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CONTINUOUSLY COMPOSED ROTATIONS

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Abstract

We discuss a continuous analogue of the traditional result to the effect that the composition of a finite number of plane rotations is either a translation or a rotation, depending on whether the sum of the rotation angles is, or is not, an integer multiple of 2π. A continuously composed rotation process would be defined by a smooth complex curve $c : [0, 1] \to \mathbb{C}$ with $c(t)$ playing the role of an 'instantaneous center' of rotation, together with a real valued function $\omega : [0, 1] \to \mathbb{R}$ representing the 'angle density'. The examples provided illustrate in various degrees an interesting Fourier series connection.

1. Introduction

In the present paper, we will investigate a continuous analogue of the following result on the composition of plane rotations [2]:

Theorem 1. *If* $R_0, R_1, \ldots, R_{n-1}$ *are n plane rotations with angles* $\theta_0, \theta_1, \ldots, \theta_{n-1}$, *respectively, then the composition* $R_{n-1} \circ R_{n-2} \circ \cdots \circ R_0$ *is either a rotation of angle* $\theta_0 + \theta_1 + \cdots + \theta_{n-1}$, *if* $\theta_0 + \theta_1 + \cdots + \theta_{n-1} \notin 2\pi\mathbb{Z}$, *or a translation, if* $\theta_0 + \theta_1 + \cdots + \theta_{n-1} \in 2\pi\mathbb{Z}$.

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The overall idea is to replace the n centers $C_0, C_2, \ldots, C_{n-1}$ of the rotations $R_0, R_1, \ldots, R_{n-1}$ with a smooth function $c : [0, 1] \to \mathbb{C}$. The rotation R_i with an angle θ_i around C_i will become an 'infinitesimal rotation' with an angle $d\theta = \omega(t)\,dt$ (where $\omega : [0, 1] \to \mathbb{R}$ is a smooth 'angle density' function) around the 'instantaneous center' of rotation $c(t)$.

2. Formulating the Main Problem

Let $c : [0, 1] \to \mathbb{R}$ be a smooth curve in the complex plane, such that $c(t)$ will play the role of an instantaneous center of rotation at the moment t. Let $\omega : [0, 1] \to \mathbb{R}$ be a continuous function and let

$$\theta(t) = \int_0^t \omega(\tau)\,d\tau,$$

so that $d\theta = \omega(t)\,dt$ will represent the angle element of an 'infinitesimal rotation' around the instantaneous center $c(t)$. Also, for $t \in [0, 1]$, let $z(t) \in \mathbb{C}$ be the position, at the moment t, of the point transformed by this continued rotation process. What we need to do is to describe, in terms of the given data $\{c, \omega\}$, the relationship between $Z = z(0)$ and $W = z(1)$:

$$W = T_{c,\,\omega}(Z).$$

Naturally, in order to get a proper generalization of Theorem 1, we expect the transformation $T_{c,\,\omega}$ to be a translation if $\Theta := \theta(1) \in 2\pi\mathbb{Z}$ and a rotation with angle Θ (around a center which is to be identified) otherwise.

3. The Main Result on Continuously Composed Rotations

The differential formulation of the fact that, at the moment t, the point $z = z(t)$ undergoes an infinitesimal rotation with an angle element $d\theta = \omega(t)\,dt$, around the instantaneous center $c(t)$ is the following:

$$dz = i(z - c)\,d\theta = i(z - c)\,\omega(t)\,dt. \tag{1}$$

From (1) it follows that governing the continued rotation process will be the following linear first-order, non-homogeneous differential equation:

$$z' - iz\omega = -ic\omega \tag{2}$$

with the initial condition $z(0) = Z$. By multiplying both sides of (2) with the integrating factor

$$\rho(t) = \exp(-i\theta(t)),$$

we get $[z\rho]' = -ic\omega\rho$ and by using $\theta' = \omega$,

$$[z\rho]' = c\rho'. \tag{3}$$

Now we integrate (3) and we get:

$$z(t)\rho(t) - z(0)\rho(0) = \int_0^t c(\tau)\rho'(\tau)\,d\tau = c(t)\rho(t) - c(0)\rho(0) - \int_0^t c'(\tau)\rho(\tau)\,d\tau$$

or, with $\rho(0) = 1$ and $z(0) = Z$,

$$z(t)\rho(t) - Z = c(t)\rho(t) - c(0) - \int_0^t c'(\tau)\rho(\tau)\,d\tau.$$

From the above relation, we obtain the following explicit formula for $z(t)$:

$$z(t) = c(t) + (Z - c(0))\exp(i\theta(t)) - \exp(i\theta(t))\int_0^t c'(\tau)\exp(-i\theta(\tau))\,d\tau. \tag{4}$$

This is the position $z(t)$, at any moment t, of the point that is continuously rotated around centers that slide along the curve given by the smooth function c, with an angular velocity $\omega(t)$ in the instantaneous rotation around the center $c(t)$, for $0 \le t \le 1$. The continued rotation process will terminate at $t = 1$, when $z = z(1) = W$, $\theta = \theta(1) = \Theta$, while (4) becomes

$$W = c(1) + (Z - c(0))\exp(i\Theta) - \exp(i\Theta)\int_0^1 c'(\tau)\exp(-i\theta(\tau))\,d\tau. \tag{5}$$

At this point we distinguish two cases.

1. If $\Theta \in 2\pi\mathbb{Z}$, then (5) is of the form

$$W = Z + B,$$

where

$$B := c(1) - c(0) - \int_0^1 c'(\tau)\exp(-i\theta(\tau))\,d\tau. \tag{6}$$

This will be a translation with the complex vector B.

2. If $\Theta \notin 2\pi\mathbb{Z}$, then (5) is of the form

$$W = Z\exp(i\Theta) + A,$$

where

$$A := c(1) - \exp(i\Theta)\left(c(0) + \int_0^1 c'(\tau)\exp(-i\theta(\tau))\,d\tau\right). \tag{7}$$

This will be a rotation with angle Θ around the center $A(1 - \exp(i\Theta))^{-1}$.

Thus, with the above notations and settings we can now state our main result, which is a continuous extension of Theorem 1.

Theorem 2. *The result of applying a continuously composed family of rotations defined as, indicated above, by the smooth center path* $c : [0, 1] \to \mathbb{C}$ *and angular velocity function* $\omega : [0, 1] \to \mathbb{R}$ *is either a translation or a rotation, depending on whether* $\Theta = \int_0^1 \omega(\tau)\,d\tau$ *is, or is not in* $2\pi\mathbb{Z}$. *In the first case, the translation is with a complex vector B given by* (6). *In the second case, the rotation is with an angle* Θ *around the center* $A(1 - \exp(i\Theta))^{-1}$, *where A is given by* (7).

4. Continuously Composed Rotations Resulting in Translations

The following examples, illustrating the synchrony between the discrete and continuous cases, represent continuous analogues of results discussed in [1], featuring translations resulting from a finite composition of plane rotations. As we will see, the discrete Fourier transforms connection noticed in [1] would appear, in the continuous limit, as a nice connection between continuously composed rotations and the Fourier coefficients of the mapping of the centers.

Example 1. According to problem, B4 in the 2004 Putnam Exam, the composition of n rotations with an angle of $2\pi/n$ each, around the centers $c(r) = r + 1$ for $r = 0, 1, \ldots, n - 1$ (in that particular order) is a translation with a vector represented by the complex number n.

For a continuous analogue, we may take $c(t) = nt$ for $0 \le t \le 1$ and $\omega(t) = 2\pi$, that is, the angle uniformly increases with an angular velocity $d\theta/dt = 2\pi$ and $\Theta = 2\pi$. Thus, the result of the continuously composed rotation process is, according to Theorem 2 above, a translation with vector

$$B = c(1) - c(0) - \int_0^1 c'(\tau)\exp(-i\theta(\tau))\,d\tau = n - \int_0^1 n\exp(-2\pi i\tau)\,d\tau = n,$$

which constitutes a continuous analogue of the above mentioned problem.

Example 2. Let $\zeta = \exp(2\pi i/n)$. In [1], it is proved that the composition of n rotations with an angle of $2\pi/n$ each, around centers $c(r) = \zeta^r$ for $r = 0, 1, ..., n-1$, (in that particular order) is a translation with a vector represented by the complex number $n(\zeta^{-1} - 1)$. If we take the limit when $n \to \infty$, this will be a translation with the complex number

$$\lim_{n\to\infty}\left[-2ni\sin\left(\frac{\pi}{n}\right)\left(\cos\left(\frac{\pi}{n}\right) - i\sin\left(\frac{\pi}{n}\right)\right)\right] = -2\pi i.$$

For a continuous analogue we will take $c(t) = \exp(2\pi it)$ for the centers path, and $\omega(t) = 2\pi$, that is uniform angular velocity distribution which gives $\theta(t) = 2\pi t$ and an overall angle $\Theta = 2\pi$. The result of the continuously composed rotation process will be, according to Theorem 2 above, and taking onto account that in this particular case we have $c(0) = c(1) = 1$, a translation with a vector represented by the complex number

$$B = -\int_0^1 c'(\tau)\exp(-i\theta(\tau))\,d\tau = -2\pi i\int_0^1 \exp(2\pi i\tau)\exp(-2\pi i\tau)\,d\tau = -2\pi i.$$

The last example generalizes the previous two and explicitly features a Fourier series connection.

Example 3. In [1], we related composition of plane rotations with discrete Fourier transforms [3], proving that if $z_0, z_1, ..., z_{n-1}$ are complex numbers, and if for all k and r with $0 \le k,\ \ r \le n-1$, we let R_r^k be the rotation around z_r with $2k\pi/n$, then for each k, the composition $R_{n-1}^k \circ R_{n-2}^k \circ \cdots \circ R_0^k$ is a translation by a vector represented by the complex number $t_k = (\zeta^{-k} - 1)\hat{z}_k$, where $\zeta = \exp(2\pi i/n)$ and $\hat{Z} = (\hat{z}_0, ..., \hat{z}_{n-1})$ is the discrete Fourier transform of $Z = (z_0, ..., z_{n-1})$.

To find a continuous model for the result in the above theorem, notice that the overall rotation angle increases with the same amount, $2k\pi/n$, as we move from one center to the next, so that the total increment will be $2k\pi$. We will model this by choosing $\theta(t) = 2k\pi t$ for $0 \le t \le 1$, which makes $\Theta = 2k\pi$. Since the (ordered) list $z_0, z_1, ..., z_{n-1}$ may be viewed as a complex-valued function defined on $\mathbb{Z}/n\mathbb{Z}$, the smooth curve of the centers, $t \mapsto c(t)$ will be set up, in this analogy, to satisfy the constraint $c(0) = c(1)$. Again, the result of the continuously composed rotation process will be a translation with a vector represented by the complex number

$$B_k := c(1) - c(0) - \int_0^1 c'(\tau)\exp(-2\pi ik\tau)\,d\tau,$$

that is,

$$B_k = -2k\pi i\int_0^1 c(\tau)\exp(-2\pi ik\tau)\,d\tau = -2k\pi i\gamma_k,$$

where γ_k are the coefficients in the Fourier expansion of the function $t \mapsto c(t)$ giving the instantaneous center of rotation:

$$c(t) = \sum_{k=-\infty}^{\infty} \gamma_k \exp(2\pi ikt).$$

Thus, we found a nice connection, in the continuous limit, between the translation vectors and the Fourier coefficients of the mapping of the centers.

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