



## **CONJUGACY AND GEOMETRY II - MOORE-PENROSE INVERSE AND FEET OF THE PERPENDICULARS**

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### **Abstract**

In this paper on space geometry, generalized inverses are used in the study of distances. Three cases are considered: distance from a point to a plane, distance from a point to a line and distance between two skew lines. Moore-Penrose inverses occur in the expressions of the feet of the perpendiculars and in the representation of the vectors materializing the distances. The results of this kind of problems fit in the cadre of approximation theory and, because best approximation problems often require the projection of the origin onto linear varieties, in order to solve the proposed problems, we make extensive use of the conjugacy principle, much present in Mathematics. The obtained results are not only useful for undergraduate Science and Engineering students but are also applicable in very practical sciences and techniques, notably on Coordinate Metrology, Photogrammetry, etc. Moreover, this paper could pave the way for more generalized problems demanding more sophisticated approaches.

### **1. Introduction**

In this paper, we are interested in problems of distances between various entities, in the ordinary space. It is known that some students begin to approach the concept of distance in pre-university studies. It is hoped that the present paper helps Science and Engineering students to successfully cope with and deepen these kinds of problems in a Calculus course or in an Introductory Linear Algebra course in their

first two years of undergraduate work. However, this text pretends to be not only confined to undergraduate students but also the provided results could be extended to a great variety of problems in the three-dimensional space, which are common in the practical sciences and engineering such as the various questions that arise in the Coordinate Metrology field.

Usually only formulae for distances are looked for. Here, instead of restricting ourselves to the computation of the distances, we focus our attention mainly on the nearest point (the foot of the perpendicular).

The main tool used in this paper, in order to determine the foot of the perpendicular, is the Moore-Penrose inverse,  $A^\dagger$ , of a real matrix  $A$ , which is the unique matrix that satisfies [3, p. 9] the following relations

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^T = AA^\dagger, \quad (A^\dagger A)^T = A^\dagger A.$$

For computing the Moore-Penrose inverse, there are formulae and algorithms. We just mention: the MacDuffee formula [1, pp. 25-26 and Theorem 5, p. 48], [3, Theorem 1.3.2, p. 14] and also [24, Theorem 1.1.5, p. 5], whose reference to [15] seems unsubstantiated; and the Decell algorithm [9].

Concerning the MacDuffee formula, we record

Let  $A = BC$ , where  $A \in \mathbb{R}^{m \times n}$ ,  $B \in \mathbb{R}^{m \times r}$ ,  $C \in \mathbb{R}^{r \times n}$ , with  $r = \text{rank}(A) = \text{rank}(B) = \text{rank}(C)$ . Then,  $A^\dagger = C^T(CC^T)^{-1}(B^TB)^{-1}B^T$ .

From the above MacDuffee formula we obtain, in particular,  $A^\dagger = A^T(AA^T)^{-1}$  and  $A^\dagger = (A^TA)^{-1}A^T$ , for a full row rank and full column rank matrix  $A$ , respectively, [2, p. 457], [5, p. 13] and [20, Exercise 5.12.16.d, p. 428].

Our proposal combines different and not so common mathematical tools specifically from linear algebra and approximation theory. These tools allow extensions to higher dimension Euclidean spaces.

The present work has also a didactical tone. This algebraic approach of a geometric problem is an appropriate instance of didactics of mathematics and could be a starting point to other researches.

In this paper, we consider distances:  $d(P, \pi)$  between a point  $P$  and a plane  $\pi$ ,

in Section 1;  $d(P, \ell)$  between a point  $P$  and a line  $\ell$ , in Section 2; and  $d(\ell_1, \ell_2)$  between two skew lines  $\ell_1$  and  $\ell_2$ , in Section 3.

Lines and planes may be defined by underdetermined systems of linear equations  $AX = B$ , where  $A$  is a full row rank matrix. We look for the minimum Euclidean norm solution of such systems. In geometric terms, the minimum Euclidean norm solution of a system, having infinitely many solutions, corresponds to the point closest to the origin of the coordinates. It is known [1, p. 109], [20, p. 423] that the Moore-Penrose inverse associated to an underdetermined system of linear equations gives the minimal least squares solution. In other words, it gives the solution nearest to the origin of the coordinates.

Thus, if the point  $P$  is at the origin and applying the Moore-Penrose inverse, then we obtain the foot of the perpendicular

$$S = A^\dagger B = A^T (AA^T)^{-1} B.$$

But this is not always the case. What happens if  $P$  is not the origin? It is well known [10, p. 25] that the distance is invariant under translation. This fact answers the preceding question: in order to apply the Moore-Penrose inverse we must perform two convenient translations. First, moving the pair  $d(P, \ell)$  of geometric objects such that the point  $P$  coincides with the origin of the coordinates and consequently falling in the previous case. Finally, performing the reverse translation. The legitimacy of the previous process is given by the conjugacy principle, much present in Mathematics, and which can be resumed by the following sentence. To solve a difficult problem  $A$ , we consider and solve an easier one  $T$ , using a transformation  $S$  and its inverse  $S^{-1}$ . The relation  $A = STS^{-1}$  is called *conjugacy* and it is an equivalence relation. The pervasiveness of the conjugacy principle is quite well documented in [19, Vol. I, pp. 53, 75; Vol. II, pp. 67, 141, 374]. For a sound and gentle approach to conjugacy classes see [4, pp. 142-147]. When using the conjugacy principle, we consider a translation  $t$  (which is not a linear application, as the null vector is not preserved) defined by

$$t_{\vec{v}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$\vec{x} \rightarrow \vec{x} + \vec{v},$$

where, of course:

$$(t_{\vec{v}})^{-1} = t_{(-\vec{v})}.$$

Some abuse of notation is patent in this paper. We use the sign “:=” to identify these situations. We consider an orthonormal referential  $\{O, (\vec{e}_1, \vec{e}_2, \vec{e}_3)\}$  and, under adequate isomorphisms, we write points and vectors in several ways, according to our needs in each moment.

## 2. Distance from a Point to a Plane

In this section, we will consider  $d(P, \pi)$  the distance from a point  $P$  to a plane  $\pi$  given by  $X = M + \alpha\vec{u} + \beta\vec{v}$ . Our goal is to determine:

- (i) the foot of the perpendicular,  $S$ , drawn from the point  $P$  onto the plane  $\pi$ ;
- (ii) the vector  $\vec{PS} = S - P$  which connects the point  $P$  to the plane  $\pi$ ;
- (iii) the distance  $d(P, \pi)$ .

In order to answer the previous questions, we state the following:

**Proposition 1.** *Let  $\pi$  be a plane given by  $X = M + \alpha\vec{u} + \beta\vec{v}$ , where*

$$M = (m_1, m_2, m_3), \quad \vec{u} = u_1\vec{e}_1 + u_2\vec{e}_2 + u_3\vec{e}_3, \quad \vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3$$

*and let  $P = (p_1, p_2, p_3)$  be a point external to  $\pi$ . Then*

- (i) *the foot of the perpendicular,  $S$ , drawn from the point  $P$  onto the plane  $\pi$  is given by*

$$S = P + A^T(AA^T)^{-1}A(M - P),$$

*where  $A$  is the row matrix whose entries are the cofactors of the elements of the third column of the matrix*

$$\begin{bmatrix} u_1 & v_1 & 1 \\ u_2 & v_2 & 1 \\ u_3 & v_3 & 1 \end{bmatrix};$$

- (ii)  *$\vec{PS}$  is a vector which achieves the distance from the point  $P$  to plane  $\pi$ ;*
- (iii) *the distance  $d(P, \pi)$  between point  $P$  and plane  $\pi$  is given by  $d(P, \pi) = \|\vec{PS}\|$ .*

**Proof.** In order to set this geometric problem in the context of Moore-Penrose inverses, we need to move the point  $P$  to the origin of the coordinates (as illustrated in Figure 1).

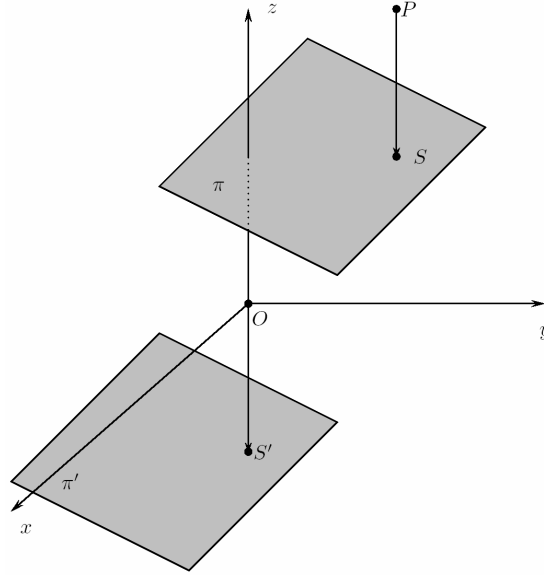
In fact, we make a translation so that the point  $P$  of the geometric pair  $(P, \pi)$  moves to the origin of the coordinates. So, the geometric pair  $(P, \pi)$  turns into the geometric pair

$$(O, \pi') := (P + \overrightarrow{PO}, \pi + \overrightarrow{PO}) = (P + O - P, \pi + O - P) := (O, \pi - P).$$

We have

$$\pi' = \pi - P := X' = (M - P) + \alpha \vec{u} + \beta \vec{v} := M' + \alpha \vec{u} + \beta \vec{v},$$

where  $M' = (m_1 - p_1, m_2 - p_2, m_3 - p_3)$ .



**Figure 1.** Foot of the perpendicular,  $S$ , drawn from the point  $P$ .

Associated with the pair  $(O, \pi')$  is the system of linear equations

$$AX' = B'.$$

Applying the Moore-Penrose inverse, we obtain the foot of the perpendicular on the plane  $\pi'$ ,

$$S' = A^\dagger B'.$$

Hence, we find

$$\overrightarrow{OS'} \text{ and } \|\overrightarrow{OS'}\| = d(O, \pi') = d(P, \pi).$$

Performing the reverse translation  $-\overrightarrow{PO} = \overrightarrow{OP}$ , we get

$$S = S' + \overrightarrow{OP} = S' + P - O = S' + P,$$

$$S' = S - P = \overrightarrow{PS}$$

and thus

$$d(P, \pi) = \|\overrightarrow{PS}\|.$$

Now, let us present the procedure when use is made of the Moore-Penrose inverse in order to determine  $S'$ . It is convenient to write the plane  $\pi'$  in cartesian notation.

We have, successively,

$$\begin{aligned} \pi' &:= X' = M' + \alpha \vec{u} + \beta \vec{v} \\ &:= \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} m'_1 \\ m'_2 \\ m'_3 \end{bmatrix} + \alpha \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \beta \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\ &:= \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} x' - m'_1 \\ y' - m'_2 \\ z' - m'_3 \end{bmatrix}. \end{aligned}$$

For the sake of eliminating the real parameters  $\alpha$  and  $\beta$ , let us assume, without loss of generality, that the principal determinant is

$$\Delta_p = \det \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \neq 0.$$

By geometric considerations and by using the Rouché-Cappeli-Kronecker-Fontené theorem, the unique characteristic determinant,  $\Delta C_3$ , corresponding to the third equation, must be null. So,

$$\Delta C_3 = 0 = \det \begin{bmatrix} u_1 & v_1 & x' - m'_1 \\ u_2 & v_2 & y' - m'_2 \\ u_3 & v_3 & z' - m'_3 \end{bmatrix} := AX' - B' = 0.$$

Hence, the plane  $\pi'$  is given by the underdetermined linear system, having infinitely many solutions,

$$AX' = B',$$

where

$$A = \begin{bmatrix} u_2v_3 - u_3v_2 & u_3v_1 - u_1v_3 & u_1v_2 - u_2v_1 \end{bmatrix}$$

and

$$B' = [(u_2v_3 - u_3v_2)m'_1 + (u_3v_1 - u_1v_3)m'_2 + (u_1v_2 - u_2v_1)m'_3],$$

with  $m'_i = m_i - p_i$ ,  $i = 1, 2, 3$ .

The looked for exact solution of minimum Euclidean norm is given by

$$S' = A^T(AA^T)^{-1}B' = A^T(AA^T)^{-1}A(M - P). \quad \square$$

### 2.1. Example

Consider the point  $P = (2, 3, 1)$  and the plane  $\pi := X = (1, 1, 1) + \alpha(-\vec{e}_1 + \vec{e}_2) + \beta(-\vec{e}_1 + \vec{e}_3)$ .

Making the translation  $\overrightarrow{PO} = -\overrightarrow{OP}$ , from the geometric pair  $(P, \pi)$  we form the geometric pair  $(O, \pi') = (O, \pi - P)$ , where

$$\pi' := X' = (-1, -2, 0) + \alpha(-\vec{e}_1 + \vec{e}_2) + \beta(-\vec{e}_1 + \vec{e}_3).$$

Writing the plane  $\pi'$  in the matrix form

$$AX' = B',$$

the foot of the perpendicular is given by

$$\begin{aligned} S' &= A^\dagger B' \\ &= A^T(AA^T)^{-1}B' \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \left( \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)^{-1} [-3] \\ &= \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}. \end{aligned}$$

Undoing the translation, we obtain

$$S = S' + \overrightarrow{OP} = (-1, -1, -1) + 2\overrightarrow{e_1} + 3\overrightarrow{e_2} + \overrightarrow{e_3} = (1, 2, 0)$$

from where we conclude that the foot of the perpendicular  $S$  onto  $\pi$  is  $S = (1, 2, 0)$ .

A vector which materializes the distance is

$$\overrightarrow{PS} = S - P = (-1, -1, -1) = -\overrightarrow{e_1} - \overrightarrow{e_2} - \overrightarrow{e_3}$$

and the distance is

$$d(P, \pi) = \|\overrightarrow{PS}\| = \|\overrightarrow{OS'}\| = \sqrt{3}.$$

### 3. Distance from a Point to a Line

In this section, we will consider  $d(P, \ell)$  the distance between a point  $P$  and a line  $\ell$  given by  $X = M + \alpha\vec{u}$ , where  $M$  is a point on the line and  $\vec{u}$  is a director vector of the line. Our goal is to determine:

- (i) the foot of the perpendicular,  $S$ , drawn from the point  $P$  onto the line  $\ell$ ;
- (ii) the vector  $\overrightarrow{PS} = S - P$  which connects the point  $P$  to the line  $\ell$ ;
- (iii) the distance  $d(P, \ell)$ .

In order to get the pretended results, we establish the following:

**Proposition 2.** *Let  $\ell$  be a line given by  $X = M + \alpha\vec{u}$ , where  $M = (m_1, m_2, m_3)$  is a point on  $\ell$  and  $\vec{u} = u_1\overrightarrow{e_1} + u_2\overrightarrow{e_2} + u_3\overrightarrow{e_3}$  is a director vector of  $\ell$  and let  $P = (p_1, p_2, p_3)$  be a point external to the line  $\ell$ . Then*

(i) *the foot of the perpendicular,  $S$ , drawn from the point  $P$  onto the line  $\ell$  is given by*

$$S = P + A^T(AA^T)^{-1}A(M - P),$$

where

$$A = \begin{bmatrix} u_2 & -u_1 & 0 \\ u_3 & 0 & -u_1 \end{bmatrix};$$

(ii)  $\overrightarrow{PS}$  is a vector which achieves the distance from the point  $P$  to the line  $\ell$ ;

(iii) the distance  $d(P, \pi)$  between point  $P$  and the line  $\ell$  is given by  $d(P, \ell) = \|\overrightarrow{PS}\|$ .

**Proof.** The distance  $d(P, \ell)$  is the norm of the vector  $\overrightarrow{PS}$ , where  $S$  is the foot of the perpendicular of the point  $P$  on the line  $\ell$  (as illustrated in Figure 2). If the point  $P$  is at the origin of the coordinates, then writing the line equation in the matrix form

$$AX = B,$$

the point  $S$  is given by

$$S = A^\dagger B,$$

where  $A^\dagger$  is the Moore-Penrose inverse of  $A$ .

Again, if the point  $P$  is not at the origin, then a translation of this point to the origin is required. Thus, translating the geometric pair  $(P, \ell)$  along the vector  $\overrightarrow{PO}$ , we obtain the geometric pair  $(O, \ell')$ , where the line  $\ell'$  is written in the matrix form

$$AX' = B',$$

which gives the foot of the perpendicular

$$S' = A^\dagger B'.$$

In order to use generalized inverses, we write the line  $\ell'$  as the intersection of two planes and hence we have to eliminate the real parameter  $\alpha$ . The equation of the line  $\ell'$ ,

$$X' = (M - P) + \alpha \vec{u},$$

gives, successively,

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} m_1 - p_1 \\ m_2 - p_2 \\ m_3 - p_3 \end{bmatrix} + \alpha \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

and

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} [\alpha] = \begin{bmatrix} x' - (m_1 - p_1) \\ y' - (m_2 - p_2) \\ z' - (m_3 - p_3) \end{bmatrix}.$$

From the interplay between algebraic and geometric considerations (Rouché-Cappelli-Kronecker-Fontené) and as the system is consistent, there are two null characteristic determinants  $\Delta C_2$  and  $\Delta C_3$ , corresponding to the second and third equations, respectively,

$$0 = \Delta C_2 = \det \begin{bmatrix} u_1 & x' - (m_1 - p_1) \\ u_2 & y' - (m_2 - p_2) \end{bmatrix}$$

and

$$0 = \Delta C_3 = \det \begin{bmatrix} u_1 & x' - (m_1 - p_1) \\ u_3 & z' - (m_3 - p_3) \end{bmatrix},$$

by assuming, without loss of generality, that the principal determinant is

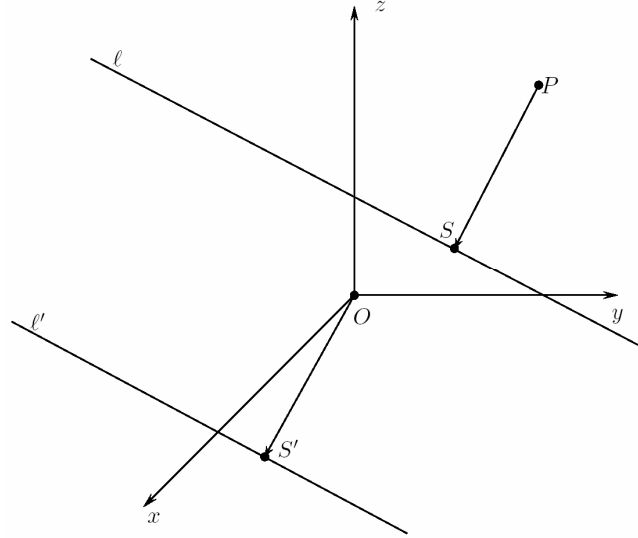
$$\Delta_p = \det[u_1] = u_1 \neq 0.$$

We form the linear system

$$AX' = B' := \begin{cases} u_2 x' - u_1 y' = u_2(m_1 - p_1) - u_1(m_2 - p_2), \\ u_3 x' - u_1 z' = u_3(m_1 - p_1) - u_1(m_3 - p_3), \end{cases}$$

which has infinitely many solutions. The (unique) one of minimum Euclidean norm is given by

$$S' = A^\dagger B' = A^T (AA^T)^{-1} B' = A^T (AA^T)^{-1} A(M - P).$$



**Figure 2.** Foot of the perpendicular,  $S$ , drawn from the point  $P$ .

Finally, under the reverse translation, we find

$$S = S' + P,$$

and

$$d(P, \ell) = \|\overrightarrow{PS}\| = \|\overrightarrow{OS'}\|. \quad \square$$

**Remark 3.1.** If the line equation is given by a system of two plane equations

$$\pi_1 := a_1x + b_1y + c_1z + d_1 = 0,$$

$$\pi_2 := a_2x + b_2y + c_2z + d_2 = 0,$$

we can write the line equation in the matricial form

$$AX = B,$$

where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}, \quad B = \begin{bmatrix} -d_1 \\ -d_2 \end{bmatrix}.$$

Otherwise, if the line equation is given in the vectorial form

$$X = M + \alpha \vec{u},$$

where  $M = (m_1, m_2, m_3) \in \ell$  and  $\vec{u} = (u_1, u_2, u_3)$  is the direction vector of  $\ell$ , in order to write the line equation in a matrix form, we need to obtain two plane equations such that the intersection is the given line. For this – instead of using characteristic determinants – we use the cross vector product. We utilize two vectors,  $\vec{v}_1, \vec{v}_2$ , such that  $\vec{u} = \vec{v}_1 \times \vec{v}_2$ , where  $\{\vec{v}_1, \vec{v}_2\}$  is a linearly independent set. If  $\vec{u} = (u_1, u_2, u_3)$ , then we can choose conveniently  $\vec{v}_1, \vec{v}_2$  such that the dot products are null:  $\vec{v}_1 \bullet \vec{u} = 0$  and  $\vec{v}_2 \bullet \vec{u} = 0$ . For example, if  $u_1 \neq 0$ , then we may choose

$$\vec{v}_1 = (u_3, 0, -u_1)$$

and

$$\vec{v}_2 = (u_2, -u_1, 0).$$

Thus, we obtain the plane equations

$$u_3x - u_1z + d_1 = 0$$

and

$$u_2x - u_1y + d_2 = 0,$$

where

$$d_1 = u_1m_3 - u_3m_1$$

and

$$d_2 = u_1m_2 - u_2m_1.$$

### 3.1. Example

Consider the point  $P = (3, 2, 1)$  and the line

$$\ell := X = (5, 2, -1) + \alpha(-\vec{e}_1 + 3\vec{e}_2 + \vec{e}_3).$$

In order to use the Moore-Penrose inverse, the point  $P$  has to be displaced to the origin. So, we perform the translation  $\vec{PO} = -3\vec{e}_1 - 2\vec{e}_2 - \vec{e}_3$ .

The geometric pair  $(P, \ell)$  is transformed into the geometric pair  $(O, \ell')$ , such that

$$\ell' := X' = (x', y', z') = (2, 0, -2) + \alpha(-\vec{e}_1 + 3\vec{e}_2 + \vec{e}_3).$$

The line can be given by the system of linear equations

$$\begin{cases} 3x' + y' = 6 \\ x' + z' = 0 \end{cases} \Leftrightarrow \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} := AX' = B'.$$

The foot of the perpendicular,  $S'$ , drawn from the origin onto the line  $\ell'$  is given by

$$\begin{aligned} S' &= A^T(AA^T)^{-1}B' \\ &= \begin{bmatrix} 3 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \left( \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 18 \\ 12 \\ -18 \end{bmatrix}. \end{aligned}$$

Reversing the translation operation  $\overrightarrow{OP}$ , we find the perpendicular foot  $S$ , on  $\ell$ ,

$$S = S' + P = \frac{1}{11} \begin{bmatrix} 18 \\ 12 \\ -18 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 51 \\ 34 \\ -7 \end{bmatrix}.$$

The vector which connects  $P$  to  $S$  is

$$\overrightarrow{PS} = S - P = \frac{1}{11} \begin{bmatrix} 18 \\ 12 \\ -18 \end{bmatrix} = \frac{18}{11} \vec{e}_1 + \frac{12}{11} \vec{e}_2 - \frac{18}{11} \vec{e}_3$$

and the distance from  $P$  to  $\ell$  is

$$d(P, \ell) = \|\overrightarrow{PS}\| = \|S - P\| = \|\overrightarrow{OS'}\| = \frac{6}{11} \sqrt{22}.$$

**Remark 3.2.** So far, in order to get two planes whose intersection is the line  $\ell$ , we used either an algebraic approach with help of characteristic determinants or a geometric approach by using the cross vector product. However, the cross product of two vectors is restricted [12, 18] to spaces of dimensions 3 and 7. So, in  $\mathbb{R}^n$  and also in more general settings, where the entries of the vectors and matrices either belong to a commutative ring or are pairwise commuting matrices, we have to use (block) determinants [6, 17]. All the same, as the inner product has less restrictions than the cross product and in order to keep the geometric flavour of this paper, we propose another way for obtaining two planes whose intersection is the line  $\ell$ . We may proceed in the following way:

(i) the line  $\ell$  is given by  $X = M + \alpha \vec{u}$ ;

(ii) construct a plane  $\pi$  perpendicular to the line  $\ell$ :

$$\pi := \overrightarrow{MX} \bullet \vec{u} = 0,$$

where  $\bullet$  stands for dot product;

(iii) choose two points  $R$  and  $T$  on the plane  $\pi$ : we build the planes  $\pi_1$  and  $\pi_2$ , such that  $\ell = \pi_1 \cap \pi_2$  and defined by three points:  $\pi_1$  is defined by the points  $M, N$  on the line  $\ell$  and  $R$  on the plane  $\pi$ ;  $\pi_2$  is defined by the same points  $M, N$  on the line  $\ell$  and the point  $T$  on the plane  $\pi$ :

$$\pi_1 := \det \begin{bmatrix} x & y & z & 1 \\ m_1 & m_2 & m_3 & 1 \\ n_1 & n_2 & n_3 & 1 \\ r_1 & r_2 & r_3 & 1 \end{bmatrix} = 0$$

and

$$\pi_2 := \det \begin{bmatrix} x & y & z & 1 \\ m_1 & m_2 & m_3 & 1 \\ n_1 & n_2 & n_3 & 1 \\ t_1 & t_2 & t_3 & 1 \end{bmatrix} = 0.$$

#### 4. Distance between Two Skew Lines

In this section, we will consider the distance,  $d(\ell_1, \ell_2)$ , between the lines  $\ell_1$  and  $\ell_2$  given by

$$\ell_1 := X = P + \alpha \vec{u} \quad \text{and} \quad \ell_2 := X = Q + \beta \vec{v}.$$

The feet of the perpendiculars  $S_1$  onto  $\ell_1$  and  $S_2$  onto  $\ell_2$  are searched. We present an algorithm to achieve the feet of the perpendiculars and the distance between the two skew lines.

So, in order to apply the technique presented in Section 3, we shall proceed in two steps:

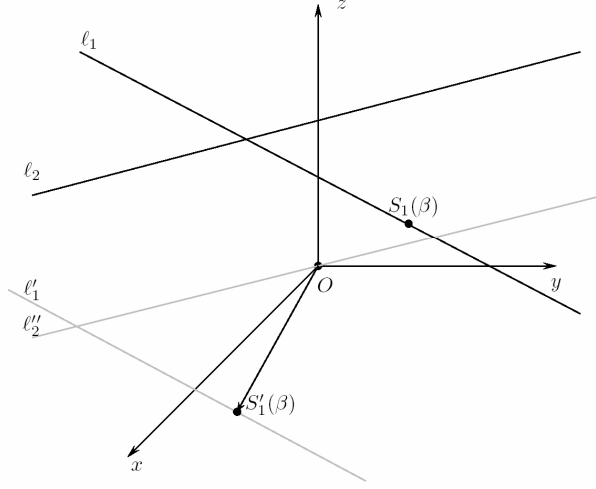
(I) First, we displace the current point  $Q(\beta)$  of the line  $\ell_2$  to the origin. We get the point  $S'_1(\beta)$  on the line  $\ell'_1$ . Performing the reverse translation, we obtain the point  $S_1(\beta)$  on the line  $\ell_1$ .

(II) Next, in an analogous way, we obtain  $S_2(\alpha)$  on the line  $\ell_2$ .

Since the point  $S_1(\beta) \in \ell_1$  we must have  $S_1(\beta) = P(\alpha)$ , where  $P(\alpha)$  is a current point of the line  $\ell_1$ . Analogously, the point  $S_2(\alpha) = Q(\beta)$ . From the consistent system of equations

$$\begin{cases} S_1(\beta) = P(\alpha), \\ S_2(\alpha) = Q(\beta), \end{cases}$$

we determine the convenient values of  $\alpha$  and  $\beta$ .



**Figure 3.** Foot of the perpendicular,  $S_1(\beta)$ , onto the line  $\ell_1$ .

The distance between the two lines is given by

$$d(\ell_1, \ell_2) = \|\overrightarrow{S_1 S_2}\|.$$

#### 4.1. Algorithm

Now, let us present the four steps procedure, illustrated by Figure 3.

- (I) Distance from the current point,  $Q(\beta)$ , of line  $\ell_2$ , to the line  $\ell_1$ .

We take the current point

$$Q(\beta) = (q_1 + \beta v_1, q_2 + \beta v_2, q_3 + \beta v_3) \text{ on } \ell_2$$

and the line

$$\ell_1 := X = P + \alpha \vec{u}.$$

As we look for the foot of the perpendicular  $S_1(\beta)$ , onto  $\ell_1$ , we apply the technique presented in Section 3. So, we perform the translation  $\overrightarrow{Q(\beta)O} = O - Q(\beta)$ . The line  $\ell'_1$  is written in vectorial form as

$$\ell'_1 := X' = P - Q(\beta) + \alpha \vec{u}.$$

The foot of the perpendicular,  $S'_1(\beta)$ , drawn from the origin onto the line

$\ell'_1$  is given by

$$S'_1(\beta) = A^T (AA^T)^{-1} B'(\beta), \text{ on } \ell'_1,$$

with

$$A = \begin{bmatrix} u_2 & -u_1 & 0 \\ u_3 & 0 & -u_1 \end{bmatrix} \text{ and } B'(\beta) = \begin{bmatrix} u_2[p_1 - q_1(\beta)] - u_1[p_2 - q_2(\beta)] \\ u_3[p_1 - q_1(\beta)] - u_1[p_3 - q_3(\beta)] \end{bmatrix}.$$

(II) Distance from the current point  $P(\alpha)$ , on line  $\ell_1$ , to the line  $\ell_2$ .

We take the current point

$$P(\alpha) = (p_1 + \alpha u_1, p_2 + \alpha u_2, p_3 + \alpha u_3) \text{ on } \ell_1$$

and the line

$$\ell_2 := X = Q + \beta \vec{v}.$$

For getting the foot of the perpendicular  $S_2(\alpha)$ , onto  $\ell_2$ , we must consider, in analogy to the preceding situation, successively: the translation  $\overrightarrow{P(\alpha)O} = O - P(\alpha)$ ; the line

$$\ell''_2 := X'' = Q - P(\alpha) + \beta \vec{v};$$

and hence the foot of the perpendicular,  $S''_2(\alpha)$ , drawn from the origin onto the line  $\ell''_2$  is given by

$$S''_2(\alpha) = C^T (CC^T)^{-1} D''(\alpha), \text{ on } \ell''_2,$$

with

$$C = \begin{bmatrix} v_2 & -v_1 & 0 \\ v_3 & 0 & -v_1 \end{bmatrix} \text{ and } D''(\alpha) = \begin{bmatrix} v_2[q_1 - p_1(\alpha)] - v_1[q_2 - p_2(\alpha)] \\ v_3[q_1 - p_1(\alpha)] - v_1[q_3 - p_3(\alpha)] \end{bmatrix}.$$

(III) Using reverse translations.

Performing the two corresponding reverse translations, we obtain

$$S_1(\beta) = S'_1(\beta) + Q(\beta) \text{ onto } \ell_1$$

$$S_2(\alpha) = S''_2(\alpha) + P(\alpha) \text{ onto } \ell_2.$$

(IV) Final step, determination of the sought concretizations  $\alpha^*$  and  $\beta^*$ .

Notice that we look for the optimum point  $Q(\beta^*)$  on  $\ell_2$  and the optimum point  $P(\alpha^*)$  on  $\ell_1$ .

After the procedure comes to an end, we must have, due to geometric reasons,

$$S_1(\beta) = P(\alpha)$$

$$S_2(\alpha) = Q(\beta).$$

The preceding relations form a consistent system of six linear equations in the two unknowns  $\alpha$  and  $\beta$ , from which we get  $\alpha = \alpha^*$  and  $\beta = \beta^*$ .

Finally, the perpendicular feet are

$$S_1 = S_1(\beta^*) = P(\alpha^*) \text{ onto the line } \ell_1$$

$$S_2 = S_2(\alpha^*) = Q(\beta^*) \text{ onto the line } \ell_2$$

and the distance between the given lines is  $d(\ell_1, \ell_2) = \|\overrightarrow{S_1 S_2}\|$ , being  $\overrightarrow{S_1 S_2}$  the vector which materializes the distance.

#### 4.2. Example

Let us be given the two lines

$$\ell_1 := X = P + \alpha \vec{u} = (1, 2, -3) + \alpha(-5\vec{e}_1 + \vec{e}_2 - 2\vec{e}_3),$$

$$\ell_2 := X = Q + \beta \vec{v} = (2, -1, -1) + \beta(\vec{e}_1 + 7\vec{e}_2 - 4\vec{e}_3).$$

(I) Consider the pair  $(\ell'_1, \ell'_2)$ , when we displace the line  $\ell_2$  to the origin of coordinates. We look for the distance from the current point  $Q(\beta)$  of line  $\ell_2$  to line  $\ell_1$ . We can write  $\ell_2$  as

$$\ell_2 := \{(2 + \beta, -1 + 7\beta, -1 - 4\beta) : \beta \in \mathbb{R}\},$$

so its current point is

$$Q(\beta) = (2 + \beta, -1 + 7\beta, -1 - 4\beta).$$

Performing the translation  $\overrightarrow{Q(\beta)O} = O - Q(\beta)$ , we get

$$\ell'_2 := X' = X - Q(\beta) = Q(\beta) - Q(\beta) + \beta \vec{v} = O + \beta \vec{v}$$

and

$$\ell'_1 := X' = X - Q(\beta) = P - Q(\beta) + \alpha \vec{u};$$

and hence the foot of the perpendicular,  $S'_1(\beta)$ , drawn from the origin onto the line  $\ell'_1$  is given by

$$S'_1(\beta) = A^T (AA^T)^{-1} B'(\beta), \text{ on } \ell'_1,$$

with

$$A = \begin{bmatrix} 1 & 5 & 0 \\ -2 & 0 & 5 \end{bmatrix} \text{ and } B'(\beta) = \begin{bmatrix} 14 - 36\beta \\ -8 + 22\beta \end{bmatrix}.$$

We compute

$$\begin{aligned} S'_1(\beta) &= \begin{bmatrix} 1 & -2 \\ 5 & 0 \\ 0 & 5 \end{bmatrix} \left( \begin{bmatrix} 1 & 5 & 0 \\ -2 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 5 & 0 \\ 0 & 5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 14 - 36\beta \\ -8 + 22\beta \end{bmatrix} \\ &= \begin{bmatrix} 1 - \frac{8\beta}{3} \\ \frac{13}{5} - \frac{20\beta}{3} \\ -\frac{6}{5} + \frac{10\beta}{3} \end{bmatrix}. \end{aligned}$$

(II) Consider, now, the pair  $(\ell'_1, \ell'_2)$ , when the line  $\ell_1$  is displaced in order to pass through the origin.

The line  $\ell_1$  may be represented as

$$\ell_1 := \{(1 - 5\alpha, 2 + \alpha, -3 - 2\alpha) : \alpha \in \mathbb{R}\}.$$

Here the translation is  $\overrightarrow{P(\alpha)O} = O - P(\alpha)$ , where

$$P(\alpha) = (1 - 5\alpha, 2 + \alpha, -3 - 2\alpha)$$

is the generic point of line  $\ell_1$ , and we have

$$\ell''_1 := X'' = X - P(\alpha) = P(\alpha) - P(\alpha) + \alpha \vec{u} = O + \alpha \vec{u}$$

and

$$\ell_2'' := X'' = X - P(\alpha) = Q - P(\alpha) + \beta \vec{v};$$

and, *mutatis mutandis*, the foot,  $S_2''(\alpha)$  onto  $\ell_2''$ , of the perpendicular drawn from the origin, is given by

$$S_2''(\alpha) = C^T (CC^T)^{-1} D''(\alpha)$$

with

$$C = \begin{bmatrix} 7 & -1 & 0 \\ -4 & 0 & -1 \end{bmatrix} \text{ and } D''(\alpha) = \begin{bmatrix} 10 + 36\alpha \\ -6 - 22\alpha \end{bmatrix}.$$

After some computations, we get

$$S_2''(\alpha) = \frac{1}{33} \begin{bmatrix} 47 + 170\alpha \\ -1 + 2\alpha \\ 10 + 46\alpha \end{bmatrix}.$$

(III) Performing the reverse translations, we write

$$S_2(\alpha) = S_2''(\alpha) + P(\alpha) = \frac{1}{33} \begin{bmatrix} 80 + 5\alpha \\ 65 + 35\alpha \\ -89 - 20\alpha \end{bmatrix} \text{ onto } \ell_2$$

and

$$S_1(\beta) = S_1'(\beta) + Q(\beta) = \begin{bmatrix} 3 - \frac{5\beta}{3} \\ \frac{8}{5} + \frac{\beta}{3} \\ -\frac{11}{5} - \frac{2\beta}{3} \end{bmatrix} \text{ onto } \ell_1.$$

(IV) Solving the system

$$(\diamond) \quad \begin{cases} P(\alpha) = S_1(\beta) & (\diamond_1) \\ Q(\beta) = S_2(\alpha) & (\diamond_2) \end{cases}$$

we obtain  $\alpha^* = -\frac{64}{235}$ ,  $\beta^* = \frac{18}{47}$ , taking, for example, one equation of the subsystem  $(\diamond_1)$  and one equation of the subsystem  $(\diamond_2)$ .

So, the feet of the perpendiculars are

$$S_1 = S_1(\beta^*) = P(\alpha^*) = \frac{1}{235} \begin{bmatrix} 555 \\ 406 \\ -577 \end{bmatrix}$$

$$S_2 = S_2(\alpha^*) = Q(\beta^*) = \frac{1}{47} \begin{bmatrix} 112 \\ 79 \\ -119 \end{bmatrix}.$$

One vector materializing the distance is

$$\overrightarrow{S_1 S_2} = S_2 - S_1 = \frac{1}{235} \begin{bmatrix} 5 \\ -11 \\ -18 \end{bmatrix} = \frac{5}{235} \vec{e}_1 - \frac{11}{235} \vec{e}_2 - \frac{18}{235} \vec{e}_3$$

and the distance is given by

$$d(\ell_1, \ell_2) = \|\overrightarrow{S_1 S_2}\| = \sqrt{\frac{2}{235}} = \frac{156}{1691}.$$

## 5. Final Remarks and Conclusions

By using the Moore-Penrose inverse as an elaborated tool for solving space geometry problems, we intended to pave the way for the formulation of questions in the context of space  $\mathbb{R}^n$ . As it is hinted in [11] and [14] some interesting problems do arise when thinking on distances involving affine subspaces.

In the present paper, our procedure is an algebraic one. For a more geometric approach, the companion paper [7], in terms of Gram determinants, also sheds some light on the usefulness of the much present yet somewhat [19, Vol. II, pp. 374-376] hidden conjugacy principle and the power of the origin of the coordinates.

As well, when studying the distance from a point to a line considered as the intersection of two given planes, we may gain hindsight for dealing with other more restrictive questions [16]. Science and Engineering readers, once they have grasped the beautiful geometric use we make of the Moore-Penrose inverse, may have the willingness to learn more about generalized inverses. Starting with the Moore-Penrose inverse may constitute a good option [3, p. 45] as for that end are enough some linear algebra preparation and mathematical maturity. Other than the faithful

Moore-Penrose inverse, there are many kinds of generalized inverses which have several applications [3], from linear programming, to electric circuit theory [3, p. 84], to control population models [3, p. 184] and even to cryptology [8]; we just mention three generalized inverses: Bott-Duffin [1, p. 92], Moore-Penrose and Drazin [1].

With regard to computational aspects of generalized inverses [24] and, in particular, concerning the Moore-Penrose inverse, some caution is in order, as it is neither continuous [20, pp. 423-424] nor numerically stable [3, p. 247], [20, pp. 423-424] and [25, pp. 885-886].

Computational experts, when dealing with matrices partitioned into blocks, may feel the necessity to go further than [23], which is a block version of Decell algorithm. They may look for a block generalization of [13] which could allow to numerically treat matrix exponentials having as an exponent the Drazin inverse of a matrix partitioned into blocks [22], as well.

More mathematically minded readers may feel rewarded when considering rather abstract settings [21]. To think on best approximation problems, in this context, may lead us to find rather sophisticated tools as we look for adequate inner products and convexity and compactness results.

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