mathcal{S}_0$ for the class of all semigroups *S* with zero such that $\mathcal{W}(S) = 0$; i.e., $\mathcal{S}_0$ is the semisimple class, in the class of semigroups with 0, associated with the radical class $\mathcal{W}$. The results obtained above on the radical $\mathcal{W}$ are analogous to those found for rings (see [1], [15, p. 197]).

**Proposition 3.4.** *The class* $\mathcal{S}_0$ *is closed under subdirect products.*

**Proof.** From [17, Section 4], we have that in the category of semigroups with zero any semisimple class of an Amitsur-Kurosh radical class is closed under subdirect products. □

**Proposition 3.5.** *The class of all r.w.r.* (*l.w.r.*) *semigroups is closed under direct products.*

**Proof.** Let $S_\lambda, \lambda \in \Lambda,$ be r.w.r. semigroups and $S = \Pi_{\lambda \in \Lambda} S_\lambda$. Then for any $x = (\dots, x_\lambda, \dots)$ in $S$, $x_\lambda \in (x_\lambda S_\lambda)^2$, for each $\lambda \in \Lambda$. So $x \in (xS)^2$, and hence *S* is r.w.r. □

Neither the class $\mathcal{W}$ nor the class $\mathcal{W}_0$ of all r.w.r. semigroups with zero is closed under subdirect products. To see this, observe that the ring of integers $\mathbb{Z}$ is isomorphic to a subdirect product of fields, and hence the semigroup $(\mathbb{Z}, \cdot)$ is isomorphic to a subdirect product of r.w.r. semigroups with zero. But $(\mathbb{Z}, \cdot)$ is not r.w.r.

Let $S^1$ be the monoid obtained from a semigroup *S* by adjoining an identity. If $x \in S$, then $x \in (xS)^2$ when $S$ is r.w.r., and hence $x \in (xS^1)^2$. Trivially $1 \in (1 \cdot S^1)^2$. So $S^1$ is r.w.r. whenever $S$ is. Conversely, since $S$ is an ideal of $S^1$, whenever $S^1$ is r.w.r., *S* is r.w.r.

Let $S^0$ be the semigroup with zero formed by adjoining a zero to the semigroup

$S$. Similar to the paragraph above, we see that if $S$ is r.w.r., then so is $S^0$. Conversely, since $S$ is an ideal of $S^0$, whenever $S^0$ is r.w.r., $S$ is r.w.r.

Let $\Lambda$ be an infinite index set and $S_\lambda$ be a semigroup with zero for each $\lambda \in \Lambda$. Let $\kappa$ be an infinite cardinal such that $\kappa \leq \text{card } \Lambda$. Define $P_\kappa$ to be the set of all $b = (\ldots, b_j, \ldots)$ in $P = \Pi_{\lambda \in \Lambda} S_\lambda$ such that $b_j = 0$ for all $j$ except in some subset with cardinality less than $\kappa$. (Thus if $\kappa = \aleph_0$, then $P_\kappa$ would be those $b$ with finite support.) Observe that each $P_\kappa$ is an ideal of $P$ and that $P_\kappa \subseteq P_\tau$ if $\kappa \leq \tau$.

**Proposition 3.6.** *Let* $\Lambda, S_\lambda, \kappa$ *and* $P_\kappa$ *be as above. If each* $S_\lambda, \lambda \in \Lambda,$ *is r.w.r., then* $P_\kappa$ *is r.w.r.*

**Proof.** Use that $P_\kappa$ is an ideal of $P = \Pi_{\lambda \in \Lambda} S_\lambda$ and that $P$ is r.w.r. Then Proposition 2.3 yields that each $P_\lambda$ is r.w.r. □

**Proposition 3.7.** *Let I be an ideal of S. If I and* $S/I$ *are r.w.r., then S is r.w.r.*

**Proof.** Let $\bar{S} = S/I$ and use the notation $\bar{x}$ to denote the image in $\bar{S}$ under the natural homomorphism. Then for any $x \in S$, $\bar{x} \in (\bar{x}\bar{S})^2$ and hence $\bar{x} = \bar{x}\bar{s}\bar{x}\bar{t}$, where $s, t \in S$. So either $x \in I$ or $x = xsxt$. The latter yields $x \in (xS)^2$. If $x \in I$, then since $I$ is r.w.r., we have $x \in (xI)^2$. But $(xI)^2 \subseteq (xS)^2$; so $x \in (xS)^2$. Thus $S$ is r.w.r. □

## 4. Rings and their Multiplicative and Adjoint Semigroups

Let $R$ denote a ring with identity and $(R, \cdot)$ be its multiplicative semigroup. Every right ideal of $R$ is a right ideal of $(R, \cdot)$ but not conversely. However, for a right ideal $B$ of the ring $R$, what is meant by $B^2$ in $R$ is not the same as what is meant by $B^2$ in $(R, \cdot)$. This comment makes the next result of interest. For necessary background on r.w.r. rings, see [13, 16].

**Proposition 4.1.** *A ring R is r.w.r. if and only if the semigroup* $(R, \cdot)$ *is r.w.r.*

**Proof.** For clarity in this proof, we will use $XY$ for the product of the ring right ideals $X$ and $Y$ and $(X, \cdot)\cdot(Y, \cdot)$ for the product of the semigroup right ideals $(X, \cdot)$ and $(Y, \cdot)$.

Let $R$ be r.w.r. and $(A, \cdot)$ be a right ideal of $(R, \cdot)$. Then for each $a \in A$, we have $a \in aR$ and $aR = (aR)^2 \subseteq (A, \cdot)\cdot(A, \cdot)$. So $(A, \cdot) \subseteq (A, \cdot)\cdot(A, \cdot)$ and hence $(R, \cdot)$ is r.w.r.

Let $(R, \cdot)$ be r.w.r. Let $A$ be a right ideal of $R$ and let $a \in A$. Then $(aR, \cdot)$ is a right ideal of $(R, \cdot)$ and so $a \in (aR, \cdot)\cdot(aR, \cdot) \subseteq A^2$. So $A = A^2$ and $R$ is r.w.r. □

It is well-known that for any ring $R$, the operation defined by $a \circ b = a + b - ab$, for each $a, b \in R$, yields a semigroup $(R, \circ)$. Furthermore, if $R$ has unity, then the function given by $\phi(x) = 1 - x$ is an isomorphism of the semigroup $(R, \cdot)$ onto the semigroup $(R, \circ)$.

**Corollary 4.2.** *If $R$ is a r.w.r. ring, then $(R, \circ)$ is a r.w.r. semigroup.*

**Proof.** Recall that a r.w.r. ring $R$ can be embedded as an ideal in a ring $R^1$ such that $R^1$ has unity 1, and such that every element has the form $\alpha 1 + a$, where $a \in R$ and $\alpha \in K$, where $K$ is a commutative r.w.r. ring with unity such that $R$ is a $K$-algebra [2]. In this case, then $(R^1, \cdot)$ is a r.w.r. semigroup and hence so is its isomorph $(R^1, \circ)$. Observe that $R$ is an ideal of $(R^1, \circ)$; so $(R, \circ)$ is also r.w.r. □

It is worth noting that if $R$ is a ring without unity, then the semigroups $(R, \cdot)$ and $(R, \circ)$ need not be isomorphic. As an extreme example, take $R^2 = 0$, $R \neq 0$. In this case, $(R, \cdot)$ is not r.w.r., but $(R, \circ)$, being a group, is a r.w.r. semigroup. Of course the ring $R$ is not r.w.r.

## 5. Simple and 0-simple Monoids and Semigroups

Let $S$ have a zero. Then $\mathbb{R}(S)$ has a zero. If no ambiguity arises, then we use 0 for each of: the zero of $S$; the zero in $\mathbb{R}(S)$; and the zero ideal of $\mathbb{R}(S)$. Observe that the set of all nilpotent right ideals of $S$ is a right ideal of $\mathbb{R}(S)$.

Recall that $S$ is a *left zero semigroup* if $ab = a$ for each $a, b \in S$, [5, p. 4].

**Proposition 5.1.** *Let S be a semigroup. If* $\mathbb{R}(S)$ *is left zero, then S is simple.*

**Proof.** Let $\mathbb{R}(S)$ be left zero. Then for any $x \in S$, we have $SxS = S(xS) = S$.

□

**Proposition 5.2.** *If S is a simple monoid, then*:

(a) $\mathbb{R}(S)$ *is left zero and S is r.w.r*;

(b) $\mathbb{L}(S)$ *is right zero and S is l.w.r.*

**Proof.** Let $A, B \in \mathbb{R}(S)$. Then $AB = (AS)B = A(SB) = AS = A$. So $\mathbb{R}(S)$ is left zero. Thus $\mathbb{R}(S)$ is a band and *S* is r.w.r. The left-sided result follows similarly.

□

**Proposition 5.3.** *If S is a* 0-*simple monoid, then S is r.w.r. and l.w.r.*

**Proof.** Proceed as in the proof of Proposition 5.2 using $A = B \neq 0$. □

**Corollary 5.4.** *Let S be a monoid and I be a maximal ideal of S. If I is r.w.r., then S is r.w.r.*

**Proof.** Since *I* is maximal, $S/I$ is a 0-simple monoid. So $S/I$ is r.w.r. by Proposition 5.3. Since *I* is r.w.r., we then have that *S* is r.w.r. by Proposition 3.7. □

By [6, Theorem 8.45], we can embed any semigroup *S* into a simple monoid $\mathcal{C}(S)$. Moreover, by [6, Theorem 8.48], the monoid $\mathcal{C}(S)$ is regular if and only if *S* is regular. This gives another method to construct r.w.r. semigroups that are not regular.

There are many regular semigroups that are not simple, and many simple monoids that are not regular. If $S_1$ is a regular semigroup which is not simple and $S_2$ is a simple monoid which is not regular, then $S_1 \times S_2$ is a r.w.r. (l.w.r.) semigroup which is neither simple nor regular.

Proposition 5.2 does not hold for simple semigroups without identity, as the next example illustrates.

**Example 5.5.** Let *T* be a Baer-Levi semigroup as defined in [6, p. 82]. Then *T* is right simple without idempotents; thus, *T* is the only right ideal of *T*, so trivially *T* is r.w.r. However, if *T* is l.w.r., then $t \in Tt$ for any $t \in T$ by Corollary 2.2. Therefore, there exists $x \in T$ such that $t = xt$. Recall that a right simple semigroup without idempotents cannot have the equation $ab = b$ which holds for any two elements $a$, $b$ [6, Lemma 8.3]. Thus $T$ is simple and $T$ is r.w.r. but not l.w.r.

**Example 5.6.** Recall that for any semigroup $T$, using the new operation $a * b = ba$, one gets the opposite semigroup $T^{\text{opp}}$. If $T$ is a Baer-Levi semigroup, then $S = T \times T^{\text{opp}}$ is simple. If $S$ is l.w.r., then every homomorphic image of $S$ is l.w.r. However, the projection map $\pi_1 : S \to T$ is a homomorphism, and $T$ is not l.w.r. Hence $S$ is not l.w.r. Similarly, using the second projection map $\pi_2 : S \to T^{\text{opp}}$, we have that $S$ is not r.w.r.

We now characterize $\mathbb{R}(S)$ for the general case, where $S$ is simple.

**Lemma 5.7.** *If $S$ is a simple semigroup and $A$, $B$ are right ideals of $S$, then* $AB = AS = A^2 = (AS)^2$.

**Proof.** $AS = A(SB) = (AS)B \subseteq AB$; so $AS = AB$. Letting $A = B$ yields $A^2 = AS$. Finally, $AS = ASA = A(SA)A \subseteq (AS)^2 \subseteq AS$. □

**Lemma 5.8.** *Let $S$ be simple. Let $\mathbb{R}_2(S) = \{A^2 \mid A \in \mathbb{R}(S)\}$. Then*

(a) *every element of $\mathbb{R}_2(S)$ is a left zero of $\mathbb{R}(S)$;*

(b) $\mathbb{R}_2(S)$ *is a two-sided ideal of* $\mathbb{R}(S)$;

(c) $(\mathbb{R}(S))^2 = \mathbb{R}_2(S)$;

(d) $\mathbb{R}_2(S)$ *is a homomorphic image of* $\mathbb{R}(S)$.

**Proof.** Let $A^2 \in \mathbb{R}_2(S)$ and $B \in \mathbb{R}(S)$. Then $A^2B = ASB$ by Lemma 5.7 and $ASB = A(SB) = AS = A^2$, again by Lemma 5.7. This proves (a) and also proves that $\mathbb{R}_2(S)$ is a right ideal of $\mathbb{R}(S)$.

(b), (c) By Lemma 5.7, we have $AB = AS = A^2$. In particular, if $B \in \mathbb{R}_2(S)$, then $AB \in \mathbb{R}_2(S)$.

(d) Define a map $\phi : \mathbb{R}(S) \to \mathbb{R}_2(S)$ by $\phi(A) = A^2$ for all $A \in \mathbb{R}(S)$. We show that this map is a homomorphism. Then $\phi(AB) = (AB)(AB) = A(BAB) = A^2$ by Lemma 5.7. By (a), we have $A^2 = A^2B^2$. But $A^2B^2 = \phi(A)\phi(B)$. □

**Proposition 5.9.** *Let S be simple. Then the following are equivalent*:

(a) *S is r.w.r.*;

(b) $x \in xS$ *for all* $x \in S$;

(c) *S is a right identity for* $\mathbb{R}(S)$;

(d) $\mathbb{R}(S)$ *is a left zero semigroup*.

**Proof.** Assume (b) and let $x \in S$. Then $S = SxS$ and hence $xS = x(SxS)$; so $x \in (xS)^2$ and $S$ is r.w.r. The converse follows immediately from Corollary 2.2.

Assume $S$ is r.w.r. and let $A \in \mathbb{R}(S)$. Since $S$ is simple, Lemma 5.7 gives $A^2 = AS$ and hence $A = AS$. So (a) implies (c) holds. The converse follows using Lemma 5.7. Observe that (a) implies (d) follows from Lemma 5.8, and the converse is trivial. □

Recall that if $S$ is isomorphic to an ideal $\overline{S}$ of a semigroup $V$ and there is a homomorphism of $V$ onto $\overline{S}$ which leaves the elements of $\overline{S}$ fixed, then $V$ is said to be a *retract extension* of $S$ [12, III.4]. The next result then follows from Lemma 5.8.

**Proposition 5.10.** *Let S be simple and not r.w.r. Then* $\mathbb{R}(S)$ *is a retract extension of a left zero semigroup by a semigroup with zero multiplication*.

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