Far East Journal of Applied Mathematics
Volume 40, Number 1, 2010, Pages 73-82
Published Online: April 22, 2010
This paper is available online at http://www.pphmj.com

A TWO-GRID METHOD FOR DECOUPLING n-DIMENSION PDEs

MIN TAN, XIANSHU JIANG and XIANZHONG ZENG

School of Mathematics and Computational Science
Hunan University of Science and Technology
Xiangtan, 411201, P. R. China
e-mail: mtan@hnust.edu.cn

Abstract

In this paper, a two-grid finite element method is proposed for solving n-d coupled partial differential equations (PDEs). With this method, we reduce the solution of the coupled system, on a fine grid, to a much coarser grid, so the equations are decoupled on the fine grid. The theoretical results are shown that the solution achieves asymptotically optimal accuracy.

1. Introduction

The two-grid finite element method was used to solve the discretizing nonsymmetric and indefinite partial differential equations (PDEs). The idea of the method was that two spaces of different scales were employed, one finite element coarse space and one finite element fine space. It was first used for symmetrization of nonsymmetric problems by Xu [6-9]. In order to solve a nonsymmetric problem on a fine grid, the solution on coarse grid was given first, and then the solution of a symmetric positive definite problem on the fine grid was gotten. Many other authors

2010 Mathematics Subject Classification: 35J57, 65N50, 65N30.

Keywords and phrases: n-d coupled PDEs, two-grid method, fine grid, coarse grid, iteration method.

This work was supported by the National Natural Science Foundations of China (10971061) and by the Hunan Province Natural Science Foundations of China (09JJ6013).

Received January 8, 2010

used this method for many different applications [1-5, 10]. In [4], Jin et al. studied two-grid discretization method for 2-dimension PDEs, e.g., *Schrödinger* equation. In [5], Tang et al. studied two-grid discretization techniques for 3-dimension PDEs, e.g., ICF system. On this basis, we research two-grid discretization techniques for *n*-dimension PDEs. First, we decouple an n-d system of PDEs by discretizing the coupled systems on the coarse grid, and solving a decoupled system on the fine coarse. We give strictly theoretical proof.

2. Preliminaries

We consider the following boundary value problem of second order elliptic problem, the problem is coupling,

$$\begin{cases} L_i U = f_i \ (i = 1, 2, \ldots, n), \quad \text{in } \Omega \in R^d, \\ U|_{\partial\Omega} = 0, \end{cases} \tag{1}$$

where unknown function $U = (u_1, u_2, \ldots, u_n)$. Ω is a polygonal domain which, for simplicity of exposition, will be assumed to be convex. The operator $L_i U$ has the following definition:

$$L_i U = -\sum_{k=1}^{d} \partial_k \sum_{l=1}^{d} a_{kl}^i \partial_l u_i + \sum_{k=1}^{d} \sum_{l=1}^{n} b_{kl}^i \partial_k u_l + \sum_{k=1}^{n} c_k^i u_k.$$

Then the equivalent variational form of (1) is defined as follows: Find $U \in (H_0^1(\Omega))^n$, such that

$$a(U, V) = (f, V), \quad \forall V \in (H_0^1(\Omega))^n, \tag{2}$$

where

$$V = (v_1, \ldots, v_n), \ f = (f_1, \ldots, f_n), \ (f, V) = \sum_{i=1}^{n} (f_i, v_i), \ a(U, V) = \sum_{i=1}^{n} a_i(u_i, v_i),$$

with

$$a_i(u_i, v_i) = \sum_{k=1}^{d} \left(\sum_{l=1}^{d} a_{kl}^i \partial_l u_i, \partial_k v_i \right) + \left(\sum_{k=1}^{d} \sum_{l=1}^{n} b_{kl}^i \partial_k u_l, v_i \right) + \left(\sum_{k=1}^{n} c_k^i u_k, v_i \right).$$

For simplicity, we denote $a(U, V) = \hat{a}(U, V) + N(U, V)$, where

$$\hat{a}(U, V) = \sum_{i=1}^{n}\sum_{k=1}^{d}\left(\sum_{l=1}^{d} a_{kl}^{i}\partial_l u_i,\ \partial_k v_i\right),$$

$$N(U, V) = \sum_{i=1}^{n}\left[\left(\sum_{k=1}^{d}\sum_{l=1}^{n} b_{kl}^{i}\partial_k u_l,\ v_i\right) + \left(\sum_{k=1}^{n} c_k^i u_k,\ v_i\right)\right].$$

We need introduce the auxiliary equation of coupling problem (1), assume $\widetilde{U} = (\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_n)$,

$$\begin{cases} \widetilde{L}_i\widetilde{U} = g_i \ (i = 1, 2, \ldots, n), & \text{in } \Omega \in R^d, \\ \widetilde{U}\,|_{\partial\Omega} = 0, \end{cases} \tag{3}$$

where

$$\widetilde{L}_i\widetilde{U} = -\sum_{k=1}^{d}\partial_k\left(\sum_{l=1}^{d} a_{lk}^{i}\partial_l\widetilde{u}_i\right) + \sum_{k=1}^{d}\sum_{l=1}^{n}\partial_k(b_{ki}^{l}\widetilde{u}_l) + \sum_{k=1}^{n} c_i^k\widetilde{u}_k,$$

then the equivalent variational form of (3) is defined as follows:

Find $\widetilde{U} = (\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_n) \in (H_0^1(\Omega))^n$, such that

$$\widetilde{a}(\widetilde{U}, \widetilde{V}) = (g, \widetilde{V}), \quad \forall \widetilde{V} \in (H_0^1(\Omega))^n, \tag{4}$$

where

$$\widetilde{V} = (\widetilde{v}_1, \ldots, \widetilde{v}_n),\ g = (g_1, \ldots, g_n),\ (g, \widetilde{V}) = \sum_{i=1}^{n}(g_i, \widetilde{v}_i),\ \widetilde{a}(\widetilde{U}, \widetilde{V}) = \sum_{i=1}^{n} a_i(\widetilde{u}_i, \widetilde{v}_i),$$

with

$$\widetilde{a}_i(\widetilde{u}_i, \widetilde{v}_i) = \sum_{k=1}^{d}\left(\sum_{l=1}^{d} a_{lk}^{i}\partial_l\widetilde{u}_i,\ \partial_k\widetilde{v}_i\right) + \left(\sum_{k=1}^{d}\sum_{l=1}^{n} b_{ki}^{l}\widetilde{u}_l,\ \partial_k v_i\right) + \left(\sum_{k=1}^{n} c_i^k\widetilde{u}_k,\ \widetilde{v}_i\right).$$

For all $\widetilde{V} \in (H_0^1(\Omega))^n$, we can easily prove that $a(\widetilde{V}, \widetilde{U}) = \widetilde{a}(\widetilde{U}, \widetilde{V})$, then we hold the following result by using (4),

$$a(\widetilde{V}, \widetilde{U}) = (g, \widetilde{V}). \tag{5}$$

Before we prove the theorem, we give four assumptions for original system (1) and its auxiliary equation (3), and we will discuss their properties.

Assumption 1. Let L_i be a uniformly elliptic operator in (1), coefficient $a_{kl}^i \in L^\infty(\Omega)$, $b_{kl}^i \in W^{1,\infty}(\Omega)$, $c_k^i \in L^\infty(\Omega)$.

Assuming the i-norm of the vector X is: $\| X \|_i = \sqrt{\| x_1 \|_i^2 + \| x_2 \|_i^2 + \cdots + \| x_n \|_i^2}$, $i = 0, 1, 2.$ Let the notation $\succ =$ and $\prec =$ be respectively equivalent to $\geq c$ and $\leq c,$ where c is some positive constant. According to Assumption 1, the following properties hold for all $W, V \in (H^1(\Omega))^n$, the proofs of these properties are gotten straightforward.

Property 1.

$$\hat{a}(W, W) \succ= \| W \|_1^2. \tag{6}$$

Property 2.

$$| a(W, V) | \prec= \| W \|_1 \| V \|_1. \tag{7}$$

Property 3.

$$| N(W, V) | \prec= \| W \|_1 \| V \|_0, \quad | N(W, V) | \prec= \| W \|_0 \| V \|_1. \tag{8}$$

Assumption 2.

$$a(W, W) \succ= \| W \|_1^2, \quad \forall W \in (H_0^1(\Omega))^n. \tag{9}$$

Assumption 3. Assume that $f \in (L^2(\Omega))^n$, the variational problem (2) has a unique solution

$$U \in (H_0^1(\Omega))^n, \text{ and } \| U \|_2 \prec= \| f \|_0. \tag{10}$$

Assumption 4. Assume that $g \in (H_0^1(\Omega))^n$, the variational problem (4) of the auxiliary problem (3) has a unique solution $\tilde{U} \in (H^1(\Omega))^n \cap (H_0^1(\Omega))^n$, and $\| U \|_2 \prec= \| g \|_0$.

Let T_h be a quasi-uniform triangulation of Ω with mesh size $h > 0$, and let n_d be the number of non-boundary nodes, and S_0^h be the corresponding piecewise linear polynomial space.

For simplicity, we assume $S_0^h \subset H_0^1(\Omega)$. Then the finite element approximation of problem (2) is defined as follows. Find a $U_h \in (S_0^h)^n$, such that

$$a(U_h, w_h) = (f, w_h), \quad \forall w_h \in (S_0^h)^n. \tag{11}$$

We denote the Algebraic system of the problem (11) to the following form:

$$A_h \hat{U}_h = b_h,$$

where $b_h = (b_1^h, b_2^h, ..., b_n^h)^T$, $\widehat{U}_h = (u_1^h, u_2^h, ..., u_n^h)^T$, $A_h = (A_{ij})$.

Theorem 1. *Let U be the solution of variational problem* (2), *and* U_h *be the solution of finite element problem* (4), *according to Assumptions* 1, 2 *and* 3. *Then the following error estimate of* U_h *holds*:

$$\| U - U_h \|_s \prec= h^{2-s} \| U \|_2, \quad s = 0, 1. \tag{12}$$

Proof. Let e_h be the error between U and U_h, that is, $e_h = U - U_h$, and we can hold the following result from (2) and (11):

$$a(e_h, w_h) = 0, \quad \forall w_h \in (S_0^h)^n. \tag{13}$$

Let $U^I \in (S_0^h)^n$ be the interpolation function of U, which conclude that $U^I - U_h \in (S_0^h)^n$, according to Assumption 1 and (13). Then

$$\| e_h \|_1^2 \prec= a(e_h, e_h) = a(e_h, U - U^I + U^I - U_h) = a(e_h, U - U^I)$$

$$\prec= \| e_h \|_1 \| U - U^I \|_1.$$

We can conclude

$$\| e_h \|_1 \prec= \| U - U^I \|_1 \prec= h \| U \|_2. \tag{14}$$

Take $g = e_h$ and $w_h = e_h$ in (6), and according to (5), (6), (10) and (13), then we have

$$\| e_h \|_0^2 = a(e_h, \widetilde{U}) = a(e_h, \widetilde{U} - \widetilde{U}^I)$$

$$\prec= \| e_h \|_1 \| \widetilde{U} - \widetilde{U}^I \|_1$$

$$\prec= h \| e_h \|_1 \| \widetilde{U} \|_2 \prec= h \| e_h \|_1 \| e_h \|_0.$$

According to the above mentioned inequality and (14), we have

$$\| e_h \|_0 \prec= h \| e_h \|_1 \prec= h^2 \| U \|_2. \tag{15}$$

Due to (14) and (15), the conclusion (12) is gotten.

3. A New Two-grid Finite Element Method

In the general case, the finite element discretization (11) apparently corresponds to a coupled system of equations. In order to reduce the computational cost, following [1, 2], we introduce another finite element space $S_0^H \subset S_0^h \subset H_0^1(\Omega)$ defined on a coarser quasiuniform triangulation of Ω, where H and h are the coarse and fine meshsize, respectively, and have the relation $H > h.$ So we propose the following algorithm.

Algorithm 1 (A two-grid finite element method)**.**

Step 1. Find $U_H \in (S_0^h)^n,$ such that

$$a(U_H, \chi) = (f, \chi), \quad \forall \chi \in (S_0^H)^n.$$

Step 2. Find $U_h^* \in (S_0^h)^n,$ so that

$$\hat{a}(U_h^*, w_h) = (f, w_h) - N(U_H, w_h), \quad \forall w_h \in (S_0^h)^n. \tag{16}$$

We note that the linear system in Step 2 is a decoupled system which involves n separate Poisson equations. The computational cost mainly focuses on two parts, one is in Step 1, which a coupled system needs to be solved on the coarser space, the other is in Step 2, which n separate Poisson equations need to be calculated on the fine grid.

The following Theorem 2 shows, U_h^* and U_h can reach the optimal accuracy under H_1-norm if $H = \sqrt{h}.$ Due to the dimension of S_0^H is much smaller than the dimension of $S_0^h,$ the efficiency of the algorithm is then evident.

Theorem 2. *Let U be the solution of variational problem* (3), U_h *be the solution of finite element problem* (4), *and* U_h^* *be the numerical solution by using*

Algorithm 1, *according to Assumptions* 1, 2 *and* 3. *Then the following error estimate of* U_h^* *holds*:

$$\| U_h - U_h^* \|_1 \prec= H^2. \tag{17}$$

So $\| U - U_h^* \|_1 \prec= h + H^2$, *if we take* $H = \sqrt{h}$, *then* U_h^* *and* U_h *have same accuracy under* H_1*-norm*.

Proof. Let $\hat{e}_h$ be the error between U_h^* and U_h, that is, $\hat{e}_h = U_h - U_h^*$, and we can hold the following result from (11) and (16):

$$\hat{a}(\hat{e}_h, w_h) + N(U_h - U_H, w_h) = 0, \quad \forall w_h \in (S_0^h)^n.$$

Take $w_h = \hat{e}_h$ in equation above, according to (6) and (8), and we have

$$\| \hat{e}_h \|_1^2 \prec= \hat{a}(\hat{e}_h, \hat{e}_h) \prec= \| U_h - U_H \|_0 \| \hat{e}_h \|_1.$$

According to the above mentioned inequality, we have $\| \hat{e}_h \|_1 \prec= \| U_h - U_H \|_0$. Due to Theorem 1,

$$\| U_h - U_H \|_0 \prec= \| U - U_h \|_0 + \| U - U_H \|_0 \prec= h^2 + H^2.$$

So that $\| U_h - U_h^* \|_1 \prec= H^2$. Consider Theorem 1 and (17), we have the following result:

$$\begin{aligned} \| U - U_h^* \|_1 &= \| U - U_h + U_h - U_h^* \|_1 \\ &\prec= \| U - U_h \|_1 + \| U_h - U_h^* \|_1 \\ &\prec= h \| U \|_2 + H^2 \| U \|_2 \prec= h + H^2. \end{aligned}$$

Algorithm 1 can be improved in a successive fashion as follows.

4. Iteration Method

Let $U_h^0 = 0$. Assume that the kth numerical solution $U_h^k \in (S_0^h)^n$ has been obtained, then the $(k+1)$th numerical solution $U_h^{k+1} \in (S_0^h)^n$ is defined as follows:

Algorithm 2 (Iteration method)**.**

Step 1. Find a correction vector $e_H \in (S_0^H)^n$ on the coarse grid, such that

$$a(e_H, \chi) = (f, \chi) - a(U_h^k, \chi), \quad \forall \chi \in (S_0^H)^n. \tag{18}$$

Step 2. Find the $(k+1)$th solution $U_h^{k+1} \in (S_0^h)^n$, it satisfies the following equation:

$$\hat{a}(U_h^{k+1}, w_h) = (f, w_h) - N(U_h^k + e_H, w_h), \quad \forall w_h \in (S_0^h)^n. \tag{19}$$

Theorem 3. *Let U be the solution of variational problem* (2), *U_h be the solution of finite element problem* (11), *and U_h^k be the kth iteration solution by using Algorithm* 2, *according to Assumptions* 1, 2 *and* 3. *Then the following error estimate of U_h^k holds*:

$$\|U_h - U_h^k\|_1 \prec= H^{k+1}, \quad k \geq 1. \tag{20}$$

Consequently,

$$\|U - U_h^k\|_1 \prec= h + H^{k+1}, \quad k \geq 1. \tag{21}$$

When $k \geq 1$ and $h = H^{k+1}$, U_h^k and U_h have the same accuracy under H_1-norm.

Proof. From (11) and (19), we have

$$\hat{a}(U_h - U_h^{k+1}, w_h) = -N(U_h - (U_h^k + e_H), w_h), \quad \forall w_h \in (S_0^h)^n. \tag{22}$$

Specially, take $w_h = U_h - U_h^{k+1}$ in (22), and according to (6), (8) and (22), we have

$$\begin{aligned} \|U_h - U_h^{k+1}\|_1^2 &\prec= \hat{a}(U_h - U_h^{k+1}, U_h - U_h^{k+1}) \\ &\prec= \|U_h - (U_h^k + e_H)\|_0 \|U_h - U_h^{k+1}\|_1. \end{aligned}$$

Thus

$$\|U_h - U_h^{k+1}\|_1 \prec= \|U_h - (U_h^k + e_H)\|_0. \tag{23}$$

From (11) and (18), we have

$$a(U_h - (U_h^k + e_H), \chi) = 0, \quad \forall \chi \in (S_0^H)^n. \tag{24}$$

Due to $e_H \in (S_0^H)^n$, so $a(U_h - (U_h^k + e_H), e_H) = 0$. From (7), (9) and (24), we have

$$\|U_h - (U_h^k + e_H)\|_1^2 \prec= a(U_h - (U_h^k + e_H), U_h - (U_h^k + e_H))$$

$$= a(U_h - (U_h^k + e_H), U_h - U_h^k)$$

$$\prec= \|U_h - (U_h^k + e_H)\|_1 \|U_h - U_h^k\|_1.$$

Thus

$$\|U_h - (U_h^k + e_H)\|_1 \prec= \|U_h - U_h^k\|_1. \tag{25}$$

Let $\tilde{U}$ be the solution of (4) when $g = U_h - (U_h^k + e_H)$, and $\tilde{U}^I \in (S_0^H)^n$ be the interpolation of $\tilde{U}$, from (7), (10), (15), (24), (25) and error estimate of finite element method. Then

$$\|U_h - (U_h^k + e_H)\|_0^2 = (g, U_h - (U_h^k + e_H))$$

$$= a(U_h - (U_h^k + e_H), \tilde{U})$$

$$= a(U_h - (U_h^k + e_H), \tilde{U} - \tilde{U}^I)$$

$$\prec= \|U_h - (U_h^k + e_H)\|_1 \|\tilde{U} - \tilde{U}^I\|_1$$

$$\prec= H\|U_h - (U_h^k + e_H)\|_1 \|\tilde{U}\|_2$$

$$\prec= H\|U_h - (U_h^k + e_H)\|_1 \|U_h - (U_h^k + e_H)\|_0.$$

Thereby,

$$\|U_h - (U_h^k + e_H)\|_0 \prec= H\|U_h - (U_h^k + e_H)\|_1. \tag{26}$$

According to (23), (25) and (26), if $k \geq 1$, we can deduce the following result:

$$\|U_h - U_h^k\|_1 \prec= H\|U_h - U_h^{k-1}\|_1 \prec= H^{k-1}\|U_h - U_h^1\|_1. \tag{27}$$

Note that U_h^1 is the solution U_h^* obtained by Algorithm 1, then according to (17) and (27), which implies that $\|U_h - U_h^k\|_1 \prec= H^{k+1}$. Because of

$$\|U - U_h^k\|_1 \prec= \|U - U_h\|_1 + \|U_h - U_h^k\|_1.$$

From (12), (20) and the inequality above, so the conclusion (21) becomes clear.

The result of Theorem 3 shows that it suffices to take $H = \sqrt[(k+1)]{h}$ to obtain the optimal approximation under H_1-norm. Because the dimension of S_0^H can be much smaller than S_0^h, so the cost of calculated amount in Algorithm 2 focuses on solving n separate Laplacian systems in step 2, which is much easier to solve than the coupled system in (11).

References

[1] O. Axelsson and W. Layton, A two-level discretization of nonlinear boundary value problems, SIAM J. Numer. Anal. 33(6) (1996), 2359-2374.

[2] O. Axelsson and A. Padiy, On a two-level Newton-type procedure applied for solving non-linear elasticity problems, Internat. J. Numer. Methods Engrg. 49(12) (2000), 1479-1493.

[3] V. Girault and J. L. Lions, Two-grid finite-element schemes for the transient Navier-Stokes problem, M2AN Math. Model. Numer. Anal. 35(5) (2001), 945-980.

[4] Jicheng Jin, Shi Shu and Jinchao Xu, A two-grid discretization method for decoupling systems of partial differential equations, Math. Comp. 75(256) (2006), 1617-1626 (electronic).

[5] Yun-mei Tang, Min Tan, Jun Jiang, A two-grid discretization techniques for simpled ICF system, Fuzzy Systems and Mathematics 11 (2005), 12-16.

[6] J. Xu, Iterative methods by SPD and small subspace solvers for nonsymmetric or indefinite problems, Proceedings of the Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations, SIAM, Philadelphia, 1992.

[7] J. Xu, A new class of iterative methods for nonselfadjoint or indefinite problems, SIAM J. Numer. Anal. 29(2) (1992), 303-319.

[8] J. Xu, A novel two-grid method for semilinear elliptic equations, SIAM J. Sci. Comput. 15(1) (1994), 231-237.

[9] J. Xu, Two-grid discretization techniques for linear and nonlinear PDEs, SIAM J. Numer. Anal. 33(5) (1996), 1759-1777.

[10] J. Xu and A. Zhou, Local and parallel finite element algorithms based on two-grid discretizations, Math. Comp. 69(231) (2000), 881-909.