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ON THE EXISTENCE AND UNIQUENESS OF THE 'WEAK' SOLUTION TO THE SIXTH PROBLEM OF THE MILLENNIUM

(This paper is a tribute to Professor Nico Sauer, Emeritus Professor at Pretoria University, South Africa, and my former mentor)

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Abstract

In the follow-up to our analysis of the non-homogeneous case of the problem in [3], in this paper, we present the analysis of the homogeneous case, leading to the existence and uniqueness of the solution to Option A in the statement of the problem as it appears in [2]. As in [3], our analysis occurs in the space, $L^2([0, T], H^2(\Omega))$. In this space, we select appropriate test functions with a compact support in the open bounded domain Ω. Our analysis differs to the one proposed by Ladyzhenskaya in [7], in the selection of some function spaces. Obviously, our test functions are 'candidates' for a 'weak' solution (in the sense of distributions), to the problem. Using 'energy methods', and some results from our previous research papers, we proceed to confirm the existence and uniqueness of the 'weak' solution to Option A of the problem as communicated in [2].

1. Symbols Used

In this paper, we use the following symbols:

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Ω : an open bounded domain in R^3;

$x = (x_1, x_2, x_3)$: spatial position in Ω;

$L^2(\Omega)$: space of the Libesque integrable functions on Ω;

$H^2(\Omega)$: the Sobolev space 2 on Ω;

$H^{3/2}(\partial\Omega)$: the Nikol'skii space on the smooth boundary $\partial\Omega$; $\partial\Omega \not\subset \Omega$;

$\gamma_0 \mathbf{v}(y, t)$: a surface velocity in $L^2([0, T], H^{3/2}(\partial\Omega))$;

$\mathbf{v}(x, t)$: a time dependent velocity field in $L^2([0, T], H^2(\Omega))$;

ρ : a constant fluid volume density due to incompressibility;

μ : a constant fluid viscosity;

$\mathbf{n}(y)$: a unit normal to the surface $\partial\Omega$; with $y \in \partial\Omega$;

$\mathbf{D}(\mathbf{v}(x, t))$: the rate of deformation tensor for the fluid.

2. Introduction

As in [3], the 'weak' solutions to the problem will be defined in $H_m^s(\Omega)$. However, for the sake of the regularity of our time derivatives, we augment the space to $L^m([0, T], H_m^s(\Omega))$; $1 \le s \le m < \infty$, $T < \infty$; with $m = s = 2$. Therefore, we seek, $\mathbf{v}(., \mathbf{t})$; $\mathbf{p}(\mathbf{x}) \in L^2((0, T), H^2(\Omega))$ such that,

$$\begin{cases} \text{(a) } \rho\partial_t \mathbf{v}(x, t) + \rho[(\mathbf{v}(x, t) \cdot \nabla)\mathbf{v}(x, t)] - \mu\Delta\mathbf{v}(x, t) + \nabla p(x) = f(x, t); \\ \qquad x \in \Omega,\ t \in (0, T),\ f(x, t) \in L^2(\Omega); \\ \text{subject to:} \\ \text{(b) } \gamma_0 \mathbf{v}(y, t) = 0;\ y \in \partial\Omega \text{ (a no-slip condition for the fluid)}; \\ \text{(c) } \mathbf{v}(x, 0) = \mathbf{v}^0(x); \\ \text{(d) } \nabla \cdot \mathbf{v}(x, t) = 0. \end{cases} \tag{1}$$

3. Weak Formulation for the Problem

We put $Y := L^2([0, T), H^2(\Omega))$.

We define a set of test functions with compact support in Ω, as follows:

$$\Theta := \{\mathbf{v}(x, t) \in Y : \gamma_0 \mathbf{v}(y, t) = 0;\ y \in \partial\Omega,\ \mathbf{v}(x, 0) = \mathbf{v}^0(x);\ \nabla \cdot \mathbf{v}(x, t) = 0\}.$$

4. Some Important Identities

Proposition 4.1. *For* $\mathbf{v} \in \Theta$, *the followings hold*:

(a) $((\mathbf{v} \cdot \nabla)\mathbf{v}, \mathbf{v})_Y = 0.$

(b) $(\nabla p, \mathbf{v})_Y = 0.$

Proof. (a) We have $\mathbf{v} \cdot \nabla\mathbf{v} = \sum_{k=1}^{3} v_k \frac{\partial v_i}{\partial x_k};\ i = 1, 2, 3.$ Then

$$((\mathbf{v} \cdot \nabla)\mathbf{v}, \mathbf{v})_Y = \sum_{m=1}^{3}\sum_{k=1}^{3}\int_\Omega v_k v_m \frac{\partial v_i}{\partial x_k} dx_m; \quad i = 1, 2, 3$$

$$= \sum_{m=1}^{3}\sum_{k=1}^{3}\frac{1}{2}\int_\Omega v_k \frac{\partial(v_m v_i)}{\partial x_k} dx_m.$$

Re-writing the previous expression and integrating by parts, we obtain

$$\sum_{m=1}^{3}\sum_{k=1}^{3}\frac{1}{2}\int_\Omega v_k \frac{\partial(v_m v_i)}{\partial x_k} dx_m = \frac{1}{2}\left[\int_{\partial\Omega}(|\gamma_0\mathbf{v} \cdot \mathbf{n}||\gamma_0\mathbf{v}|^2)ds - \int_\Omega(|\mathbf{v}|^2\nabla \cdot \mathbf{v})dx\right].$$

With the choice of $\mathbf{v}$ in Θ; integration by parts and the use of the Gauss’s divergence theorem, the result follows.

Using the Gauss’s divergence theorem, we have

(b)

$$\int_\Omega (\nabla p \cdot \mathbf{v})_Y dx = \int_\Gamma \gamma_0 p \gamma_0 \mathbf{v} \cdot \mathbf{ds} - \int_\Omega p(\nabla \cdot \mathbf{v})_{L^2(\Omega)} dx.$$

By 1(b) and 1(d), it follows that, $(\nabla p, \mathbf{v})_Y = 0.$ This, in turn, implies that $\nabla p \in Y^\perp.$

5. The Energy Statement for the Problem

To derive the energy statement for the problem, we take the scalar product of 1(a) with $\mathbf{v} \in \Theta,$ as follows:

$$\rho(\partial_t \mathbf{v}, \mathbf{v})_Y + \rho((\mathbf{v} \cdot \nabla)\mathbf{v}, \mathbf{v})_{H^2(\Omega)} - (\mu\Delta\mathbf{v}, \mathbf{v})_{H^2(\Omega)} + (\nabla p, \mathbf{v})_Y = (\mathbf{f}, \mathbf{v})_Y. \quad (2)$$

By Proposition 4.1, we have

$$((\mathbf{v} \cdot \nabla)\mathbf{v}, \mathbf{v})_{H^2(\Omega)} = 0 \text{ and } (\nabla p, \mathbf{v})_Y = 0.$$

Simplifying (2), we obtain

$$E^/(t) + \mu\| \nabla v(t) \|^2_{L^2(\Omega)} = \int_\Omega fvdx, \quad (3)$$

as the energy identity for the problem. From the no-slip condition $(\gamma_0 \mathbf{v} = 0),$ for $c_\rho > 0,$ and since Ω is bounded, we deduce the Poincare inequality:

$$\| \nabla \mathbf{v}(t) \|^2_{L^2(\Omega)} \geq c_\rho \| \mathbf{v}(t) \|^2_{L^2(\Omega)} \quad \text{(see pp. 248-249 of [1]).} \quad (4)$$

In view of (4), we rewrite (3) as follows:

$$E^/(t) + \mu c_\rho \| v(t) \|^2_{L^2(\Omega)} \leq \int_\Omega fvdx,$$

that is,

$$E^/(t) + 2\mu c_\rho E(t) \leq \int_\Omega fvdx. \quad (5)$$

The solution to (5) is given by

$$E(t) \leq \exp(-2\mu c_\rho t)\int_0^t \left[\exp(2\mu c_\rho \zeta)\int_\Omega fvdx\right] d\zeta + C\exp(-2\mu c_\rho t);\ t \in [0, T), \quad (6)$$

from which we deduce that as $t \to \infty,$ $E(t) \to 0;$ in the sense of an exponential decay for the energy of the problem.

By (6), we observe that, at $t = 0,$ $E(0) \leq C,$ that is,

$$\frac{1}{2}\| \mathbf{v}^0(x) \|^2_{H^2(\Omega)} \leq C; \quad (7)$$

thus pointing to the uniform boundedness of $\mathbf{v}^0(x).$

We therefore, rewrite (6) in the following form:

$$\frac{1}{2}\|\mathbf{v}(x, t)\|_Y^2 \leq \exp(-2\mu c_\rho t)\int_0^t\left[\exp(2\mu c_\rho \zeta)\int_\Omega fvdx\right]d\zeta$$

$$+\frac{1}{2}\|\mathbf{v}^0(x)\|_{H^2(\Omega)}^2\exp(-2\mu c_\rho t);\quad \text{for } t \in [0, T). \tag{8}$$

The rate of deformation for the problem is defined by the following equation:

$$\|D(\mathbf{v})\|_{L^2(\Omega)}^2 = \frac{1}{2}\|\nabla\mathbf{v}\|_{L^2(\Omega)}^2 \quad \text{(see p. 27 of [5], for } \gamma_0\mathbf{v}\cdot\mathbf{n} = \eta_v = 0). \tag{9}$$

We also have that

$$E^/(t) \leq -2\mu\|D(\mathbf{v})\|_{L^2(\Omega)}^2\beta(t) \quad \text{(see (12) on p. 9 of [4])}$$

for $\beta(t) := 1 - C_2\sqrt{2E(t)} - C_2^2E(t) > 0$ (see pp. 36-37 of [5]).

In view of (9), therefore,

$$E^/(t) \leq -\mu\|\nabla\mathbf{v}\|_{L^2(\Omega)}^2\beta(t). \tag{10}$$

From (3), (9) and (10), we obtain the following energy inequality:

$$\mu[1-\beta(t)]\|\nabla\mathbf{v}\|_{L^2(\Omega)}^2 \leq \int_\Omega fvdx, \quad \text{for } \beta(t) \in (0, 1/2],$$

where (see p. 9 of [4], for $\beta(t) \in (0, 3/2]$).

Eventually, we have

$$\frac{1}{2}\mu\|\nabla\mathbf{v}\|_{L^2(\Omega)}^2 \leq \int_\Omega fvdx, \quad \text{for } 1-\beta(t) \geq 1/2. \tag{11}$$

Adding the inequalities (8) and (11), we obtain

$$\frac{1}{2}\|\mathbf{v}\|_{L^2(\Omega)}^2 + \frac{1}{2}\mu\|\nabla\mathbf{v}\|_{L^2(\Omega)}^2$$

$$\leq \exp(-2\mu c_\rho t)\int_0^t\left[\exp(2\mu c_\rho\zeta)\int_\Omega fvdx\right]d\zeta + \int_\Omega fvdx$$

$$+\frac{1}{2}\|\mathbf{v}^0(x)\|_{L^2(\Omega)}^2\exp(-2\mu c_\rho t). \tag{12}$$

As our attention is focused on the solution to Option A as stipulated by the statement in [2], we take $f \equiv 0$, and from (12), we obtain

$$\| \mathbf{v}(x, t) \|_Y^2 + \mu \| \nabla \mathbf{v}(x, t) \|_Y^2 \leq \| \mathbf{v}^0(x) \|_{H^2(\Omega)}^2 \exp(-2\mu c_\rho t). \tag{13}$$

We now rewrite (13) in the form that will reveal an operator that will play a crucial role in the conclusion of the existence and uniqueness of the 'weak' solution to the problem:

We now rewrite (13) in the form

$$(\mathbf{v}, \mathbf{v})_Y - \mu(\Delta \mathbf{v}, \mathbf{v})_Y \leq \| \mathbf{v}^0(x) \|_{H^2(\Omega)}^2 \exp(-2\mu c_\rho t). \tag{14}$$

6. The Riesz's Representation for the Problem

We construct the following map: $\Phi : \Theta \times \Theta \to R$, defined by

$$(\mathbf{v}, \mathbf{w})_Y \mapsto (\mathbf{v}, \mathbf{w})_Y - \mu(\Delta \mathbf{v}, \mathbf{w})_Y; \quad \text{for } \mathbf{v}, \mathbf{w} \in \Theta.$$

It is not hard to show that Φ is a bounded sesquelinear form on $\Theta \times \Theta$. Hence, by the Riesz's representation theorem (p. 192 of [6]), there exists a bounded linear operator $A : \Theta \to \Theta$, such that, for $\mathbf{v} = \mathbf{w}$,

$$\Phi(\mathbf{v}, \mathbf{v}) = (A\mathbf{v}, \mathbf{v})_Y \quad \text{and} \quad \| \Phi \| = \| A \|. \tag{15}$$

In view of the preceding deduction, we then have $((I - \mu\Delta)\mathbf{v}, \mathbf{v})_Y = (A\mathbf{v}, \mathbf{v})_Y$; from which we deduce that

$$I - \mu\Delta = A. \tag{16}$$

7. The Characterization of the Operator $I - \mu\Delta$

Proposition 7.1. *The operator $I - \mu\Delta$ is self-adjoint and positive on* Θ.

Proof. For $\mathbf{v}, \mathbf{w} \in \Theta$,

$$(\mu\Delta\mathbf{v}, \mathbf{w})_Y = \mu\left(\int_{\partial\Omega} \gamma_1 \mathbf{v} \cdot \gamma_0 \mathbf{w} ds - \int_\Omega | \nabla \mathbf{v} \cdot \nabla \mathbf{w} | dx\right); \quad \gamma_1 \mathbf{v} = \nabla \mathbf{v} \cdot \mathbf{n},$$

$$(\mu\Delta\mathbf{w}, \mathbf{v})_Y = \mu\left(\int_{\partial\Omega} \gamma_1 \mathbf{w} \cdot \gamma_0 \mathbf{v} ds - \int_\Omega | \nabla \mathbf{w} \cdot \nabla \mathbf{v} | dx\right); \quad \gamma_1 \mathbf{w} = \nabla \mathbf{w} \cdot \mathbf{n},$$

from which we conclude

$$(\mu\Delta\mathbf{v},\ \mathbf{w})_Y = (\mu\Delta\mathbf{w},\ \mathbf{v})_Y; \quad \text{with} \quad \gamma_0\mathbf{w} = \gamma_0\mathbf{v} = 0, \text{ by 1(b).} \tag{17}$$

Further,

$$((I - \mu\Delta)\mathbf{v},\ \mathbf{w})_Y = (\mathbf{v},\ \mathbf{w})_Y - (\mu\Delta\mathbf{v},\ \mathbf{w})_Y,$$

$$((I - \mu\Delta)\mathbf{w},\ \mathbf{v})_Y = (\mathbf{w},\ \mathbf{v})_Y - (\mu\Delta\mathbf{w},\ \mathbf{v})_Y.$$

By (17) and the inner product property,

$$((I - \mu\Delta)\mathbf{v},\ \mathbf{w})_Y = ((I - \mu\Delta)\mathbf{w},\ \mathbf{v})_Y;$$

thus proving that $(I - \mu\Delta)$ is self-adjoint. Also,

$$((I - \mu\Delta)\mathbf{v},\ \mathbf{v})_Y = \|\mathbf{v}\|_Y^2 + \mu\|\nabla\mathbf{v}\|_Y^2 \geq 0;$$

thus proving the positive property of $I - \mu\Delta$ on Θ.

Proposition 7.2. *The operator* $I - \mu\Delta$ *is invertible and* $(I - \mu\Delta)^{-1}$ *is a bounded linear operator on* Θ.

Proof. By (16), $\mu\Delta$ is bounded on Θ.

Hence, there exists, $C_1 > 0$, such that,

$$\|(\mu\Delta)\mathbf{v}\|_Y \leq C_1\|\mathbf{v}\|_Y \quad \text{(see p. 91 of [6]).}$$

That is,

$$-\|(\mu\Delta)\mathbf{v}\|_Y \geq -C_1\|\mathbf{v}\|_Y.$$

However,

$$\|(I - \mu\Delta)\mathbf{v}\|_Y \geq \|\mathbf{v}\|_Y - \|\mu\Delta\|_Y \geq \|\mathbf{v}\|_Y - C_1\|\mathbf{v}\|_Y = (1 - C_1)\|\mathbf{v}\|_Y.$$

Therefore, provided, $1 - C_1 > 0$,

$$\|(I - \mu\Delta)\mathbf{v}\|_Y \geq (1 - C_1)\|\mathbf{v}\|_Y. \tag{18}$$

On the other hand, since $I - \mu\Delta$ is bounded by (16), there exists $C_1^{/} > 0$, such that,

$$\|(I - \mu\Delta)\mathbf{v}\|_Y \leq C_1^{/}\|\mathbf{v}\|_Y. \tag{19}$$

Combining (18) and (19), we obtain the inequality,

$$(1 - C_1)\| \mathbf{v} \|_Y \leq \| (I - \mu\Delta)\mathbf{v} \|_Y \leq C_1^{/} \| \mathbf{v} \|_Y. \tag{20}$$

From the preceding inequality, $(I - \mu\Delta)\mathbf{v} = \mathbf{0}$, implies that, $\mathbf{v} = \mathbf{0}$, that is, $Ker(I - \mu\Delta) = \{\mathbf{0}\}$, and hence $(I - \mu\Delta)^{-1}$ existed.

By Theorem 2.6-10(a) in [6], $(I - \mu\Delta)^{-1} : \Theta \to \Theta$ is a linear operator, and, by Theorem 2.6-10(b), $(I - \mu\Delta)^{-1}$ is bounded.

Proposition 7.3. $I - \mu\Delta$ *is a compact operator on* Θ.

Proof. Since $\Omega \subset R^3$ (according to Option A in [2]), Θ is finite dimensional. By Theorems 8.1-4(a) and 8.1-4(b) in [6], $I - \mu\Delta$ is compact on Θ. By (16), we subsequently conclude that A is also compact on Θ.

Proposition 7.4. *The operator* $(I - \mu\Delta)^{-1}A$ *is compact on* Θ.

Proof. By (16) and Proposition 7.3, A is a compact operator on Θ. By Proposition 7.2, $(I - \mu\Delta)^{-1} : \Theta \to \Theta$ is a bounded linear operator. By Theorem 8.3-2 in [6], then $(I - \mu\Delta)^{-1}A$ is also compact and the result follows.

With the results in Propositions 7.1, 7.2, 7.3 and 7.4, we are now ready to set up the problem (Option A in [2]) in order to deduce the existence and uniqueness of the 'weak' solution to problem (1), noting that pressure was eliminated in the energy statement. However, if the velocity solution existed and is unique, the same would apply to the pressure. Mathematically, once the velocity has been proven to exist and is unique, we could use the Navier-Stokes equation to calculate the corresponding pressure.

8. The Existence and Uniqueness for the Solution to the Problem (Option A in [2])

By (16), we have

$$(I - \mu\Delta)\mathbf{v} = A\mathbf{v}. \tag{21}$$

Using (21), we then set up the following equation for the problem having demonstrated the existence; linearity and boundedness of $(I - \mu\Delta)^{-1}$, in Proposition 7.2.

Hence, we put

$$\mathbf{v} = \alpha(I - \mu\Delta)^{-1}A\mathbf{v};\ \alpha \in (0, 1), \quad \text{for } \mathbf{v} \in \Theta. \tag{22}$$

Lemma 8.1. *For* $\mathbf{v} \in \Theta$, *the solution to* $\mathbf{v} = \alpha(I - \mu\Delta)^{-1}A\mathbf{v}$; $\alpha \in (0, 1)$, *is uniformly bounded*.

Proof. By (8) for $f \equiv 0$ (according to Option A in [2])

$$\begin{aligned}\| \mathbf{v}(x, t) \|_Y &\leq \| \mathbf{v}^0(x) \|_{H^2(\Omega)} \exp(-\mu c_\rho t) \\ &\leq C \exp(-\mu c_\rho t) < C,\end{aligned}$$

since $\exp(-\mu c_\rho t) < 1$, for all $t \in (0, T)$; and the result follows.

Main Theorem 8.2. *For* $\mathbf{v} \in \Theta$, *the equation* $\mathbf{v} = \alpha(I - \mu\Delta)^{-1}A\mathbf{v};\ \alpha \in (0, 1)$ *has at most one solution*.

Proof. By Lemma 8.1 the solution to the equation, $\mathbf{v} = \alpha(I - \mu\Delta)^{-1}A\mathbf{v}$; $\alpha \in (0, 1)$, for $\mathbf{v} \in \Theta$, is uniformly bounded.

By Proposition 7.4, the operator $(I - \mu\Delta)^{-1}A$ is compact. Then, by the Leray-Schaueder fixed-point Theorem (p. 245 in [8]), the solution to the equation does exist.

To prove uniqueness, let $\mathbf{w} \in \Theta$ be another solution to the equation. Then,

$$\begin{aligned}\| \mathbf{v} - \mathbf{w} \|_Y &= \| \alpha(I - \mu\Delta)^{-1}A(\mathbf{v} - \mathbf{w}) \|_Y \\ &\leq \| \alpha(I - \mu\Delta)^{-1}A \|_Y \| \mathbf{v} - \mathbf{w} \|_Y.\end{aligned}$$

Since by (16), $\| \alpha(I - \mu\Delta)^{-1}A \|_Y < 1$, $\alpha \in (0, 1)$, uniqueness follows.

Remarks 8.3.

(a) By (7), we have

$$\frac{1}{2}\| \mathbf{v}^0(x) \|^2_{H^2(\Omega)} \leq C.$$

Since

$$\| \mathbf{v}^0(x) \|_{H^2(\Omega)} = \left(\sum_{0 \le \alpha \le 2} \left| \frac{\partial^{|\alpha|} \mathbf{v}^0(x)}{\partial x_{\alpha_0} \partial x_{\alpha_1} \partial x_{\alpha_2}} \right| \right)^{1/2}; \quad |\alpha| = \alpha_0 + \alpha_1 + \alpha_2,$$

$\left| \dfrac{\partial^{|\alpha|} \mathbf{v}^0(x)}{\partial x_{\alpha_0} \partial x_{\alpha_1} \partial x_{\alpha_2}} \right| \le C$, and the converse also holds. In a way, we can claim that our $\mathbf{v}^0(x)$ satisfies, to a certain extent, condition (4) as per the requirements of Option A in the statement of the problem in [2].

(b) By (8), we have

$$\frac{1}{2} \| \mathbf{v}(x, t) \|_Y^2 \le \frac{1}{2} \| \mathbf{v}^0(x) \|_{H^2(\Omega)}^2 \exp(-2\mu c_\rho t) < \frac{1}{2} \| \mathbf{v}^0(x) \|_{H^2(\Omega)}^2 \le C,$$

since $\exp(-\mu c_\rho t) < 1$, for all $t \in (0, T)$. Therefore, for $t \in (0, T)$, $\frac{1}{2} \| \mathbf{v}(x, t) \|_Y^2 < C$. Hence, our energy satisfies condition (7) in the stipulation of Option A of the problem in [2].

(c) The fixed-point of (22), which is the solution to Option A, satisfies the inequality (8). Therefore, we have

$$\| \mathbf{v}(x, t) \|_Y \le \| \mathbf{v}^0(x) \|_{H^2(\Omega)} \exp(-\mu c_\rho t), \quad \text{for all } t \in [0, T). \tag{23}$$

Should we "stretch" the interval $[0, T)$ to $[0, \infty)$ the inequality (23) still holds.

Hence, we assert that the fixed-point of (22) is global on time, and, therefore, it satisfies condition (6) of the stipulations to the solution to Option A of the problem.

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