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ON THE $H(S)$-PART IN BCH-ALGEBRAS

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Abstract

We consider special subsets $H(S)$ in a BCH-algebra X, where S is a non-empty subset of X. We give related properties of them and provide an equivalent condition that the special $H(S)$-part of X is an ideal.

1. Introduction

In 1966, Imai and Iséki ([7]) and Iséki ([8]) introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In 1983, Hu and Li ([5, 6]) introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. They have studied some properties of these algebras.

As we know, the primary aim of the theory of BCH-algebras is to determine the structure of all BCH-algebras. The main task of a structure theorem is to find a complete system of invariants describing the BCH-algebra up to isomorphism, or to establish some connection with other mathematics branches. In addition, the ideal theory plays an important role in studying BCH-algebras, and some interesting

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results have been obtained by several authors ([1, 2, 3, 4, 12, 13]). In this paper, we construct special subsets $T(S)$ in a BCH-algebra *X*, where *S* is a non-empty subset of *X*. We give related properties of them and provide an equivalent condition that the special $H(S)$-part of *X* is an ideal.

2. Preliminaries

A *BCH-algebra* is a non-empty set *X* with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(1) $x * x = 0,$

(2) $x * y = 0$ and $y * x = 0$ imply $x = y,$

(3) $(x * y) * z = (x * z) * y,$

for all *x*, *y*, *z* in *X*. A BCH-algebra *X* satisfying the identity $((x * y) * (x * z)) * (z * y) = 0$ and $0 * x = 0,$ for all $x, y, z \in X$ is called a *BCK-algebra*. A BCH-algebra *X* is said to be *medial* ([2]) if it satisfies $(x * y) * (a * b) = (x * a) * (y * b)$ for all $x, y, a, b \in X$. We defined the relation $\leq$ in a BCH-algebra by: $x \leq y$ if and only if $x * y = 0.$

In any BCH-algebra *X*, the following hold: for all $x, y \in X$,

(4) $(x * (x * y)) \leq y,$

(5) $x \leq 0$ implies $x = 0,$

(6) $0 * (x * y) = (0 * x) * (0 * y),$

(7) $x * 0 = x,$

(8) $0 * (0 * (0 * x)) = 0 * x.$

A non-empty subset S of BCH-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S.$ A non-empty subset *I* of BCH-algebra *X* is called an *ideal* of *X* if $0 \in I$ and if $x * y, y \in I$ imply that $x \in I.$ Note that an ideal of a BCH-algebra may not be a subalgebra. An ideal *I* of BCH-algebra *X* is said to be *closed* if $0 * x \in I$ for all $x \in I.$ Note that every closed ideal in BCH-algebra *X* is a subalgebra, but converse is not true.

In a BCH-algebra X, the set $A^+ := \{x \in X \mid 0 \leq x\}$ is called a *positive part* of X and the set $A(X) := \{x \in X \mid 0 * (0 * x) = x\}$ is called an *atom part* of X. Note that $A(X) = \{0 * (0 * x) \mid x \in X\} = \{0 * x \mid x \in X\}$ and $A^+ \cap A(X) = \{0\}$ ([10]).

For any elements x, y in a BCH-algebra X, let us write $x * y^n$ for $(\cdots((x * y) * y) * \cdots) * y$, where y occurs n times.

In what follows, the letter X denotes a BCH-algebra unless otherwise specified.

3. Main Results

Definition 3.1. Let S be a subset of X. The set

$$H(S) := \{y \in S \mid y = 0 * x^2 \text{ for some } x \in S\}$$

is called the $H(S)$*-part* of X.

Clearly, $0 \in H(S)$ if S containing 0.

Theorem 3.2. *If S is a subalgebra of X, then $H(S)$ is a subalgebra of X.*

Proof. Let $a, b \in H(S)$. Then $a = 0 * x^2$ and $b = 0 * y^2$ for some $x, y \in S$. Thus, we have

$$\begin{aligned} a * b &= ((0 * x) * x) * ((0 * y) * y) \\ &= ((0 * ((0 * y) * y)) * x) * x \\ &= (((0 * (0 * y)) * (0 * y)) * x) * x \\ &= (((0 * x) * (0 * y)) * (0 * y)) * x \\ &= ((0 * (x * y)) * (0 * y)) * x \\ &= ((0 * (0 * y)) * x) * (x * y) \\ &= ((0 * x) * (0 * y)) * (x * y) \\ &= (0 * (x * y)) * (x * y) \\ &= 0 * (x * y)^2, \end{aligned}$$

and $x * y \in S$. Hence $a * b \in H(S)$. □

Corollary 3.3. *$H(X)$ is a subalgebra of X.*

Theorem 3.4. *If S is a subset of X, then $H(S) \subseteq A(S)$, where $A(S) := \{x \in S \mid 0 * (0 * x) = x\}$.*

Proof. Let $a \in H(S)$. Then $a = 0 * x^2$ for some $x \in S$. Thus, we have

$$0 * (0 * a) = 0 * (0 * (0 * x^2)) = 0 * x^2 = a.$$

Hence $H(S) \subseteq A(S)$. □

Lemma 3.5 ([11])**.** *Let S be a subalgebra of X. Then $A(S)$ is a subalgebra of X.*

Note that every subalgebra of a medial BCH-algebra *X* is an ideal in *X* (see [2]). By Theorem 3.2, Theorem 3.4 and Lemma 3.5, we have

Corollary 3.6. *Let S be a subalgebra of X. Then $H(S)$ is an ideal of $A(S)$.*

In general, the $H(S)$-part $H(S)$ of *X* may not be an ideal of *X* as shown in the following example.

Example 3.7. Let $X := \{0, a, b, c\}$ be a BCH-algebra in which $*$-operation is defined by:

$*$	0	a	b	c
0	0	c	0	a
a	a	0	a	c
b	b	c	0	a
c	c	a	c	0

Taking an ideal $S := X$, then $H(S) = \{0, a, c\}$ is not an ideal of *X* since $b * a = c \in H(S)$ and $b \notin H(S)$.

Now we give equivalent conditions that $H(S)$ is an ideal of *X*.

Theorem 3.8. *Let S be a closed ideal of X. The following are equivalent*:

(i) *$H(S)$ is an ideal of X.*

(ii) *$x * a = y * a$ implies $x = y$ for all $x, y \in A^+$ and $a \in H(S)$.*

(iii) *$x * a = 0 * a$ implies $x = 0$ for all $x \in A^+$ and $a \in H(S)$.*

Proof. (i) $\Rightarrow$ (ii) Let $H(S)$ be an ideal of X and $x * a = y * a$, for all $x, y \in A^+$ and $a \in H(S)$. Then by Theorem 3.2, we have

$$(x * y) * a = (x * a) * y = (y * a) * y = (y * y) * a = 0 * a \in H(S).$$

Since $H(S)$ is an ideal of X, it follows that $x * y \in H(S)$. On the other hand, note that $x * y \in A^+$ and $A^+ \cap H(S) \subseteq A^+ \cap A(S) \subseteq A^+ \cap A(X) = \{0\}$. Thus, we have $x * y = 0$. Similarly, we get $y * x = 0$, and hence $x = y$.

(ii) $\Rightarrow$ (iii) Since $0 \in A^+$, it is clear.

(iii) $\Rightarrow$ (i) Assume that (iii) holds. Let $s, t \in S$ such that $s * t \in H(S)$ and $t \in H(S)$. Then since S is an ideal of X, we have $s \in S$. Denote $u = 0 * (0 * s)$. Then we get

$$u * t = (0 * (0 * s)) * t = 0 * (t * s) \in A(S)$$

and $s * t \in H(S) \subseteq A(S)$ and

$$\begin{aligned}(u * t) * (s * t) &= ((0 * (0 * s)) * (s * t)) * t \\ &= (((0 * s) * (0 * t)) * (0 * s)) * t \\ &= (0 * (0 * t)) * t = 0.\end{aligned}$$

Thus by Theorem 3 in [2] we have $u * t = s * t$. Hence

$$(s * u) * t = (s * t) * u = (u * t) * u = 0 * t,$$

which implies from (iii) that $s * u = 0$, i.e., $s * (0 * (0 * s)) = 0$. Therefore $s = 0 * (0 * s) \in A(S)$ since $s \in S$. As $H(S)$ is an ideal of $A(S)$, we get $s \in H(S)$, and $H(S)$ is an ideal of X. □

By Theorem 3.8, we have equivalent condition that $H(X)$ is an ideal of X.

Corollary 3.9. *The following are equivalent*:

(i) $H(X)$ *is an ideal of* X.

(ii) $x * a = y * a$ *implies* $x = y$ *for all* $x, y \in A^+$ *and* $a \in H(X)$.

(iii) $x * a = 0 * a$ *implies* $x = 0$ *for all* $x \in A^+$ *and* $a \in H(X)$.

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