Advances in Differential Equations and Control Processes
Volume 5, Number 1, 2010, Pages 37-47
Published Online: May 15, 2010
This paper is available online at http://www.pphmj.com

SOLUTION OF THE LINEAR DIFFUSION EQUATION WITH A TIME DEPENDENT DIFFUSION COEFFICIENT, IN TWO AND THREE DIMENSIONS

GEORGIOS AIM. SKIANIS* **and EFTHIMIA VERIKIOU**

Department of Geography and Climatology
Faculty of Geology and Geo-Environment
University of Athens
157 84, Athens, Greece
e-mail: skianis@geol.uoa.gr

Abstract

In the present paper, the diffusion equation for a time dependent diffusion coefficient $D(t)$ is solved, in two and three dimensions. It is pointed out that the behavior of the solution is controlled by the definite integral of $D(t)$ for time, assuming that the diffusion process is taking place at the time interval $[0, t]$. The physical meaning of the solution is that at time *t*, the value of the quantity which is involved in the diffusion process depends on the mean value of $D(t)$ at $[0, t]$ and not on its specific variation with time. The conclusions of this paper may be useful in modeling diffusion processes in various fields of geosciences such as hydrology, geomorphology and soil or atmospheric pollution.

0. Introduction

The diffusion equation has been extensively used to describe heat and mass

2010 Mathematics Subject Classification: 35-XX.

Keywords and phrases: diffusion equation, diffusion coefficient, Fourier transform, radial Fourier transform.

*Corresponding author

Received February 18, 2010

transfer in various physical systems (Zerefos [10], Gupta [2], Menke and Abbot [4], Streeter et al. [8]). It has also been used in geomorphology to describe a landform evolution by erosion processes (Culling [1], Kirkby [3] and Scheidegger [5]).

In all these approaches, the diffusion coefficient D is considered to be time independent. In the context of our research in theoretical geomorphology, however, we have solved the diffusion equation with a time varying coefficient in one dimension (Skianis et al. [6]). The physical meaning of a time varying D in geomorphologic processes is that climate variations as well as human activities such as cutting of trees or ploughing, may change the soil erodibility and, consequently, the value of the diffusion coefficient. Skianis et al. [6] showed that the development of the landform with a time varying D is controlled by a function $g(t)$ which is defined as the integral of Ddt for a time interval from zero to t.

A time varying D may also have a physical meaning in other systems where diffusion processes take place such as atmospheric pollution or transportation of a liquid with contaminants through the subsoil. In such cases, the diffusion equation has to be solved in two or in three dimensions. We can intuitively assume that in the multi-dimensional case the solution of the diffusion equation should behave in a similar way, as in the one dimensional case. Since intuition by its own is not enough to come to reliable conclusions, in this paper, the diffusion equation in two and three dimensions is solved and its behavior for certain simple geometries is discussed. These models may have a broader interest in various fields of earth sciences such as hydrology, soil and atmospheric pollution, geomorphology and geothermy.

1. The Diffusion Equation in Two Dimensions

In Cartesian coordinates x, y, the diffusion equation is (Gupta [2])

$$\frac{\partial f}{\partial t} = D(t) \cdot \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) = D(t) \cdot \nabla^2 f(x, y) \tag{1}$$

with an initial condition

$$f(x, y, t = 0) = \varphi(x, y) \tag{2a}$$

and a boundary condition

$$\lim_{x, y \to \pm\infty} f(x, y, t) = 0 \tag{2b}$$

where f is the time and space dependent variable $f(x, y, t)$.

The Fourier transform $F(u_x, u_y, t)$ of $f(x, y, t)$ is given by

$$F(u_x, u_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y) \cdot \exp[-i(xu_x + yu_y)]dxdy \tag{3}$$

where u_x and u_y are the wave numbers at x and y directions, respectively, i is the imaginary unity.

The Fourier transform of $\nabla^2 f(x, y)$ is defined as $\text{F.T.}(\nabla^2 f(x, y))$ and it is given by (Gupta [2])

$$\text{F.T.}(\nabla^2 f(x, y)) = -w^2 F(u_x, u_y) \tag{4}$$

and w^2 is given by

$$w^2 = u_x^2 + u_y^2 \tag{5}$$

Combining relations (1), (3) and (4) gives

$$\frac{\partial F}{\partial t} = -D(t)w^2 F(u_x, u_y, t) \tag{6}$$

The solution of this ordinary differential equation is

$$F(u_x, u_y) = \text{F.T.}(\varphi) \cdot \exp[-w^2 g(t)] \tag{7}$$

where $\text{F.T.}(\varphi)$ is the Fourier transform of $\varphi(x, y)$ and $g(t)$ is given by

$$g(t) = \int_0^t D(a)da \tag{8}$$

Relation (7) gives the solution of the diffusion equation at the wave number domain. According to the convolution theorem (Gupta [2]) and well-known Fourier transforms (Spiegel [7]), the product of $\text{F.T.}(\varphi) \cdot \exp[-w^2 g(t)]$ at wave number domain is the convolution of $\varphi(x, y)$ with $\exp[-(x^2 + y^2)/(4g(t))]$, which gives the solution $f(x, y, t)$ at space domain. Therefore the solution of the diffusion equation with a time varying diffusion coefficient is given by

$$f(x, y, t) = \varphi(x, y)^* \exp\left[-\frac{x^2 + y^2}{4g(t)}\right]$$

$$= \frac{1}{4\pi g(t)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(s_x, s_y) \cdot \exp\left[-\frac{(x - s_x)^2 + (y - s_y)^2}{4g(t)}\right] ds_x ds_y \quad (9)$$

If D is time constant, then according to relation (8), $g(t) = Dt$ and, according to relation (9), the solution of the diffusion equation with a time constant diffusion coefficient is

$$f(x, y, t) = \varphi(x, y)^* \exp\left[-\frac{x^2 + y^2}{4Dt}\right]$$

$$= \frac{1}{4\pi Dt} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(s_x, s_y) \cdot \exp\left[-\frac{(x - s_x)^2 + (y - s_y)^2}{4Dt}\right] ds_x ds_y \quad (10)$$

Relation (9) shows that the solution of the diffusion equation with a time varying diffusion coefficient is controlled by the function $g(t)$, which is the integral of $D(t)$ for time t.

On the other hand, if we define a mean value D_m of the diffusion coefficient t for a time interval $[0, t]$, then according to relation (8), $g(t)$ becomes

$$g(t) = D_m t \quad (11)$$

Combining relations (9) and (11) gives

$$f(x, y, t) = \varphi(x, y)^* \exp\left[-\frac{x^2 + y^2}{4D_m t}\right]$$

$$= \frac{1}{4\pi D_m t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(s_x, s_y) \cdot \exp\left[-\frac{(x - s_x)^2 + (y - s_y)^2}{4D_m t}\right] ds_x ds_y \quad (12)$$

Comparing relations (10) and (12), it can be concluded that the behavior of the solution of the diffusion equation with a time varying diffusion coefficient $D(t)$ depends on the mean value of $D(t)$ for the time interval during which the diffusion process is developed. The solution does not depend on the specific variation of the diffusion coefficient during this time interval.

1.a. Special case: the diffusion equation in a 2-dimensional space with radial symmetry

The 2-dimensional diffusion equation with radial symmetry has a particular importance in theoretical geomorphology, when the hill or the mountain upon which the erosion process takes place presents a symmetry along a vertical axis. The diffusion equation is (Zauderer [9]):

$$\frac{\partial f}{\partial t} = D(t) \cdot \left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} \right) \tag{13}$$

with $r \geq 0$.

There is the initial condition

$$f(r, t = 0) = \varphi(r) \tag{14a}$$

and the boundary condition

$$\lim_{r \to \infty} f(r) = 0 \tag{14b}$$

The zero order Hankel transform $F(\lambda, t)$ of $f(r, t)$ is given by

$$F(\lambda, t) = \int_0^\infty r J_0(\lambda r) f(r, t) dr \tag{15}$$

where J_0 is the Bessel function of first kind and zero order.

Applying the Hankel transform on relation (13) and taking into account its properties (Zauderer [9]), the following ordinary differential equation is obtained

$$\frac{\partial F}{\partial t} + D(t) \cdot \lambda^2 F = 0 \tag{16}$$

The solution of equation (16) is

$$F(\lambda, t) = HT\varphi(\lambda) \cdot \exp[-\lambda^2 g(t)] \tag{17}$$

where $\varphi(\lambda)$ is the Hankel transform of $\varphi(r)$.

Applying the inverse Hankel transform on relation (17) and taking into account certain properties of the function J_0 (Zauderer [9]), it can be obtained that

$$f(r, t) = \frac{1}{2g(t)} \int_0^\infty \exp\left[-\frac{r^2 + s^2}{4g(t)} \right] I_0\left(\frac{rs}{2g(t)} \right) s f(s) ds \tag{18}$$

where I_0 is the modified Bessel function of first kind and zero order.

For a time constant *D*, the solution $f(r, t)$ of the diffusion equation has the form of relation (18), however, $g(t)$ has to be replaced by *Dt*.

If the initial condition is represented by a function $\varphi(r)$ which has a constant value f_0 for $0 \leq r \leq b$ and a zero value for $r > b$, relation (18) becomes

$$f(r, t) = \frac{1}{2g(t)} \int_0^b f_0 \exp\left[-\frac{r^2 + s^2}{4g(t)}\right] I_0\left(\frac{rs}{2g(t)}\right) sds \tag{19}$$

If *D* is time constant, then $f(r, t)$ is given by

$$f(r, t) = \frac{1}{2Dt} \int_0^b f_0 \exp\left[-\frac{r^2 + s^2}{4Dt}\right] I_0\left(\frac{rs}{2Dt}\right) sds \tag{20}$$

For $r = 0$ and a time dependent *D*, relation (19) becomes

$$f(r = 0, t) = f_1(t) = f_0\left[1 - \exp\left(-\frac{b^2}{4g(t)}\right)\right] \tag{21}$$

For $r = 0$ and a time constant *D*, relation (20) becomes

$$f(r = 0, t) = f_2(t) = f_0\left[1 - \exp\left(-\frac{b^2}{4Dt}\right)\right] \tag{22}$$

If $D(t)$ presents a periodical variation with time which is given by

$$D(t) = 1 + \sin(\pi t) \tag{22a}$$

or alternatively by

$$D(t) = 1 - \sin(\pi t) \tag{22b}$$

then according to relation (8), $g(t)$ is given by

$$g(t) = t \mp \frac{\cos(\pi t)}{\pi} - \frac{1}{\pi} \tag{23}$$

The minus sign corresponds to relation (22a) and the plus sign to relation (22b).

Combining relations (21) and (23), the variations of $f_1(1)$ and $f_1(-1)$ against t may be easily calculated. $f_1(1)$ and $f_1(-1)$ are defined to be equal to the function $f_1(t)$, for a $D(t)$ given by relations (22a) and (22b), respectively.

In Figure 1, the time variation of the function f_2, for a time constant $D = 1$ is represented, together with functions $f_1(1)$ and $f_1(-1)$. In all cases, f_0 is put equal to unity.

It can be observed that a time dependent diffusion coefficient introduces significant deviations to the values of *f*. On the other hand, at time instances *t*, where $g(t)$ equals *Dt*, which means that the mean value of the time dependent $D(t)$ is equal to the time constant value *D*, the value of f_2 is equal to the value of $f_1(1)$ and that of $f_1(-1)$.

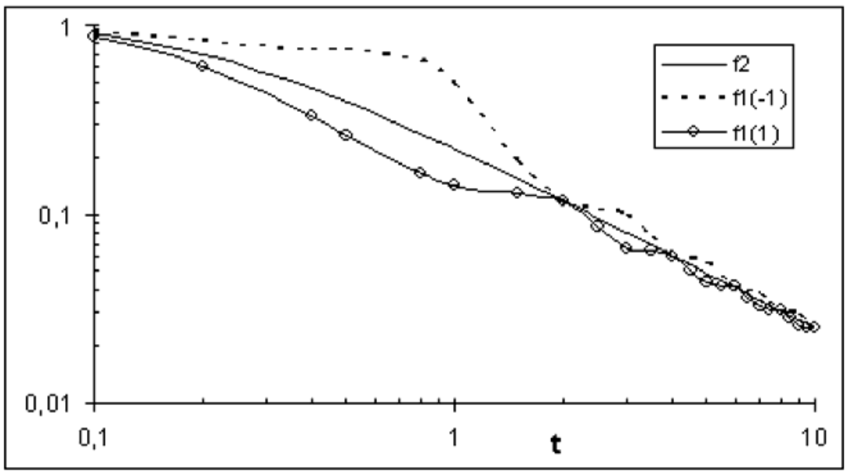

Figure 1. The time variation of the solution *f*, in a two dimensional space with radial symmetry at $r = 0$, for a time constant and a time dependent diffusion coefficient, $D = 1$, $b = 1$, $f_0 = 1$, $c = -1$.

2. The Diffusion Equation in Three Dimensions

In three dimensions, the diffusion equation at space domain is

$$\frac{\partial f}{\partial t} = D(t) \cdot \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) = D(t) \cdot \nabla^2 f(x, y, z) \tag{24}$$

The boundary and initial conditions are the same as in the 2-dimensional case. Function φ depends on x, y, z.

At wave number domain, the diffusion equation takes the form

$$\frac{\partial F}{\partial t} = -D(t)\, w^2 F(u_x,\, u_y,\, t) \tag{25}$$

In the 3-dimensional case, w^2 is given by

$$w^2 = u_x^2 + u_y^2 + u_z^2 \tag{26}$$

Working in the same way as in the 2-dimensional case, it can be proved that the solution of the partial differential equation (24) is given by

$$f(x,\, y,\, z,\, t) = \frac{1}{[4\pi g(t)]^{3/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(s_x,\, s_y,\, s_z)$$

$$\times \exp\left[-\frac{(x-s_x)^2 + (y-s_y)^2 + (z-s_z)^2}{4g(t)}\right] ds_x ds_y ds_z \tag{27}$$

If the diffusion process is due to a point source f_0 at time $t = 0$, then the initial condition is expressed by a Dirac-delta function according to the expression:

$$f(x,\, y,\, z,\, t = 0) = \varphi(x,\, y,\, z) = f_0 \cdot \delta(x) \cdot \delta(y) \cdot \delta(z) \tag{28}$$

Taking into account well-known properties of the delta function (Menke and Abbott [4]), relation (27) becomes

$$f(x,\, y,\, z,\, t) = \frac{f_0}{[4\pi g(t)]^{3/2}} \cdot \exp\left[-\frac{x^2 + y^2 + z^2}{4g(t)}\right] \tag{29}$$

Working in the same way, it can be easily proved that in two dimensions the solution f for a point source is given by

$$f(x,\, y,\, t) = \frac{f_0}{4\pi g(t)} \cdot \exp\left[-\frac{x^2 + y^2}{4g(t)}\right] \tag{30}$$

For a time constant D, the function $g(t)$ in relations (29) and (30) is substituted by Dt and the following well-known expressions are obtained, in three and two dimensions, respectively,

$$f(x, y, z, t) = \frac{f_0}{[4\pi Dt]^{3/2}} \cdot \exp\left[-\frac{x^2 + y^2 + z^2}{4Dt}\right] \tag{31}$$

$$f(x, y, t) = \frac{f_0}{4\pi Dt} \cdot \exp\left[-\frac{x^2 + y^2}{4Dt}\right] \tag{32}$$

The temporal variation of f for a time dependent and a time independent diffusion coefficient is, in qualitative terms, similar to that of the curves of (Figure 1). A measure of the influence of the temporal variation of the diffusion coefficient on the behaviour of f may be obtained by the ratio $f_1(t)/f_2(t)$; $f_1(t)$ is the function f at the source of the function for a time variable diffusion coefficient and $f_2(t)$ is the respective function f for a time constant diffusion coefficient.

Taking into account relations (29) and (31) and putting $(x, y, z) = (0, 0, 0)$, the quantity $f_1(t)/f_2(t)$ in three dimensions is given by

$$\frac{f_1(t)}{f_2(t)} = \left[\frac{Dt}{4\pi g(t)}\right]^{3/2} \tag{33}$$

Combining relations (30) and (32) and putting $(x, y) = (0, 0)$, the quantity $f_1(t)/f_2(t)$ in two dimensions is given by

$$\frac{f_1(t)}{f_2(t)} = \frac{Dt}{4\pi g(t)} \tag{34}$$

For a time constant D equal to unity and $D(t)$ defined by the relation (22b), the expression for $g(t)$ is given by relation (23) with the plus sign and, taking into account the relations (33) and (34), the ratio $f_1(t)/f_2(t)$ can be calculated for various t values. In Figure 2, the curves $f_1(t)/f_2(t)$ for the 2-dimensional and the 3-dimensional cases are presented. It can be observed that the deviations between f values with a time constant D and f values with a time dependent D are quite high at small t values and tend to diminish as long as t increases. The deviations are higher in the 3-dimensional case than in a 2-dimensional case.

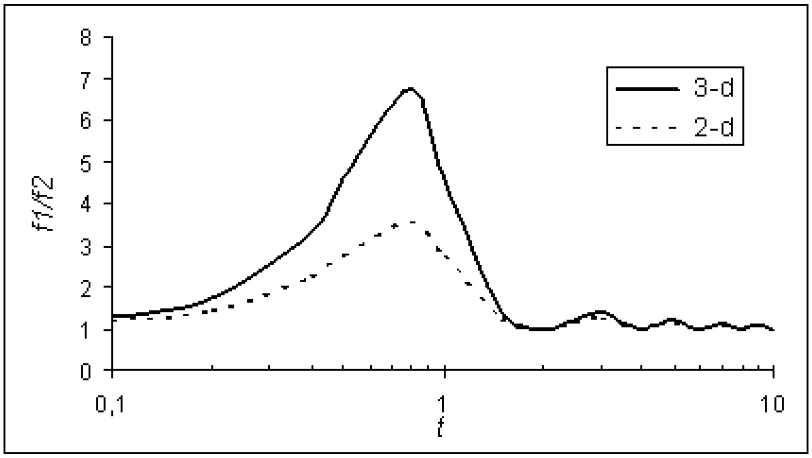

Figure 2. The time variation of the ratio f_1/f_2, in two and in three dimensions, $D = 1$.

The solution of the 1-dimensional diffusion equation with a time dependent diffusion coefficient has been found by Skianis et al. [6]. In that case, the solution f is controlled by $\sqrt{[4\pi g(t)]}$. In qualitative terms, the temporal variation $f_1(t)/f_2(t)$ is the same with that of the 2-dimensional and 3-dimensional cases but the deviation between the solution for a time dependent D and that of a time constant D is smaller. Therefore a time dependent diffusion coefficient controls the behaviour of the solution of the diffusion equation. The deviations between the solution for a time constant D and that of a time dependent D become higher as long as the space dimension increases.

3. Conclusions

The solution of the diffusion equation with a time dependent diffusion coefficient is controlled by the quantity $\sqrt{[4\pi g(t)]^n}$, where n is the space dimension and takes values 1, 2, 3 in geological and environmental processes.

On the other hand, the value of the solution f at time t does not depend on the specific temporal variation of the time diffusion coefficient but only on its mean value for the time interval $[0, t]$. This remark may be useful in studying the diffusion of a contaminant in the subsoil or in the atmosphere, when the hydraulic or diffusion properties of the medium are not time constant. Knowledge of the mean value of these parameters at the time interval $[0, t]$ is sufficient in order to make reliable calculations of the concentration of the contaminant in space and time. Furthermore, the results and conclusions of this paper may be also useful in modelling geomorphological processes or heat conduction at ground surface, when the diffusion coefficient varies with time.

References

[1] W. E. H. Culling, Theory of erosion on soil-covered slopes, J. Geology 73(2) (1965), 230-254.

[2] B. D. Gupta, Mathematical Physics, Vikas Publishing House Pvt. Ltd., 1987.

[3] M. J. Kirkby, Hillslope process response models based on the continuity equation, Slopes: Form and Processes, Vol. 3, D. Brunsden, ed., Institute of British Geographers, Special Publication, 1971, pp. 15-30.

[4] W. Menke and D. Abbott, Geophysical Theory, Columbia University Press, New York, 1991.

[5] A. E. Scheidegger, Theoretical Geomorphology, Springer-Verlag, 1991.

[6] G. Aim. Skianis, D. Vaiopoulos and N. Evelpidou, Solution of the linear diffusion equation for modelling erosion processes with a time varying diffusion coefficient, Earth Surface Processes and Landforms 33(10) (2007), 1491-1501.

[7] M. R. Spiegel, Fourier Analysis, McGraw-Hill, New York, ESPI, Athens, 1978.

[8] V. L. Streeter, E. B. Wylie and K. W. Bedford, Fluid Mechanics, WCB, McGraw-Hill, 1998.

[9] E. Zauderer, Partial Differential Equations of Applied Mathematics, John Wiley & Sons, Inc., New York, 1989.

[10] C. Zerefos, A Course in Atmospheric and Environmental Physics, Thessaloniki, Greece, 1984 (in Greek).