# NODAL SETS AND SINGULAR SETS OF SOLUTIONS FOR SEMI-LINEAR ELLIPTIC EQUATIONS ASSOCIATED WITH SUPERCONDUCTIVITY 

JUNICHI ARAMAKI<br>Department of Mathematical Sciences<br>Faculty of Science and Engineering<br>Tokyo Denki University<br>Hatoyama-machi, Saitama 350-0394, Japan<br>e-mail: aramaki@mail.dendai.ac.jp


#### Abstract

In the present paper, we study the structure of the nodal sets and the singular sets of solutions for a semi-linear elliptic equation in general dimensional Euclidean space $\mathbb{R}^{n}$. We shall show that both the nodal sets and singular sets are on unions of countable $C^{1}$ manifolds. Especially, the highest dimensional subsets of nodal sets and singular sets are on unions of countable $C^{1, \alpha}$ manifolds for some $0<\alpha<1$. Thus the nodal sets and singular sets are countably $(n-1)$ and $(n-2)$ rectifiable, respectively.


## 1. Introduction

We consider a semi-linear elliptic equation

$$
\begin{equation*}
-\nabla_{\boldsymbol{A}}^{2} \psi=f\left(|\psi|^{2}\right) \psi \text { in } \Omega \tag{1.1}
\end{equation*}
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^{n}$ and a function $f$ is real-valued, bounded on 2010 Mathematics Subject Classification: 82D55, 47F05, $35 J 15$.
Keywords and phrases: nodal sets, singular sets, semi-linear elliptic equations, GinzburgLandau system.

Received November 18, 2009
$[0, \infty)$. Here $\boldsymbol{A}$ is a real vector-valued function (called magnetic potential), $\psi$ is a complex-valued function. $\nabla_{\boldsymbol{A}}$ and $\nabla_{\boldsymbol{A}}^{2}$ are defined by $\nabla_{\boldsymbol{A}}=\nabla-i \boldsymbol{A}, \quad \nabla$ is the gradient operator and

$$
\nabla_{\boldsymbol{A}}^{2} \psi=\Delta \psi-i[2 \boldsymbol{A} \cdot \nabla \psi+(\operatorname{div} \boldsymbol{A}) \psi]-|\boldsymbol{A}|^{2} \psi
$$

In this paper, we shall clarify the structure of the nodal set and singular set of any non-trivial solution $\psi$ of (1.1).

Let us recall that superconductivity in three dimensional space can be described by a pair of $(\psi, \mathcal{A})$, where $\psi$ is a complex-valued function called the order parameter and $\mathcal{A}$ is a real vector-valued function called the magnetic potential, which is a minimizer of the Ginzburg-Landau functional

$$
\int_{\Omega}\left\{\left|\nabla_{\kappa \mathcal{A}} \psi\right|^{2}-\kappa^{2}|\psi|^{2}+\frac{\kappa^{2}}{2}|\psi|^{4}\right\} d x+\kappa^{2} \int_{\mathbb{R}^{3}}|\operatorname{curl} \mathcal{A}-\mathcal{H}|^{2} d x .
$$

Here $\mathcal{H}$ is the applied magnetic field, $\kappa$ is the Ginzburg-Landau parameter. Then the Euler equations for $\psi$ become

$$
\begin{cases}-\nabla_{\kappa \mathcal{A}}^{2} \psi=\kappa^{2}\left(1-|\psi|^{2}\right) \psi & \text { in } \Omega,  \tag{1.2}\\ \mathbf{v} \cdot \nabla_{\kappa \mathcal{A}} \psi=0 & \text { on } \partial \Omega,\end{cases}
$$

where $\mathbf{v}$ is the outer unit normal vector at $\partial \Omega$. It is well known that the solution of (1.2) satisfies $|\psi| \leq 1$ in $\Omega$. If we put $\boldsymbol{A}=\kappa \mathcal{A}$ and

$$
f(t)=\left\{\begin{array}{lll}
\kappa^{2}(1-t) & \text { if } & |t| \leq 1 \\
0 & \text { if } & |t|>1
\end{array}\right.
$$

then the first equation for $\psi$ in (1.2) is of the form (1.1).
In the superconductivity theory or Landau-de Gennes model of liquid crystal, if we want to know the property of the third critical field $H_{c_{3}}$ or $Q_{c_{3}}$, we need the estimate of the lowest eigenvalue $\mu(q A)$ of the Schrödinger operator $-\nabla_{q A}^{2}$, i.e.,

$$
\begin{cases}-\nabla_{q A}^{2} \phi=\mu(q A) \phi & \text { in } \Omega  \tag{1.3}\\ \mathbf{v} \cdot \nabla_{q A} \phi=0 & \text { on } \partial \Omega\end{cases}
$$

If we put $f(t)=\mu(q \boldsymbol{A})$ which is a constant, equation (1.3) is also of the form (1.1).

For the theory of liquid crystal, $\boldsymbol{A}=\boldsymbol{n}$ is a unit vector field. See Lu and Pan [17, $18]$ and Pan [20, 21].

For $n \geq 3$, the eigenvalue problem

$$
\begin{equation*}
-\nabla_{A}^{2} \psi=\mu \psi \quad \text { in } \Omega \tag{1.4}
\end{equation*}
$$

with the Dirichlet boundary condition or the Neumann boundary condition is considered by Helffer and Mohamed [15] and Helffer and Morame [16]. This problem is also of the form (1.1).

First, we note an important property that equation (1.1) is the gauge invariant. That is to say, if $\psi$ is a solution of (1.1) and $\chi$ is a smooth real-valued function, then $\phi=e^{i \chi} \psi$ is a solution of

$$
\begin{equation*}
-\nabla_{A-\nabla \chi}^{2} \phi=f\left(|\phi|^{2}\right) \phi \text { in } \Omega \tag{1.5}
\end{equation*}
$$

Let $B(x)$ be an anti-symmetric $n \times n$ matrix with $(i, j)$ component $B_{i j}=$ $\partial A_{i} / \partial x_{j}-\partial A_{j} / \partial x_{i}$, where $\boldsymbol{A}=\left(A_{1}, A_{2}, \ldots, A_{n}\right)$. If $B(x) \equiv 0$ in $\Omega$ and $\Omega$ is simply connected, there exists a unique real-valued function $\chi$ up to an additive constant such that $\boldsymbol{A}=\nabla \chi$. Thus $\phi=e^{i \chi} \psi$ is a solution of

$$
-\Delta \phi=f\left(|\phi|^{2}\right) \phi
$$

In this case, we can apply the results in the papers which treat in the frame of realvalued functions. However, if $B(x) \not \equiv 0$, the nodal set or singular set of the complex-valued solution $\psi$ cannot be reduced to the case of real-valued functions. See Pan [22].

Here the nodal set of $\psi$ is defined by

$$
\mathcal{N}(\psi)=\{x \in \Omega ; \psi(x)=0\}
$$

and the singular set of $\psi$ is defined as a subset of the nodal set

$$
\mathcal{S}(\psi)=\{x \in \Omega ; \psi(x)=0, \nabla \psi(x)=0\} .
$$

The structures of $\mathcal{N}(\psi)$ and $\mathcal{S}(\psi)$ for real-valued solution were analyzed by many authors, see Garofalo and Lin [8], Han [10], Han et al. [12] and Han [11]. For
$n=3$, the structure of $\mathcal{S}(\psi)$ of complex-valued solution of (1.1) is investigated by [22].

In this paper, we attempt the extension of the results of [22] to the general dimensional case with some improvements.

In the following, we say that a function $\psi \in L_{\text {loc }}^{2}(\Omega)$ vanishes of infinite order at $x_{0} \in \Omega$ if for any integer $m \geq 0$,

$$
\int_{B_{r}\left(x_{0}\right)}|\psi| d x=O\left(r^{m+n}\right) \text { as } r \rightarrow 0
$$

where $B_{r}\left(x_{0}\right)=\left\{x \in \mathbb{R}^{n} ;\left|x-x_{0}\right|<r\right\}$. We use the symbols $W^{1,2}(\Omega ; \mathbb{C})$ or $L^{p}\left(\Omega ; \mathbb{R}^{n}\right)$, etc. for the usual Sobolev spaces of complex-valued functions or realvalued vector functions, etc., respectively.

Our main results are following.
Theorem 1.1. Let $\Omega \subset \mathbb{R}^{n}(n \geq 3)$ be a bounded domain. Assume that
(a) $\boldsymbol{A} \in L^{\infty}\left(\Omega ; \mathbb{R}^{n}\right)$, $\operatorname{div} \boldsymbol{A} \in L_{\mathrm{loc}}^{q}(\Omega)$ with $q>n / 2$ if $n \geq 4$ and $q \geq 2$ if $n=3$, and $B \in L^{\infty}\left(\Omega ; \mathbb{R}^{n^{2}}\right)$.
(b) $\psi \in W_{\text {loc }}^{1,2}(\Omega ; \mathbb{C})$ is a non-trivial complex-valued solution of (1.1) with $\nabla_{A} \psi \in W_{\text {loc }}^{1,2}\left(\Omega ; \mathbb{C}^{n}\right)$ and $\left\|f\left(|\psi|^{2}\right)\right\|_{L^{\infty}(\Omega)}<\infty$.

Then the following hold:
(1) (Doubling property) There exists a constant $C>0$ such that for any $B_{2 R}\left(x_{0}\right) \Subset \Omega$,

$$
\int_{B_{2 R}\left(x_{0}\right)}|\psi|^{2} d x \leq C \int_{B_{R}\left(x_{0}\right)}|\psi|^{2} d x
$$

(2) $\psi$ cannot vanish of infinite order at any point in $\Omega$.

For the nodal sets and singular sets of the solution $\psi$, we have
Theorem 1.2. Under the conditions of Theorem 1.1, we have the following decompositions of $\mathcal{N}(\psi)$ and $\mathcal{S}(\psi)$. For any $\Omega^{\prime} \Subset \Omega$,

$$
\mathcal{N}(\psi) \cap \Omega^{\prime}=\bigcup_{j=0}^{n-1}\left(\mathcal{N}^{j}(\psi) \cap \Omega^{\prime}\right)
$$

where $\mathcal{N}^{j}(\psi) \cap \Omega^{\prime}$ is on a union of countable $j$-dimensional $C^{1}$ manifolds for $j \leq n-2$ and $\mathcal{N}^{n-1}(\psi) \cap \Omega^{\prime}$ is on a union of countable ( $n-1$ )-dimensional $C^{1, \alpha}$ manifolds with any $\alpha \in(0,1) \cap(0,2-n / q]$, and

$$
\mathcal{S}(\psi) \cap \Omega^{\prime}=\bigcup_{j=0}^{n-2}\left(\mathcal{S}^{j}(\psi) \cap \Omega^{\prime}\right),
$$

where $\mathcal{S}^{j}(\psi) \cap \Omega^{\prime}$ is on a union of countable $j$-dimensional $C^{1}$ manifolds for $j \leq n-3$ and $\mathcal{S}^{n-2}(\psi) \cap \Omega^{\prime}$ is on a union of countable ( $n-2$ )-dimensional $C^{1, \alpha}$ manifolds for some $\alpha \in(0,1)$. Thus $\mathcal{N}^{j}(\psi) \cap \Omega^{\prime}$ and $\mathcal{S}^{j}(\psi) \cap \Omega^{\prime}$ are countably j-rectifiable for $0 \leq j \leq n-1$ and $0 \leq j \leq n-2$, respectively.

Remark 1.3. For $n=3$, [22] analyzed only the singular set. [22] showed that $\mathcal{S}^{j}(\psi) \cap \Omega^{\prime}$ for $j=0,1$, have finite decompositions

$$
\mathcal{S}^{j}(\psi) \cap \Omega^{\prime}=\bigcup_{m \geq 2}\left(\mathcal{S}_{m}^{j}(\psi) \cap \Omega^{\prime}\right) \quad(j=0,1)
$$

where $\mathcal{S}_{m}^{0}(\psi) \cap \Omega^{\prime}$ is at most countable set and $\mathcal{S}_{m}^{1}(\psi) \cap \Omega^{\prime}$ is on a countable union of 1 -dimensional $C^{1, \alpha}$ manifolds for some $\alpha \in(0,1)$.

## 2. Preliminary Remarks

Though we do not assume the boundary condition for solutions of (1.1), we note some remarks on solutions of (1.1) with the boundary conditions in this section.

First, as long as we consider equation (1.1) with the Neumann boundary condition and $f$ is smooth, decreasing function on $[0, \infty)$ and $f(0) \geq 0$, we can slightly relax the boundedness assumption on $f$ to that $f$ is bounded on $\{t \geq 0 ; f(t) \geq 0\}$. In fact, we have

Proposition 2.1. Let $\Omega$ be smooth, simply connected bounded domain in $\mathbb{R}^{n}$ and $f$ be a smooth, decreasing function in $\overline{\mathbb{R}}_{+}=\{t \in \mathbb{R} ; t \geq 0\}$ satisfying $f(0) \geq 0$.

Assume that $A \in W^{1,2}\left(\Omega ; \mathbb{R}^{n}\right)$. Then if the magnetic Neumann problem

$$
\begin{cases}-\nabla_{\boldsymbol{A}}^{2} \psi=f\left(|\psi|^{2}\right) \psi & \text { in } \Omega  \tag{2.1}\\ \mathbf{v} \cdot \nabla_{\boldsymbol{A}} \psi=0 & \text { on } \partial \Omega\end{cases}
$$

has a solution $\psi \in W^{1,2}(\Omega ; \mathbb{C})$, then $f\left(|\psi|^{2}\right) \geq 0$.
Proof. It is well known that we may assume the Coulomb gauge condition: $\operatorname{div} \boldsymbol{A}=0$ in $\Omega$ and $\mathbf{v} \cdot \boldsymbol{A}=0$ on $\partial \Omega$. In fact, the problem

$$
\begin{cases}-\Delta \varphi=\operatorname{div} \boldsymbol{A} & \text { in } \Omega \\ \frac{\partial}{\partial \mathbf{v}} \varphi=-\mathbf{v} \cdot \boldsymbol{A} & \text { on } \partial \Omega\end{cases}
$$

has a unique solution $\varphi$ up to an additive constant since

$$
\int_{\Omega} \operatorname{div} \boldsymbol{A} d x=\int_{\partial \Omega} \mathbf{v} \cdot \boldsymbol{A} d S
$$

where $d S$ is the surface area of $\partial \Omega$. Then it suffices to replace $\boldsymbol{A}$ with $\boldsymbol{A}+\nabla \varphi$. By the elliptic regularity theory, we have $\psi \in C^{\infty}(\bar{\Omega})$ (cf. Sandier and Serfaty [24]). We can write the first equation in (2.1) in the form

$$
-\Delta \psi+2 i \boldsymbol{A} \cdot \nabla \psi+|\boldsymbol{A}|^{2} \psi=f\left(|\psi|^{2}\right) \psi
$$

Multiplying $\bar{\psi}$ to this equation, we have

$$
\begin{aligned}
\frac{1}{2} \Delta|\psi|^{2} & =\mathfrak{R}[(\Delta \psi) \bar{\psi}]+|\nabla \psi|^{2} \\
& =-f\left(|\psi|^{2}\right)|\psi|^{2}+2 \mathfrak{R}[i(\boldsymbol{A} \cdot \nabla \psi) \bar{\psi}]+|\boldsymbol{A}|^{2}|\psi|^{2}+|\nabla \psi|^{2}
\end{aligned}
$$

Since

$$
\left|\nabla_{\boldsymbol{A} \psi}\right|^{2}=|(\nabla-i \boldsymbol{A}) \psi|^{2}=|\nabla \psi|^{2}+2 \mathfrak{R}[i(\boldsymbol{A} \cdot \nabla \psi) \bar{\psi}]+|\boldsymbol{A}|^{2}|\psi|^{2}
$$

we have

$$
\begin{equation*}
-\frac{1}{2} \Delta|\psi|^{2}=f\left(|\psi|^{2}\right)|\psi|^{2}-\left|\nabla_{\boldsymbol{A}} \psi\right|^{2} \tag{2.2}
\end{equation*}
$$

Choose $x_{0} \in \bar{\Omega}$ so that $\max _{x \in \bar{\Omega}}|\psi(x)|=\left|\psi\left(x_{0}\right)\right|$. If $\left|\psi\left(x_{0}\right)\right|=0$, then $\psi \equiv 0$ in $\bar{\Omega}$ and so $f(0) \geq 0$ by assumption. If $\left|\psi\left(x_{0}\right)\right|>0,|\psi|$ is a smooth function near
$x_{0}$. When $x_{0} \in \Omega$, by the maximality of $|\psi|$ at $x_{0}$, we have $\nabla|\psi|\left(x_{0}\right)=0$ and $\Delta|\psi|\left(x_{0}\right) \leq 0$. From (2.2), we get $f\left(\left|\psi\left(x_{0}\right)\right|^{2}\right) \geq 0$. Since $|\psi(x)|^{2} \leq\left|\psi\left(x_{0}\right)\right|^{2}$ and $f$ is decreasing, we see that $f\left(|\psi(x)|^{2}\right) \geq 0$. When $x_{0} \in \partial \Omega$, we have $(\partial|\psi| / \partial \boldsymbol{\tau})\left(x_{0}\right)=0$, where $\boldsymbol{\tau}$ is the tangent unit vector at $x_{0}$ to $\partial \Omega$. On the other hand, it follows from the boundary condition that $\partial \psi / \partial \boldsymbol{v}=0$ on $\partial \Omega$. Multiplying $\bar{\psi}$, we have

$$
\frac{\partial}{\partial \mathbf{v}}|\psi|^{2}=2 \Re\left[\left(\frac{\partial}{\partial \mathbf{v}} \psi\right) \bar{\psi}\right]=0
$$

on $\partial \Omega$. Thus we have $2|\psi| \partial|\psi| / \partial \mathbf{v}=0$ on $\partial \Omega$. Since $\left|\psi\left(x_{0}\right)\right|>0$, we have $(\partial|\psi| / \partial v)\left(x_{0}\right)=0$. Therefore, $\nabla|\psi|\left(x_{0}\right)=0$. Since $\Delta|\psi|\left(x_{0}\right) \leq 0$, we have $f\left(\left|\psi\left(x_{0}\right)\right|^{2}\right) \geq 0$.

Remark 2.2. Under the above condition, if we consider the solution of (2.1), we have $f\left(|\psi|^{2}\right)$ is bounded in $\Omega$.

In our previous papers Aramaki [1, 2, 3] and Aramaki et al. [4] associated with superconductivity, we considered the equation of type (1.1) under appropriate hypotheses on $f$ (cf. Pan and Kwek [23]). Let $\Omega \subset \mathbb{R}^{2}$ be a smooth, simply connected domain. We consider the equations associated with superconductivity with the de Gennes boundary condition:

$$
\begin{cases}-\nabla_{\boldsymbol{A}}^{2} \psi=\frac{1}{\varepsilon^{2}} f\left(|\psi|^{2}\right) \psi  \tag{2.3}\\ \operatorname{curl}^{2} \boldsymbol{A}=-\frac{i}{2}(\bar{\psi} \nabla \psi-\psi \nabla \bar{\psi})-|\psi|^{2} \boldsymbol{A} & \text { in } \Omega \\ \mathbf{v} \cdot \nabla_{\boldsymbol{A}} \psi+\gamma \psi=0, & \text { on } \partial \Omega \\ \operatorname{curl} \boldsymbol{A} \times \mathbf{v}=0 & \end{cases}
$$

[3] and [4] showed that when $\gamma>0$ and $\operatorname{curl} \boldsymbol{A} \equiv 0$, the non-trivial solution $\psi$ of (2.3) does not vanish in $\Omega$.

When $f(t)=1-t$, equations (2.3) represent superconductivity. For this equation, there are many articles. For example, when $\gamma=0$ and $\Omega \subset \mathbb{R}^{2}$ is simply connected domain, Elliot et al. [6] showed that the non-trivial solution $\psi$ of (2.2) has only isolated zeros. For non-simply connected case, see Helffer et al. [13] and [14].

## 3. Regularity of the Solution

In this section, we give an $L^{p}$-estimate for the weak solution $\psi$ of (1.1). We follow the arguments in [22] with necessary modifications. Let $f_{\infty}$ := $\left\|f\left(|\psi|^{2}\right)\right\|_{L^{\infty}(\Omega)}$.

Lemma 3.1. Let $\boldsymbol{A} \in L^{\infty}\left(\Omega ; \mathbb{R}^{n}\right)$, $\operatorname{div} \boldsymbol{A} \in L_{\mathrm{loc}}^{q}(\Omega)$ with $q>n / 2$ and let $\psi \in$ $W_{\text {loc }}^{1,2}(\Omega ; \mathbb{C})$ be any solution of (1.1). Then $\psi \in W_{\operatorname{loc}}^{2, q}(\Omega ; \mathbb{C})$ and for any $B_{2 R}\left(x_{0}\right)$ $\Subset \Omega$ and any $1<p \leq q$, there exists a constant $C>0$ which depends only on $p$, q, $\|\boldsymbol{A}\|_{L^{\infty}(\Omega)}$ such that

$$
\begin{align*}
& R^{2}\left\|D^{2} \psi\right\|_{L^{p}\left(B_{R}\left(x_{0}\right)\right)}+R\|\nabla \psi\|_{L^{p}\left(B_{R}\left(x_{0}\right)\right)} \\
\leq & C\left\{\|\psi\|_{L^{p}\left(B_{2 R}\left(x_{0}\right)\right)}+R^{2}\left\|f\left(|\psi|^{2}\right) \psi-i(\operatorname{div} \boldsymbol{A}) \psi\right\|_{L^{p}\left(B_{2 R}\left(x_{0}\right)\right)}\right\} . \tag{3.1}
\end{align*}
$$

Proof. Step 1. We show that $\psi$ is locally Hölder continuous in $\Omega$.
Write (1.1) in the form

$$
\begin{equation*}
-\Delta \psi+2 i \boldsymbol{A} \cdot \nabla \psi+\left[|\boldsymbol{A}|^{2}-f\left(|\psi|^{2}\right)\right] \psi=-i(\operatorname{div} \boldsymbol{A}) \psi \tag{3.2}
\end{equation*}
$$

in $\Omega$. For any fixed $R>0$ such that $B_{2 R}\left(x_{0}\right) \Subset \Omega$, since we want the local estimate, we may assume that $\operatorname{div} \boldsymbol{A} \in L^{q}(\Omega)$ and $\psi \in W^{1,2}(\Omega ; \mathbb{C})$. By the Sobolev imbedding theorem,

$$
\psi \in W^{1,2}(\Omega) \hookrightarrow L^{2 n /(n-2)}(\Omega)
$$

Since $2 n /(n-2)>2$ if $n \geq 3, \quad \psi \in L^{p_{1}}(\Omega)$ with $p_{1}=2 n /(n-2)$. Put $p_{2}=$ $q p_{1} /\left(q+p_{1}\right)$ and applying the Hölder inequality,

$$
\|(\operatorname{div} \boldsymbol{A}) \psi\|_{L^{p_{2}}(\Omega)} \leq\|\operatorname{div} \boldsymbol{A}\|_{L^{q}(\Omega)}\|\psi\|_{L^{p_{1}}(\Omega)}
$$

Since $\boldsymbol{A}$ and $f\left(|\psi|^{2}\right)$ are bounded, it follows from the elliptic regularity theory (cf. Gilbarg and Trudinger [9, Theorem 9.11]) that we have $\psi \in W^{2,} p_{2}\left(B_{2 R^{\prime}}\left(x_{0}\right)\right)$ for any $0<R^{\prime}<R$. If $2-n / p_{2}>0$, by Sobolev imbedding theorem, for any $0<\alpha<$
$2-n / p_{2}, \psi \in C^{\alpha}\left(B_{2 R^{\prime}}\left(x_{0}\right)\right)$. If $2-n / p_{2}=0$, for any $p>p_{2}, W^{2, p_{2}}\left(B_{2 R^{\prime}}\left(x_{0}\right)\right)$ $\hookrightarrow L^{p}\left(B_{2 R^{\prime}}\left(x_{0}\right)\right)$. Similarly as above, we have $(\operatorname{div} \boldsymbol{A}) \psi \in L^{p_{3}}\left(B_{2 R^{\prime}}\left(x_{0}\right)\right)$ with $p_{3}=q p /(q+p)$. Thus $\psi \in W^{2, p_{3}}\left(B_{2 R^{\prime \prime}}\left(x_{0}\right) ; \mathbb{C}\right)$ for any $0<R^{\prime \prime}<R^{\prime}$. If we choose $p$ large enough, we see that $2-n / p_{3}>0$, so $\psi \in C^{\alpha}\left(B_{2 R^{\prime \prime}}\left(x_{0}\right)\right)$ with $\alpha=2-n / p_{3}$. If $2-n / p_{2}<0$, again applying the Sobolev imbedding theorem, $W^{2, p_{2}}\left(B_{2 R^{\prime \prime}}\left(x_{0}\right) ; \mathbb{C}\right) \hookrightarrow L^{p_{3}}\left(B_{2 R^{\prime \prime}}\left(x_{0}\right) ; \mathbb{C}\right) \quad$ with $\quad p_{3}=q p_{2} n /\left(q n+p_{2}(n-2 q)\right)$.
Therefore, we have $\psi \in W^{2, p_{3}}\left(B_{2 R^{\prime \prime \prime}}\left(x_{0}\right) ; \mathbb{C}\right)$ for any $0<R^{\prime \prime \prime}<R^{\prime \prime}$. Since clearly

$$
p_{3}-p_{2}=\frac{(2 q-n) q p_{1}^{2}}{\left(q+p_{1}\right)\left(q n+2(n-q) p_{1}\right)}>0
$$

if we repeat the above arguments finitely many times, we see that $\psi \in$ $C^{\alpha}\left(\bar{B}_{R}\left(x_{0}\right) ; \mathbb{C}\right)$ for some $\alpha \in(0,1)$, where $\bar{B}_{r}\left(x_{0}\right)=\left\{x ;\left|x-x_{0}\right| \leq r\right\}$ for $r>0$.

Step 2. Let $B_{2 R}\left(x_{0}\right) \Subset \Omega$ and write (1.1) in the form

$$
\begin{equation*}
-\Delta \psi+2 i \boldsymbol{A} \cdot \nabla \psi+|\boldsymbol{A}|^{2} \psi=f\left(|\psi|^{2}\right) \psi-i(\operatorname{div} \boldsymbol{A}) \psi \tag{3.3}
\end{equation*}
$$

By the argument of Step 1, we may assume that $\psi \in C^{\alpha}\left(\bar{B}_{2 R}\left(x_{0}\right) ; \mathbb{C}\right)$. Noting that for any $1<p \leq q, \quad f\left(|\psi|^{2}\right) \psi-i(\operatorname{div} \boldsymbol{A}) \psi \in L^{p}\left(B_{2 R}\left(x_{0}\right)\right)$ and so applying [9, Theorem 9.11 and its proof], we have

$$
\begin{aligned}
& R^{2}\left\|D^{2} \psi\right\|_{L^{p}\left(B_{R}\left(x_{0}\right)\right)}+R\|\nabla \psi\|_{L^{p}\left(B_{R}\left(x_{0}\right)\right)} \\
\leq & C\left\{\|\psi\|_{L^{p}\left(B_{2 R}\left(x_{0}\right)\right)}+R^{2}\left\|f\left(|\psi|^{2}\right) \psi-i(\operatorname{div} \boldsymbol{A}) \psi\right\|_{L^{p}\left(B_{2 R}\left(x_{0}\right)\right)}\right\} .
\end{aligned}
$$

## 4. Doubling Property

In this section, we shall discuss on the doubling property of non-trivial solutions of (1.1). We follow the ideas in [8] and [22]. Let $0 \not \equiv \psi \in W_{\text {loc }}^{1,2}(\Omega)$ be a solution of (1.1) and $x_{0} \in \Omega$. Then for any $r>0$ with $B_{r}\left(x_{0}\right) \Subset \Omega$, define some quantities:

$$
I\left(x_{0}, r\right)=\int_{B_{r}\left(x_{0}\right)}\left\{\left|\nabla_{\boldsymbol{A}} \psi\right|^{2}-f\left(|\psi|^{2}\right)|\psi|^{2}\right\} d x
$$

$$
\begin{align*}
& H\left(x_{0}, r\right)=\int_{\partial B_{r}\left(x_{0}\right)}|\psi|^{2} d S_{r} \\
& D\left(x_{0}, r\right)=\int_{B_{r}\left(x_{0}\right)}|\psi|^{2} d x \\
& M\left(x_{0}, r\right)=\frac{r I\left(x_{0}, r\right)}{H\left(x_{0}, r\right)}, \quad N\left(x_{0}, r\right)=\frac{r D\left(x_{0}, r\right)}{H\left(x_{0}, r\right)} \text { if } H\left(x_{0}, r\right) \neq 0 \tag{4.1}
\end{align*}
$$

where $d S_{r}$ is the surface area of the sphere $\partial B_{r}\left(x_{0}\right)$.
Lemma 4.1. Under the hypotheses of Theorem 1.1, let $\psi \in W_{\mathrm{loc}}^{1,2}(\Omega ; \mathbb{C})$ be any non-trivial complex-valued solution of (1.1). Then there exist constants $r_{0}, c, N$, where $r_{0}$ depends only on $f_{\infty}$ and $n$, moreover $c$ and $N$ depend only on $\Omega, \psi, f_{\infty}$, $\|B\|_{L^{\infty}(\Omega)}$ and $n$ such that for any $0<r \leq r_{0} / 2$ with $B_{2 r}\left(x_{0}\right) \Subset \Omega$, we have

$$
M\left(x_{0}, r\right) \leq c
$$

and

$$
\int_{B_{2 r}\left(x_{0}\right)}|\psi|^{2} d x \leq 4^{N} \int_{B_{r}\left(x_{0}\right)}|\psi|^{2} d x
$$

Proof. By Lemma 3.1 and the hypothesis of Theorem 1.1, we see that $\psi \in$ $W_{\text {loc }}^{2,2}(\Omega ; \mathbb{C})$ and $\nabla_{\boldsymbol{A}} \psi \in W_{\text {loc }}^{1,2}\left(\Omega ; \mathbb{C}^{n}\right)$. Without loss of generality, we may assume $x_{0}=0$ and so for brevity of notations, we write $I\left(x_{0}, r\right), H\left(x_{0}, r\right), D\left(x_{0}, r\right)$, $M\left(x_{0}, r\right), N\left(x_{0}, r\right)$ and $B_{r}\left(x_{0}\right)$ by $I(r), H(r), D(r), M(r), N(r)$ and $B_{r}$, respectively.

Step 1. We compute $H^{\prime}(r)$ and $I^{\prime}(r)$.
Let $(\rho, \omega) \in \mathbb{R}^{+} \times S^{n-1}$ be the polar coordinates where $S^{n-1}$ is the surface of the unit sphere $B_{1}$ in $\mathbb{R}^{n}$. Since we can write

$$
H(r)=r^{n-1} \int_{\partial B_{1}}|\psi(r, \omega)|^{2} d S
$$

where $d S$ is the surface area on $\partial B_{1}=S^{n-1}$, we have

$$
\begin{aligned}
H^{\prime}(r) & =(n-1) r^{n-2} \int_{\partial B_{1}}|\psi(r, \omega)|^{2} d S+2 r^{n-1} \mathfrak{R} \int_{\partial B_{1}} \bar{\psi} \frac{\partial \psi}{\partial \rho} d S \\
& =\frac{n-1}{r} H(r)+2 \mathfrak{R} \int_{\partial B_{r}}\left\langle\bar{\psi} \nabla \psi, \frac{x}{r}\right\rangle d S_{r} .
\end{aligned}
$$

Here it follows from the divergence theorem that

$$
\begin{aligned}
\mathfrak{R} \int_{\partial B_{r}}\left\langle\bar{\psi} \nabla \psi, \frac{x}{r}\right\rangle d S_{r} & =\mathfrak{R} \int_{\partial B_{r}}\left\langle\bar{\psi} \nabla_{\boldsymbol{A}} \psi, \frac{x}{r}\right\rangle d S_{r} \\
& =\mathfrak{R} \int_{B_{r}} \operatorname{div}\left(\bar{\psi} \nabla_{\boldsymbol{A}} \psi\right) d x \\
& =\mathfrak{R} \int_{B_{r}}\left\{\bar{\psi} \operatorname{div}\left(\nabla_{\boldsymbol{A}} \psi\right)+\nabla \bar{\psi} \cdot \nabla_{\boldsymbol{A}} \psi\right\} d x \\
& =\mathfrak{R} \int_{B_{r}}\left\{\bar{\psi} \nabla_{\boldsymbol{A}}^{2} \psi+\left|\nabla_{\boldsymbol{A}} \psi\right|^{2}\right\} d x \\
& =\mathfrak{R} \int_{B_{r}}\left\{-f\left(|\psi|^{2}\right)|\psi|^{2}+\left|\nabla_{\boldsymbol{A}} \psi\right|^{2}\right\} d x=I(r) .
\end{aligned}
$$

Thus we have

$$
\begin{equation*}
H^{\prime}(r)=\frac{n-1}{r} H(r)+2 I(r) . \tag{4.2}
\end{equation*}
$$

Next, we compute $I^{\prime}(r)$. Clearly, we have

$$
I^{\prime}(r)=\int_{\partial B_{r}}\left\{\left|\nabla_{\boldsymbol{A}} \psi\right|^{2}-f\left(|\psi|^{2}\right)|\psi|^{2}\right\} d S_{r}
$$

By the divergence theorem,

$$
\begin{aligned}
\int_{\partial B_{r}}\left|\nabla_{\boldsymbol{A}} \psi\right|^{2} d S_{r} & \left.=\left.\frac{1}{r} \int_{\partial B_{r}}\langle | \nabla_{\boldsymbol{A}} \psi\right|^{2} x, \frac{x}{r}\right\rangle d S_{r} \\
& =\frac{1}{r} \int_{B_{r}} \operatorname{div}\left(\left|\nabla_{\boldsymbol{A}} \psi\right|^{2} x\right) d x \\
& \left.=\frac{1}{r} \int_{B_{r}} n\left|\nabla_{\boldsymbol{A}} \psi\right|^{2} d x+\left.\frac{1}{r} \int_{B_{r}}\langle x, \nabla| \nabla_{\boldsymbol{A}} \psi\right|^{2}\right\rangle d x .
\end{aligned}
$$

In order to compute the last term in the above equality, we have

$$
\begin{aligned}
J & \left.:=\left.\int_{B_{r}}\langle x, \nabla| \nabla_{\boldsymbol{A}} \psi\right|^{2}\right\rangle d x \\
& =2 \Re \int_{B_{r}}\left\langle x, \nabla\left(\nabla_{\boldsymbol{A}} \psi\right) \cdot \overline{\nabla_{\boldsymbol{A}} \psi} d x\right\rangle \\
& =2 \Re \int_{B_{r}} \sum_{j, k=1}^{n} x_{j}\left[\partial_{j}\left(\partial_{k} \psi-i A_{k} \psi\right)\right]\left(\partial_{k} \bar{\psi}+i A_{k} \bar{\psi}\right) d x \\
& =: 2\left(J_{1}-J_{2}\right)
\end{aligned}
$$

where $J_{1}$ and $J_{2}$ are defined by

$$
\begin{aligned}
& J_{1}=\Re \int_{B_{r}} \sum_{j, k=1}^{n} x_{j}\left[\partial_{k}\left(\partial_{j} \psi-i A_{j} \psi\right)\right]\left(\partial_{k} \bar{\psi}+i A_{k} \bar{\psi}\right) d x \\
& J_{2}=\mathfrak{J} \int_{B_{r}} \sum_{j, k=1}^{n} x_{j}\left[\partial_{k}\left(A_{j} \psi\right)-\partial_{j}\left(A_{k} \psi\right)\right]\left(\partial_{k} \bar{\psi}+i A_{k} \bar{\psi}\right) d x .
\end{aligned}
$$

We continue the computation of $J_{1}$ and $J_{2}$,

$$
\begin{aligned}
J_{1}= & \Re \int_{B_{r}} \sum_{j, k=1}^{n}\left\{\partial_{k}\left[x_{j}\left(\partial_{j} \psi-i A_{j} \psi\right)\right]\left(\partial_{k} \bar{\psi}+i A_{k} \bar{\psi}\right)\right. \\
& -x_{j}\left(\partial_{j} \psi-i A_{j} \psi\right) \partial_{k}\left(\partial_{k} \bar{\psi}+i A_{k} \bar{\psi}\right) \\
& \left.-\delta_{j k}\left(\partial_{j} \psi-i A_{j} \psi\right)\left(\partial_{k} \bar{\psi}+i A_{k} \bar{\psi}\right)\right\} d x \\
= & \sum_{j, k=1}^{n} \Re \int_{\partial B_{r}} \frac{x_{j} x_{k}}{r}\left(\partial_{j} \psi-i A_{j} \psi\right)\left(\partial_{k} \bar{\psi}+i A_{k} \bar{\psi}\right) d x \\
& -\Re \int_{B_{r}}\left\{\left(x \cdot \overline{\nabla_{\boldsymbol{A}} \psi}\right) \operatorname{div}\left(\nabla_{\boldsymbol{A}} \psi\right)+\left|\nabla_{\boldsymbol{A}} \psi\right|^{2}\right\} d x \\
= & \frac{1}{r} \int_{\partial B_{r}}\left|x \cdot \nabla_{\boldsymbol{A}} \psi\right|^{2} d S_{r}
\end{aligned}
$$

$$
\begin{aligned}
& -\mathfrak{R} \int_{B_{r}}\left\{\left(x \cdot \overline{\nabla_{\boldsymbol{A}} \psi}\right)\left[\nabla_{\boldsymbol{A}}^{2} \psi+i \boldsymbol{A} \cdot \nabla_{\boldsymbol{A}} \psi\right]+\left|\nabla_{\boldsymbol{A}} \psi\right|^{2}\right\} d x \\
= & \frac{1}{r} \int_{\partial B_{r}}\left|x \cdot \nabla_{\boldsymbol{A}} \psi\right|^{2} d S_{r}+\mathfrak{R} \int_{B_{r}}\left(x \cdot \overline{\nabla_{\boldsymbol{A}} \psi}\right) f\left(|\psi|^{2}\right) \psi d x \\
& +\mathfrak{J} \int_{B_{r}}\left(x \cdot \overline{\nabla_{\boldsymbol{A}} \psi}\right)\left(\boldsymbol{A} \cdot \nabla_{\boldsymbol{A}} \psi\right) d x-\int_{B_{r}}\left|\nabla_{\boldsymbol{A}} \psi\right|^{2} d x .
\end{aligned}
$$

In order to compute $J_{2}$, we note that

$$
\begin{aligned}
& \sum_{j, k=1}^{n} x_{j}\left[\partial_{k}\left(A_{j} \psi\right)-\partial_{j}\left(A_{k} \psi\right)\right]\left(\partial_{k} \bar{\psi}+i A_{k} \bar{\psi}\right) \\
= & \sum_{j, k=1}^{n} B_{j k} \psi x_{j}\left(\overline{\nabla_{\boldsymbol{A}} \psi}\right)_{k}+(\boldsymbol{A} \cdot x)\left|\nabla_{\boldsymbol{A}} \psi\right|^{2}-\left(x \cdot \nabla_{\boldsymbol{A}} \psi\right)\left(\boldsymbol{A} \cdot \overline{\nabla_{\boldsymbol{A}} \psi}\right)
\end{aligned}
$$

Therefore, we have

$$
J_{2}=\mathfrak{J} \int_{B_{r}}\left\{\left(B \nabla_{\boldsymbol{A}} \psi\right) \cdot x\right\} \psi d x+\mathfrak{J} \int_{B_{r}}\left(x \cdot \overline{\nabla_{\boldsymbol{A}} \psi}\right)\left(\boldsymbol{A} \cdot \nabla_{\boldsymbol{A}} \psi\right) d x
$$

Thus we see that

$$
\begin{align*}
I^{\prime}(r)= & \frac{n-2}{r} \int_{B_{r}}\left|\nabla_{\boldsymbol{A}} \psi\right|^{2} d x+\int_{\partial B_{r}}\left\{2\left|\frac{x}{r} \cdot \nabla_{\boldsymbol{A}} \psi\right|^{2}-f\left(|\psi|^{2}\right)|\psi|^{2}\right\} d S_{r} \\
& +2 \Re \int_{B_{r}}\left(\frac{x}{r} \cdot \overline{\nabla_{\boldsymbol{A}} \psi}\right) f\left(|\psi|^{2}\right) \psi d x-\frac{2}{r} \mathfrak{J} \int_{B_{r}}\left\{\left(B \nabla_{\boldsymbol{A}} \psi\right) \cdot x\right\} \psi d x . \tag{4.3}
\end{align*}
$$

Step 2. We show that there exists $r_{0}$ depending only on $f_{\infty}$ and $n$ such that for any $x_{0} \in \Omega$ and for all $0<r \leq r_{0}$ with $B_{r} \Subset \Omega$,

$$
\begin{equation*}
\int_{B_{r}\left(x_{0}\right)}|\psi|^{2} d x \leq r \int_{\partial B_{r}\left(x_{0}\right)}|\psi|^{2} d S_{r} \tag{4.4}
\end{equation*}
$$

We shall prove that (4.4) holds. We may assume that $x_{0}=0$. Multiplying (1.1) by $\left(r^{2}-|x|^{2}\right) \bar{\psi}$ and integrating over $B_{r}$, we have

$$
\int_{B_{r}}-\nabla_{A}^{2} \psi\left(r^{2}-|x|^{2}\right) \bar{\psi} d x=\int_{B_{r}} f\left(|\psi|^{2}\right)\left(r^{2}-|x|^{2}\right)|\psi|^{2} d x
$$

By the integraton by parts, the left hand side is equal to

$$
\begin{aligned}
& \mathfrak{R} \int_{B_{r}} \overline{\nabla_{\boldsymbol{A}} \psi} \cdot \nabla_{\boldsymbol{A}}\left(\left(r^{2}-|x|^{2}\right) \psi\right) d x \\
= & \int_{B_{r}}\left(r^{2}-|x|^{2}\right)\left|\nabla_{\boldsymbol{A}} \psi\right|^{2} d x-2 \Re \int_{B_{r}} \psi\left(x \cdot \overline{\nabla_{\boldsymbol{A}} \psi}\right) d x .
\end{aligned}
$$

Here we note

$$
\begin{aligned}
& 2 \mathfrak{R} \int_{B_{r}} \psi\left(x \cdot \overline{\nabla_{A} \psi}\right) d x \\
= & \int_{B_{r}}\left\{\operatorname{div}\left(|\psi|^{2} x\right)-n|\psi|^{2}\right\} d x \\
= & \int_{\partial B_{r}} \frac{x}{r} \cdot|\psi|^{2} x d S_{r}-n \int_{B_{r}}|\psi|^{2} d x .
\end{aligned}
$$

Therefore, we see that

$$
\begin{aligned}
& r^{2} \int_{B_{r}} f\left(|\psi|^{2}\right)|\psi|^{2} d x \\
\geq & \int_{B_{r}} f\left(|\psi|^{2}\right)\left(r^{2}-|x|^{2}\right)|\psi|^{2} d x \\
= & \int_{B_{r}}\left(r^{2}-|x|^{2}\right)\left|\nabla_{A} \psi\right|^{2} d x-r \int_{\partial B_{r}}|\psi|^{2} d S_{r}+n \int_{B_{r}}|\psi|^{2} d x \\
\geq & -r \int_{\partial B_{r}}|\psi|^{2} d S_{r}+n \int_{B_{r}}|\psi|^{2} d x .
\end{aligned}
$$

From this inequality, we have

$$
r \int_{\partial B_{r}}|\psi|^{2} d S_{r} \geq \int_{B_{r}}\left[n-r^{2} f\left(|\psi|^{2}\right)\right]|\psi|^{2} d x
$$

Choose $r_{0}>0$ so that $r_{0} \leq \sqrt{\frac{n-1}{f_{\infty}}}$. Then, for any $0<r \leq r_{0}$ with $B_{r} \Subset \Omega$,

$$
\int_{B_{r}}|\psi|^{2} d x \leq r \int_{\partial B_{r}}|\psi|^{2} d S_{r}
$$

Step 3. Let $\psi \not \equiv 0$ in $\Omega$ and $r_{0}$ be as in Step 2. Then for any $0<r \leq r_{0}$ with
$B_{r}\left(x_{0}\right) \Subset \Omega, \quad H\left(x_{0}, r\right)>0$. In fact, if $H\left(x_{0}, r\right)=0$ for some $0<r \leq r_{0}$, then $\psi \equiv 0$ on $\partial B_{r}\left(x_{0}\right)$. By Step 2, we see that $\psi \equiv 0$ in $B_{r}\left(x_{0}\right)$. Then it follows from the unique continuation theorem of Aronszajn [5] that $\psi \equiv 0$ in $\Omega$.

Assume that $x_{0}=0$ as before and define

$$
J\left(r_{0}\right)=\left\{r \in\left(0, r_{0}\right) ; M(r)>\max \left\{1, M\left(r_{0}\right)\right\}\right\}
$$

If $r \in J\left(r_{0}\right)$, then we have $I(r)>0$ and $H(r)<r I(r)$. Therefore, it follows from Step 2 that

$$
\begin{equation*}
\int_{B_{r}}|\psi|^{2} d x \leq r H(r)<r^{2} I(r) \tag{4.5}
\end{equation*}
$$

We shall show that
Claim. There exists a constant $\lambda>0$ depending only on $f_{\infty},\|B\|_{L^{\infty}(\Omega)}, \Omega$ and $n$ such that

$$
M^{\prime}(r) \geq-\lambda M(r) \text { for all } r \in J\left(r_{0}\right)
$$

In fact, since

$$
M^{\prime}(r)=\frac{I(r)+r I^{\prime}(r)}{H(r)}-\frac{r I(r) H^{\prime}(r)}{H(r)^{2}}
$$

it follows from (4.2) and (4.3) that

$$
\begin{aligned}
\frac{M^{\prime}(r)}{M(r)}= & \frac{I^{\prime}(r)}{I(r)}-\frac{1}{r}-\frac{2 I(r)}{H(r)} \\
= & \frac{1}{I(r)} \frac{n-2}{r} \int_{B_{r}}\left|\nabla_{\boldsymbol{A}} \psi\right|^{2} d x+\frac{2}{I(r)} \int_{\partial B_{r}}\left|\frac{x}{r} \cdot \nabla_{\boldsymbol{A}} \psi\right|^{2} d S_{r} \\
& -\frac{1}{I(r)} \int_{\partial B_{r}} f\left(|\psi|^{2}\right)|\psi|^{2} d S_{r} \\
& +\frac{2}{I(r)} \mathfrak{R} \int_{B_{r}}\left(\frac{x}{r} \cdot \overline{\nabla_{\boldsymbol{A}} \psi}\right) f\left(|\psi|^{2}\right) \psi d x \\
& -\frac{2}{I(r)} \Im \int_{B_{r}}\left(B \overline{\nabla_{A} \psi} \cdot \frac{x}{r}\right) \psi d x-\frac{1}{r}-\frac{2 I(r)}{H(r)}
\end{aligned}
$$

$$
\begin{align*}
& -\frac{1}{r I(r)} \int_{B_{r}}\left|\nabla_{A} \psi\right|^{2} d x+\frac{1}{r I(r)} \int_{B_{r}} f\left(|\psi|^{2}\right)|\psi|^{2} d x \\
= & \frac{n-3}{r I(r)} \int_{B_{r}}\left|\nabla_{\boldsymbol{A}} \psi\right|^{2} d x+\frac{2}{I(r)} \int_{\partial B_{r}}\left|\frac{x}{r} \cdot \nabla_{A} \psi\right|^{2} d S_{r} \\
& -\frac{2 I(r)}{H(r)}+\frac{1}{I(r)}\left\{\frac{1}{r} \int_{\partial B_{r}} f\left(|\psi|^{2}\right)|\psi|^{2} d x-\int_{\partial B_{r}} f\left(|\psi|^{2}\right)|\psi|^{2} d S_{r}\right\} \\
& +\frac{2}{I(r)} \Re \int_{B_{r}}\left(\frac{x}{r} \cdot \overline{\nabla_{A} \psi}\right) f\left(|\psi|^{2}\right) \psi d x \\
& -\frac{2}{I(r)} \mathfrak{J} \int_{B_{r}}\left(B \overline{\nabla_{A} \psi} \cdot \frac{x}{r}\right) \psi d x . \tag{4.6}
\end{align*}
$$

Here from the definition of $I(r)$ and the Schwarz inequality,

$$
I(r)^{2} \leq \int_{\partial B_{r}}|\psi|^{2} d S_{r} \int_{\partial B_{r}}\left|\frac{x}{r} \cdot \nabla_{A} \psi\right|^{2} d S_{r}
$$

Therefore

$$
\begin{aligned}
& \frac{2}{I(r)} \int_{\partial B_{r}}\left|\frac{x}{r} \cdot \nabla_{A} \psi\right|^{2} d S_{r}-\frac{2 I(r)}{H(r)} \\
= & \frac{2}{H(r) I(r)}\left[\int_{\partial B_{r}}|\psi|^{2} d S_{r} \int_{\partial B_{r}}\left|\frac{x}{r} \cdot \nabla_{A} \psi\right|^{2} d S_{r}-I(r)^{2}\right] \geq 0 .
\end{aligned}
$$

By Step 2 and the fact that $r \in J\left(r_{0}\right)$, we see that

$$
\int_{\partial B_{r}} f\left(|\psi|^{2}\right)|\psi|^{2} d S_{r} \leq f_{\infty} H(r) \leq r I(r) f_{\infty}
$$

We estimate the last two terms in (4.6). By using the Schwarz inequality and (4.5),

$$
\begin{aligned}
& \left|2 \mathfrak{R} \int_{B_{r}}\left(\frac{x}{r} \cdot \overline{\nabla_{\boldsymbol{A}} \psi}\right) f\left(|\psi|^{2}\right) \psi d x\right| \\
\leq & 2 f_{\infty} \int_{B_{r}}\left|\nabla_{\boldsymbol{A}} \psi\right||\psi| d x \\
\leq & f_{\infty}\left\{\int_{B_{r}}\left|\nabla_{\boldsymbol{A}} \psi\right|^{2} d x+\int_{B_{r}}|\psi|^{2} d x\right\}
\end{aligned}
$$

$$
\begin{aligned}
& \leq f_{\infty}\left\{I(r)+\int_{B_{r}}\left(f\left(|\psi|^{2}\right)+1\right)|\psi|^{2} d x\right\} \\
& \leq f_{\infty} I(r)\left\{1+\left(f_{\infty}+1\right) r^{2}\right\} .
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
& \quad\left|2 \mathfrak{J} \int_{B_{r}}\left\{\left(B \overline{\nabla_{A} \psi}\right) \cdot \frac{x}{r}\right\} \psi d x\right| \\
& \leq 2\|B\|_{L^{\infty}(\Omega)} \int_{B_{r}}\left|\nabla_{A} \psi \| \psi\right| d x \\
& \leq\|B\|_{L^{\infty}(\Omega)} I(r)\left\{1+\left(f_{\infty}+1\right) r^{2}\right\} .
\end{aligned}
$$

Thus since $n \geq 3$, we get

$$
\begin{aligned}
\frac{M^{\prime}(r)}{M(r)} \geq & -\frac{1}{I(r)}\left[r I(r) f_{\infty}+r I(r) f_{\infty}-f_{\infty} I(r)\left\{1+\left(1+f_{\infty}\right) r^{2}\right\}\right. \\
& \left.-\|B\|_{L^{\infty}(\Omega)} I(r)\left\{1+\left(1+f_{\infty}\right) r^{2}\right\}\right] \\
= & -2 r f_{\infty}-\left(f_{\infty}+\|B\|_{L^{\infty}(\Omega)}\right)\left[1+\left(1+f_{\infty}\right) r^{2}\right] \geq-\lambda
\end{aligned}
$$

where

$$
\lambda=2 r_{0} f_{\infty}+\left(f_{\infty}+\|B\|_{L^{\infty}(\Omega)}\right)\left[1+\left(1+f_{\infty}\right) r_{0}^{2}\right]
$$

Since $r \in J\left(r_{0}\right), M(r)>1$. Thus the claim holds.
Step 4. Define $\hat{M}(r)=e^{\lambda r} M(r)$. Then by Step 3, we see that

$$
\hat{W}^{\prime}(r)=\left(\lambda+M^{\prime}(r)\right) e^{\lambda r} \geq 0 \text { in } J\left(r_{0}\right)
$$

If $(a, b)$ is a constituent interval of $J\left(r_{0}\right)$, then we see that $M(b)=\max \left\{1, M\left(r_{0}\right)\right\}$. Since $\hat{M}(r)$ is monotone increasing in $(a, b)$, for any $r \in(a, b), \hat{M}(r) \leq \hat{M}(b)$ $\leq \max \left\{e^{\lambda r_{0}}, e^{\lambda r_{0}} M\left(r_{0}\right)\right\}$. On $\left(0, r_{0}\right) \backslash J\left(r_{0}\right)$, we also have the same inequality. Thus we see that for any $0<r \leq r_{0}$,

$$
\begin{equation*}
M(r) \leq \hat{M}(r) \leq c_{3}:=\max \left\{e^{\lambda r_{0}}, e^{\lambda r_{0}} M\left(r_{0}\right)\right\} \tag{4.7}
\end{equation*}
$$

where $c_{3}$ depends only on $f_{\infty},\|B\|_{L^{\infty}(\Omega)}, r_{0}$ and $n$.

In this stage, we can also prove that if $\psi \not \equiv 0$ in $B_{r_{1}}$ with $B_{r_{1}} \Subset \Omega$, then $H(r)$ $>0$ for any $0<r \leq r_{1}$ without using the unique continuation theorem [5]. In fact, suppose it were false. Then there exists $0<r_{2} \leq r_{1}$ such that $H\left(r_{2}\right)=0$. Since $M(r) \leq c_{3} e^{-\lambda r}, I(r) \leq c_{3} H(r) / r$. Therefore, by (4.2),

$$
H^{\prime}(r)=\frac{n-1}{r} H(r)+2 I(r) \leq \frac{H(r)}{r}\left(n-1+2 c_{3}\right) .
$$

If we choose $r_{0}>0$ so that $r_{0} \leq \min \left\{1, \sqrt{(n-1) / f_{\infty}}\right\}$, then $H^{\prime}(r) \leq k H(r) / r$, where $k=n-1+2 c_{3}$. Thus

$$
\left(r^{-k} H(r)\right)^{\prime}=r^{-k}\left(H^{\prime}(r)-\frac{k}{r} H(r)\right) \leq 0 .
$$

This implies that for any $r_{2} \leq r \leq r_{1}, r^{-k} H(r) \leq r_{2}^{-k} H\left(r_{2}\right)=0$. Hence $H(r)=0$ for any $r_{2} \leq r \leq r_{1}$. In particular, we have

$$
\int_{\partial B_{r_{1}}}|\psi|^{2} d S_{r_{1}}=0
$$

By Step 2, we get $\int_{B_{r_{1}}}|\psi|^{2} d x=0$. Thus we get $\psi \equiv 0$ in $B_{r_{1}}$. This is a contradiction.

Now since $I(r)=\frac{1}{r} e^{-\lambda r} \hat{M}(r) H(r)$, it follows from (4.2) that

$$
\begin{equation*}
\frac{H^{\prime}(r)}{H(r)}=\frac{n-1}{r}+\frac{2}{r} e^{-\lambda r} \hat{M}(r) \tag{4.8}
\end{equation*}
$$

From this equality, we have

$$
\left(\log \frac{H(r)}{r^{n-1}}\right)^{\prime}=\frac{H^{\prime}(r)}{H(r)}-\frac{n-1}{r}=\frac{2}{r} e^{-\lambda r} \hat{M}(r)
$$

For $\rho \leq r_{0} / 2$ with $B_{2 \rho} \Subset \Omega$, integrate from $\rho$ to $2 \rho$. Then we get

$$
\log \left(\frac{H(2 \rho)}{2^{n-1} H(\rho)}\right)=\int_{\rho}^{2 \rho} \frac{2}{r} e^{-\lambda r} \hat{M}(r) d r \leq 2 c_{3} \log 2
$$

Thus we have

$$
H(2 \rho) \leq 4^{c_{3}} 2^{n-1} H(\rho)=4^{N} H(\rho),
$$

where $N=c_{3}+(n-1) / 2$ depends only on $f_{\infty},\|B\|_{L^{\infty}(\Omega)}, \psi, n$. That is to say,

$$
\int_{\partial B_{2 \rho}}|\psi|^{2} d S_{2 \rho} \leq 4^{N} \int_{\partial B_{\rho}}|\psi|^{2} d S_{\rho} \text { for all } 0<\rho \leq r_{0} / 2 \text { with } B_{2 \rho} \Subset \Omega .
$$

When $R \leq r_{0} / 2$ with $B_{2 R} \Subset \Omega$, integrating this inequality from $\rho=0$ to $\rho=R$, we get

$$
\begin{equation*}
\int_{B_{2 R}}|\psi|^{2} d x \leq 4^{N} \int_{B_{R}}|\psi|^{2} d x \tag{4.9}
\end{equation*}
$$

This inequality holds for the case $\psi \equiv 0$ in $B_{2 R}$.
Step 5. Let $\psi \not \equiv 0$ in $\Omega$. In (4.7) and (4.9), the constants $c_{3}$ and $N$ in fact depend on $x_{0}$. We must show that the constants are independent of $x_{0}$. In order to do so, it suffices to prove that for fixed $0<r \leq r_{0}$ with $B_{r}\left(x_{0}\right) \Subset \Omega$,

$$
\begin{equation*}
\tilde{M}(r)=\sup _{B_{r}\left(x_{0}\right) \Subset \Omega} M\left(x_{0}, r\right)<\infty . \tag{4.10}
\end{equation*}
$$

We may assume that $\psi \in L^{2}(\Omega), \nabla_{\boldsymbol{A}} \psi \in L^{2}(\Omega)$ and $f_{\infty}<\infty$. Therefore, since

$$
\begin{aligned}
\sup _{B_{r}\left(x_{0}\right) \Subset \Omega} I\left(x_{0}, r\right) & =\sup _{B_{r}\left(x_{0}\right) \Subset \Omega} \int_{B_{r}\left(x_{0}\right)}\left\{\left|\nabla_{\boldsymbol{A}} \psi\right|^{2}-f\left(|\psi|^{2}\right)|\psi|^{2}\right\} d x \\
& \leq\left\|\nabla_{\boldsymbol{A}} \psi\right\|_{L^{2}(\Omega)}+f_{\infty}\|\psi\|_{L^{2}(\Omega)}
\end{aligned}
$$

it suffices to prove that for any fixed $0<r \leq r_{0}$, with $B_{r}\left(x_{0}\right) \Subset \Omega, \inf _{B_{r}\left(x_{0}\right)} \Subset \Omega$ $H\left(x_{0}, r\right)>0$. Since $H\left(x_{0}, r\right)$ is continuous in $x_{0}$, it suffices to prove that

$$
\begin{equation*}
H\left(x_{0}, r\right)>0 \text { as long as } 0<r \leq r_{0}, B_{r}\left(x_{0}\right) \Subset \Omega . \tag{4.11}
\end{equation*}
$$

Suppose that (4.11) were false. Then there exists $0<r_{1} \leq r_{0}$ and $x_{0} \in \Omega$ such that $B_{r_{1}}\left(x_{0}\right) \Subset \Omega$ and $H\left(x_{0}, r_{1}\right)=0$. Then repeating the proof of Step 4, we get $\psi \equiv 0$ in $B_{r_{1}}\left(x_{0}\right)$. By the doubling property or the unique continuation theorem, we have $\psi \equiv 0$ in $\Omega$. Thus (4.11) holds. If we put $c_{3}=e^{\lambda r_{0}} \max \left\{1, \tilde{M}\left(r_{0}\right)\right\}$ and $N=c_{3}+$ $(n-1) / 2, \quad c_{3}$ and $N$ are depending only on $f_{\infty},\|B\|_{L^{\infty}(\Omega)}, \psi, \Omega$ but independent of $x_{0}$. This completes the proof of Lemma 4.1.

Proof of Theorem 1.1. (i) follows from Lemma 4.1. (ii) is a result of the doubling property or the unique continuation theorem. Here we shall prove (ii) using the doubling property. Assume that $\psi$ vanishes of infinite order at $x_{0} \in \Omega$, i.e., for any integer $m \geq 0$,

$$
\int_{B_{R}\left(x_{0}\right)}|\psi|^{2} d x=O\left(R^{n+m}\right) \text { as } R \rightarrow 0
$$

We may assume that $x_{0}=0$ as before. We simply put $c_{2}=4^{N}$ and choose $R_{0}>0$ so that $B_{R_{0}} \Subset \Omega$. Then using the doubling property, we have

$$
\begin{aligned}
\int_{B_{R_{0}}}|\psi|^{2} d x & \leq c_{2} \int_{B_{R_{0} / 2}}|\psi|^{2} d x \\
& \leq c_{2}^{k} \int_{B_{R_{0} / 2^{k}}}|\psi|^{2} d x \\
& =c_{2}^{k}\left|B_{R_{0} / 2^{k}}\right|^{\alpha} \frac{1}{\left|B_{R_{0} / 2^{k}}\right|^{\alpha}} \int_{B_{R_{0} / 2^{k}}}|\psi|^{2} d x,
\end{aligned}
$$

where $\left|B_{R_{0} / 2^{k}}\right|$ is the volume of $B_{R_{0} / 2^{k}}$, and $\alpha>0$ is to be chosen so that $c_{2} 2^{-n \alpha}=1$, i.e., $\alpha=\log c_{2} /(n \log 2)$. Then

$$
c_{2}^{k}\left|B_{R_{0} / 2^{k}}\right|^{\alpha}=\omega_{n}^{\alpha} R_{0}^{n \alpha}
$$

where $\omega_{n}$ is the volume of the unit sphere $B_{1}$. Therefore, we have

$$
\begin{aligned}
\int_{B_{R_{0}}}|\psi|^{2} d x & \leq \omega_{n}^{\alpha} R_{0}^{n \alpha} \frac{1}{\left|B_{R_{0} / 2^{k}}\right|^{\alpha}} \int_{B_{R_{0} / 2^{k}}}|\psi|^{2} d x \\
& =\omega_{n}^{\alpha} R_{0}^{n \alpha} \frac{1}{\left|B_{R_{0} / 2^{k}}\right|^{\alpha}} O\left(\left(\frac{R_{0}}{2^{k}}\right)^{n+m}\right) \\
& \leq C \omega_{n}^{\alpha} R_{0}^{n \alpha}\left(\frac{R_{0}}{2^{k}}\right)^{-n \alpha+n+m}
\end{aligned}
$$

where $C$ is a constant independent of $k$. Thus if we choose $m>0$ large enough and let $k \rightarrow 0$, we see that $\psi \equiv 0$ in $B_{R_{0}}$. Since $\Omega$ is arcwise connected, this implies
$\psi \equiv 0$ in $\Omega$. We can also use the unique continuation theorem of [5]. This completes the proof of Theorem 1.1.

Lemma 4.2. If the conditions of Theorem 1.1 hold and if $\psi \in C^{1}(\Omega ; \mathbb{C})$, then for any $x_{0} \in \mathcal{N}(\psi)=\{x \in \Omega ; \psi(x)=0\}$,

$$
\limsup _{r \rightarrow 0} \frac{M\left(x_{0}, r\right)}{N\left(x_{0}, r\right)}=1
$$

and the limit holds uniformly in $x_{0} \in \mathcal{N}(\psi)$. Moreover, there exists $c_{3}>0$ depending only on $f_{\infty},\|A\|_{L^{\infty}(\Omega)},\|B\|_{L^{\infty}(\Omega)}, \psi$ and $n$ such that for any $0<r \leq r_{0} / 2$ with $B_{2 r}\left(x_{0}\right) \Subset \Omega$, where $r_{0}$ is as in Lemma 4.1,

$$
N\left(x_{0}, r\right) \leq c_{3}
$$

Proof. As before, we may assume that $x_{0}=0$. We see from (4.4) that

$$
\begin{aligned}
I(r) & =\int_{B_{r}}\left\{\left|\nabla_{\boldsymbol{A}} \psi\right|^{2}-f\left(|\psi|^{2}\right)|\psi|^{2}\right\} d x \\
& \leq \int_{B_{r}}\left\{(|\nabla \psi|+|A \psi|)^{2}+f_{\infty}|\psi|^{2}\right\} d x \\
& \leq(1+r) \int_{B_{r}}|\nabla \psi|^{2} d x+\left[\left(1+\frac{1}{r}\right)\|\boldsymbol{A}\|_{L^{\infty}(\Omega)}^{2}+f_{\infty}\right] \int_{B_{r}}|\psi|^{2} d x \\
& \leq(1+r) D(r)+\left[r f_{\infty}+(1+r)\|\boldsymbol{A}\|_{L^{\infty}(\Omega)}^{2}\right] H(r) .
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
D(r) & =\int_{B_{r}}|\nabla \psi|^{2} d x \\
& \leq \int_{B_{r}}\left(\left|\nabla_{\boldsymbol{A}} \psi\right|+|A \psi|\right)^{2} d x \\
& \leq(1+r) \int_{B_{r}}\left|\nabla_{A} \psi\right|^{2} d x+\left(1+\frac{1}{r}\right)\|A\|_{L^{\infty}(\Omega)}^{2} \int_{B_{r}}|\psi|^{2} d x \\
& =(1+r)\left[\int_{B_{r}}\left(\left|\nabla_{\boldsymbol{A}} \psi\right|^{2}-f(|\psi|)^{2}|\psi|^{2}\right) d x+\int_{B_{r}} f(|\psi|)^{2}|\psi|^{2} d x\right.
\end{aligned}
$$

$$
\begin{aligned}
& \left.+\frac{1}{r}\|\boldsymbol{A}\|_{L^{\infty}(\Omega)} \int_{B_{r}}|\psi|^{2} d x\right] \\
\leq & (1+r)\left\{\left[I(r)+\left[r f_{\infty}+\|\boldsymbol{A}\|_{L^{\infty}(\Omega)}^{2}\right] H(r)\right\} .\right.
\end{aligned}
$$

Thus we have

$$
I(r) \geq \frac{D(r)}{1+r}-\left[r f_{\infty}+\|\boldsymbol{A}\|_{L^{\infty}(\Omega)}^{2}\right] H(r)
$$

Therefore, we have

$$
\begin{aligned}
& \frac{1}{1+r} \frac{r D(r)}{H(r)}-r\left[r f_{\infty}+\|\boldsymbol{A}\|_{L^{\infty}(\Omega)}^{2}\right] \\
\leq & \frac{r I(r)}{H(r)} \leq(1+r) \frac{r D(r)}{H(r)}+r\left[r f_{\infty}+\|\boldsymbol{A}\|_{L^{\infty}(\Omega)}^{2}\right]
\end{aligned}
$$

This shows that

$$
\begin{align*}
& \frac{1}{1+r} N\left(x_{0}, r\right)-r\left[r f_{\infty}+\|\boldsymbol{A}\|_{L^{\infty}(\Omega)}^{2}\right] \\
\leq & M\left(x_{0}, r\right) \leq(1+r) N\left(x_{0}, r\right)+r\left[r f_{\infty}+(1+r)\|\boldsymbol{A}\|_{L^{\infty}(\Omega)}^{2}\right] . \tag{4.12}
\end{align*}
$$

In particular, using Lemma 4.1, we see that

$$
\begin{equation*}
N\left(x_{0}, r\right) \leq(1+r) M\left(x_{0}, r\right)+r(1+r)\left[r f_{\infty}+\|\boldsymbol{A}\|_{L^{\infty}(\Omega)}^{2}\right] \leq c_{3}, \tag{4.13}
\end{equation*}
$$

where $c_{3}$ depends only on $f_{\infty},\|A\|_{L^{\infty}(\Omega)}, \Omega,\|B\|_{L^{\infty}(\Omega)}$ and $n$. Hence the second conclusion of Lemma 4.2 holds.

Claim. For any $x_{0} \in \mathcal{N}(\psi), \quad \lim \sup _{r \rightarrow 0} M\left(x_{0}, r\right) \geq 1$.
Suppose that it were false. Then there exists $x_{0} \in \mathcal{N}(\psi), r_{1}>0$ and $c<1$ such that $M\left(x_{0}, r\right)<c$ for all $0<r \leq r_{1}$. Put $2 r_{2}=\operatorname{dist}\left(x_{0}, \partial \Omega\right)$. Similarly as Step 4 in the proof of Lemma 4.1, we have

$$
\begin{equation*}
\left(\log H\left(x_{0}, r\right)\right)^{\prime}=\frac{H^{\prime}\left(x_{0}, r\right)}{H\left(x_{0}, r\right)}=\frac{1}{r}\left(n-1+2 M\left(x_{0}, r\right)\right) . \tag{4.14}
\end{equation*}
$$

Integrating (4.14) from $r$ to $r_{1}$, we have

$$
\log \frac{H\left(x_{0}, r_{1}\right)}{H\left(x_{0}, r\right)}=\int_{r}^{r_{1}} \frac{n-1+2 M\left(x_{0}, t\right)}{t} d t \leq(n-1+2 c) \log \left(\frac{r_{1}}{r}\right)
$$

where $c$ is a constant as in Lemma 4.1. Thus we have

$$
H\left(x_{0}, r\right) \geq H\left(x_{0}, r_{1}\right)\left(\frac{r}{r_{1}}\right)^{n-1+2 c}
$$

Since $\psi \in C^{1}(\Omega ; \mathbb{C})$ and $x_{0} \in \mathcal{N}(\psi)$, by the mean value theorem,

$$
\begin{aligned}
|\psi(x)| & =\left|\left(x-x_{0}\right) \cdot \nabla \psi\left(x_{0}+\theta\left(x-x_{0}\right)\right)\right| \quad(0<\theta<1) \\
& \leq c_{r_{2}} r
\end{aligned}
$$

where $c_{r_{2}}=\max _{d(x, \partial \Omega) \geq r_{2}}|\nabla \psi|$. Therefore, we have

$$
\begin{equation*}
\frac{H\left(x_{0}, r\right)}{r^{n+1}} \leq \frac{1}{r^{n+1}} c_{r_{2}}^{2} r^{2}\left|S^{n-1}\right| r^{n-1}=c_{r_{2}}^{2}\left|S^{n-1}\right|<\infty \tag{4.15}
\end{equation*}
$$

On the other hand, since

$$
\frac{H\left(x_{0}, r\right)}{r^{n-1+2 c}} \geq H\left(x_{0}, r_{1}\right)\left(\frac{1}{r_{1}}\right)^{n-1+2 c}
$$

we have

$$
\frac{H\left(x_{0}, r\right)}{r^{n+1}} \geq H\left(x_{0}, r_{1}\right)\left(\frac{1}{r_{1}}\right)^{n-1+2 c} r^{2(c-1)}
$$

Since $c<1$, letting $r \rightarrow 0$, this leads to a contradiction to (4.15). Hence the claim holds.

Since from (4.13),

$$
\frac{N\left(x_{0}, r\right)}{M\left(x_{0}, r\right)} \leq(1+r)\left(1+\frac{r\left[r f_{\infty}+\|\boldsymbol{A}\|_{L^{\infty}(\Omega)}\right]}{M\left(x_{0}, r\right)}\right)
$$

and from (4.12),

$$
M\left(x_{0}, r\right) \geq-r\left[r f_{\infty}+\|\boldsymbol{A}\|_{L^{\infty}(\Omega)}\right] \geq-c_{2}
$$

for some constant $c_{2}>0$. We note that we can choose the constant $c_{2}$ to be
independent of $x_{0} \in \mathcal{N}(\psi)$. Thus we see that

$$
\limsup _{r \rightarrow 0} \frac{N\left(x_{0}, r\right)}{M\left(x_{0}, r\right)} \leq 1
$$

uniformly in $x_{0} \in \mathcal{N}(\psi)$. On the other hand, we have

$$
\begin{aligned}
\frac{N\left(x_{0}, r\right)}{M\left(x_{0}, r\right)} & \geq \frac{1}{1+r}-\frac{r}{1+r} \frac{r f_{\infty}+(1+r)\|\boldsymbol{A}\|_{L^{\infty}(\Omega)}}{M\left(x_{0}, r\right)} \\
& \geq \frac{1}{1+r}+\frac{r}{1+r} \frac{r f_{\infty}+(1+r)\|\boldsymbol{A}\|_{L^{\infty}(\Omega)}}{c_{2}}
\end{aligned}
$$

Thus we see that

$$
\limsup _{r \rightarrow 0} \frac{N\left(x_{0}, r\right)}{M\left(x_{0}, r\right)} \geq 1
$$

uniformly in $x_{0} \in \mathcal{N}(\psi)$. This completes the proof of Lemma 4.2.

## 5. Structure of the Singular Sets

In this section, we study the structure of the level sets, particularly the singular set of any solution of (1.1). First, we give an important theorem (cf. [10] and [22]).

Theorem 5.1. Under the conditions of Theorem 1.1, let $B_{2 R}\left(x_{0}\right) \Subset \Omega$ with $R \leq r_{0}$, where $r_{0}$ is as in Lemma 4.1. Then there exists an integer $m \geq 0$ such that we can write

$$
\begin{equation*}
\psi(x)=P_{m}\left(x-x_{0}\right)+\phi(x) \quad \text { for } \quad x \in B_{R}\left(x_{0}\right), \tag{5.1}
\end{equation*}
$$

where $P_{m}$ is non-zero complex-valued homogeneous, harmonic polynomial of degree $m$, and $\phi$ satisfies the following. For any $1<p \leq q$, there exists a constant $C>0$ depending on $m, p, q, f_{\infty},\|\boldsymbol{A}\|_{L^{\infty}(\Omega)},\|\operatorname{div} \boldsymbol{A}\|_{L^{q}\left(B_{2 R}\left(x_{0}\right)\right)}$ and $n$ such that for any $0<r \leq R / 2$,

$$
|\phi(x)| \leq C\left|x-x_{0}\right|^{m+2-n / q}
$$

and

$$
r^{2}\left\|D^{2} \phi\right\|_{L^{p}\left(B_{r}\left(x_{0}\right)\right)}+r\|\nabla \phi\|_{L^{p}\left(B_{r}\left(x_{0}\right)\right)} \leq C r^{m+(2-n / q)+n / p} .
$$

For the proof, we need a lemma.
Lemma 5.2. Under the hypotheses of Theorem 5.1, let $\psi \in W_{\text {loc }}^{1,2}(\Omega ; \mathbb{C})$ be any complex-valued weak solution of (1.1). Assume that there exists $m \geq 0$ such that

$$
\begin{equation*}
|\psi(x)| \leq C\left|x-x_{0}\right|^{m} \quad \text { for all } \quad x \in B_{R}\left(x_{0}\right) \subset \Omega . \tag{5.2}
\end{equation*}
$$

Then we have the following:
(i) For any $1<p \leq q$, there exists a constant $C_{1}$ depending only on $\|\boldsymbol{A}\|_{L^{\infty}(\Omega)}, p, q$ and $n$ such that for any $0<r \leq \min \{1, R / 2\}$,

$$
\begin{aligned}
& r^{2}\left\|D^{2} \psi\right\|_{L^{p}\left(B_{r}\left(x_{0}\right)\right)}+r\|\nabla \psi\|_{L^{p}\left(B_{r}\left(x_{0}\right)\right)} \\
\leq & C_{1} r^{m+n / p}\left\{1+r^{2} f_{\infty}+r^{2-n / q}\|\operatorname{div} \boldsymbol{A}\|_{L^{q}\left(B_{2 R}\left(x_{0}\right)\right)}\right\},
\end{aligned}
$$

and

$$
\|\Delta \psi\|_{L^{p}\left(B_{r}\left(x_{0}\right)\right)} \leq C_{1} r^{m+(n / p)-(n / q)}\left\{\|\operatorname{div} \boldsymbol{A}\|_{L^{q}\left(B_{2 r}\left(x_{0}\right)\right)}+f_{\infty}+1\right\}
$$

(ii) When $m$ in (5.2) is a non-negative integer, there exists a complex-valued homogeneous, harmonic polynomial $P_{m}$ of degree $m$ such that

$$
\psi(x)=P_{m}\left(x-x_{0}\right)+\phi(x) \quad \text { for } \quad x \in B_{R}\left(x_{0}\right)
$$

where $\phi$ satisfies

$$
|\phi(x)| \leq c_{2}\left|x-x_{0}\right|^{m+2-n / q}
$$

where $c_{2}$ depends only on $p, q, m, C_{1}, f_{\infty},\|\operatorname{div} \boldsymbol{A}\|_{L^{q}\left(B_{R}\left(x_{0}\right)\right)}$ and $n$.
Proof. The proof is essentially due to [22]. We may assume that $x_{0}=0$. By the hypothesis (5.2) and a simple computation, $\|\psi\|_{L^{p}\left(B_{r}\right)} \leq C_{1} r^{m+n / p}$. Using the Hölder inequality and (5.2),

$$
\begin{aligned}
\|(\operatorname{div} \boldsymbol{A}) \psi\|_{L^{p}\left(B_{r}\right)} & \leq\left\{\int_{B_{r}}|\operatorname{div} \boldsymbol{A}|^{q} d x\right\}^{1 / q}\left\{\int_{\mathrm{B}_{r}}|\psi|^{p q /(q-p)} d x\right\}^{(q-p) / p q} \\
& \leq C r^{m+(n / p)-(n / q)}\|\operatorname{div} \boldsymbol{A}\|_{L^{q}\left(B_{r}\right)} .
\end{aligned}
$$

From Lemma 3.1,

$$
\begin{align*}
& r^{2}\left\|D^{2} \psi\right\|_{L^{p}\left(B_{r}\right)}+r\|\nabla \psi\|_{L^{p}\left(B_{r}\right)} \\
\leq & C\left\{\|\psi\|_{L^{p}\left(B_{2 r}\right)}+r^{2}\left\|f\left(|\psi|^{2}\right) \psi-i(\operatorname{div} \boldsymbol{A}) \psi\right\|_{L^{p}\left(B_{2 r}\right)}\right\} \\
\leq & C r^{m+n / p}\left\{1+r^{2} f_{\infty}+r^{2-n / q}\|\operatorname{div} \boldsymbol{A}\|_{L^{q}\left(B_{2 r}\right)}\right\} . \tag{5.3}
\end{align*}
$$

Since $-\Delta \psi+2 i \boldsymbol{A} \cdot \nabla \psi+|\boldsymbol{A}|^{2} \psi=f\left(|\psi|^{2}\right) \psi$, it follows from (5.3) that

$$
\begin{aligned}
\|\Delta \psi\|_{L^{p}\left(B_{r}\right)} \leq & 2\|\boldsymbol{A} \cdot \nabla \psi\|_{L^{p}\left(B_{r}\right)}+\left\||\boldsymbol{A}|^{2} \psi\right\|_{L^{p}\left(B_{r}\right)} \\
& +\left\|f\left(|\psi|^{2}\right) \psi\right\|_{L^{p}\left(B_{r}\right)}+\|(\operatorname{div} \boldsymbol{A}) \psi\|_{L^{p}\left(B_{r}\right)} \\
\leq & 2\|\boldsymbol{A}\|_{L^{\infty}(\Omega)}\|\nabla \psi\|_{L^{p}\left(B_{r}\right)}+\|\boldsymbol{A}\|_{L^{\infty}(\Omega)}^{2}\|\psi\|_{L^{p}\left(B_{r}\right)} \\
& +f_{\infty}\|\psi\|_{L^{p}\left(B_{r}\right)}+\|(\operatorname{div} \boldsymbol{A}) \psi\|_{L^{p}\left(B_{r}\right)} \\
\leq & C r^{m+(n / p)-(n / q)}\left\{r^{n / q-1}\left[1+r^{2} f_{\infty}+r^{2-n / q}\|\operatorname{div} \boldsymbol{A}\|_{L^{q}\left(B_{2 r}\right)}\right]\right. \\
& \left.+r^{n / q}\left(1+f_{\infty}\right)+\|\operatorname{div} \boldsymbol{A}\|_{L^{q}\left(B_{2 r}\right)}\right\} \\
\leq & C r^{m+(n / p)-(n / q)}\left\{1+f_{\infty}+\|\operatorname{div} \boldsymbol{A}\|_{L^{q}\left(B_{2 r}\right)}\right\} .
\end{aligned}
$$

Thus (i) holds. For the proof of (ii), we apply [10, Lemma 3.3] with $d=m$, $p>n / 2$. If we put $\alpha=2-n / q(>0)$ by the hypothesis of Theorem 1.1, we can write

$$
m+\frac{n}{p}-\frac{n}{q}=m+\frac{n}{p}-2+\alpha
$$

By [10], there exists a homogeneous, harmonic polynomial $P_{m}$ of degree $m$ such that

$$
\psi(x)=P_{m}\left(x-x_{0}\right)+\phi(x)
$$

and

$$
|\phi(x)| \leq C\left|x-x_{0}\right|^{m+\alpha} \quad \text { in } \quad B_{r}\left(x_{0}\right)
$$

This completes the proof of Lemma 5.2.
Proof of Theorem 5.1. As before, we may assume that $x_{0}=0$. Since from

Theorem 1.1 we see that $\psi$ has no zero of infinite order, so there exists an integer $m \geq 0$ such that

$$
|\psi(x)| \leq C|x|^{m} \quad \text { in } \quad B_{R} \Subset \Omega,
$$

and

$$
\begin{equation*}
\lim _{x \rightarrow 0} \frac{|\psi(x)|}{|x|^{m+1}}=\infty \tag{5.4}
\end{equation*}
$$

By Lemma 5.2, there exists a complex-valued homogeneous, harmonic polynomial $P_{m}$ of degree $m$ such that

$$
\psi(x)=P_{m}(x)+\phi(x)
$$

and

$$
|\phi(x)| \leq C|x|^{m+\alpha}
$$

where $\alpha=2-n / q$.
Claim. $P_{m} \not \equiv 0$.
Suppose that it were false. Then $\psi(x)=\phi(x)$. By Lemma 5.2(i) with $m$ replaced by $m+\alpha$, we have

$$
\|\Delta \psi\|_{L^{p}\left(B_{r}\right)} \leq C_{1} r^{m+n / p-2+2 \alpha}\left\{\|\operatorname{div} \boldsymbol{A}\|_{L^{q}\left(B_{2 r}\right)}+f_{\infty}+1\right\}
$$

Again applying [10, Lemma 3.3] with $d=m, \varepsilon=2 \alpha$, there exists a complexvalued homogeneous, harmonic polynomial $\tilde{P}_{m}$ of degree $m$ such that

$$
\psi(x)=\tilde{P}_{m}(x)+\tilde{\phi}(x)
$$

and

$$
|\tilde{\phi}(x)| \leq C|x|^{m+2 \alpha}
$$

On the other hand, since $|\psi(x)|=|\phi(x)| \leq C_{1}|x|^{m+2 \alpha}, \quad \tilde{P}_{m}(x) \equiv 0$. Therefore,

$$
|\psi(x)|=|\tilde{\phi}(x)| \leq C|x|^{m+2 \alpha}
$$

Repeating this procedure we have $|\psi(x)| \leq C|x|^{m+1}$ in $B_{R}$. This implies

$$
\limsup _{x \rightarrow 0} \frac{|\psi(x)|}{|x|^{m+1}} \leq C
$$

This contradicts to (5.4).

Let $1<p \leq q$. Applying the $L^{p}$ estimate to the equation $\Delta \phi=\Delta \psi$, we have

$$
r^{2}\left\|D^{2} \phi\right\|_{L^{p}\left(B_{r}\right)}+r\|\nabla \phi\|_{L^{p}\left(B_{r}\right)} \leq C_{0}\left\{\|\phi\|_{L^{p}\left(B_{2 r}\right)}+r^{2}\|\Delta \psi\|_{L^{p}\left(B_{2 r}\right)}\right\}
$$

where $C_{0}$ depends on $p, q, n$ and

$$
\|\phi\|_{L^{p}\left(B_{2 r}\right)} \leq C_{2} r^{m+\alpha+n / p}
$$

From Lemma 5.2,

$$
r^{2}\|\Delta \psi\|_{L^{p}\left(B_{2 r}\right)} \leq C_{3} r^{m+\alpha+n / p}
$$

Thus we get

$$
r^{2}\|\Delta \phi\|_{L^{p}\left(B_{r}\right)}+r\|\nabla \phi\|_{L^{p}\left(B_{r}\right)} \leq C_{4} r^{m+\alpha+n / p}
$$

This completes the proof of Theorem 5.1.
Corollary 5.3. Under the conditions of Theorem 1.1, we have

$$
\lim _{r \rightarrow 0} M\left(x_{0}, r\right)=\lim _{r \rightarrow 0} N\left(x_{0}, r\right) \quad \text { for every } \quad x_{0} \in \Omega
$$

Proof. We may assume that $x_{0}=0$ and $\psi$ has the form (5.1). Since $P_{m}$ is homogeneous of degree $m$, we see that $x \cdot \nabla P_{m}=m P_{m}$. Also, since $P_{m}$ is harmonic,

$$
\begin{align*}
\int_{B_{r}}\left|\nabla P_{m}\right|^{2} d x & =\mathfrak{\Re} \int_{\partial B_{r}} \overline{P_{m}} \frac{\partial P_{m}}{\partial \mathbf{v}} d S_{r} \\
& =\mathfrak{R} \int_{\partial B_{r}} \overline{P_{m}}\left(\frac{x}{r} \cdot \nabla P_{m}\right) d S_{r} \\
& =\frac{m}{r} \int_{\partial B_{r}}\left|P_{m}\right|^{2} d S_{r} \\
& =m r^{2 m+n-2} \int_{S^{n-1}}\left|P_{m}(\omega)\right|^{2} d S . \tag{5.5}
\end{align*}
$$

Since we have

$$
\int_{B_{r}}|\nabla \phi|^{2} d x \leq C r^{2 m+n-2+2 \alpha}, \quad \int_{\partial B_{r}}|\phi|^{2} d S_{r} \leq C_{1} r^{2 m+n-1+2 \alpha},
$$

we see that from (5.5),

$$
\begin{aligned}
\lim _{r \rightarrow 0} N(r) & =\lim _{r \rightarrow 0} \frac{r \int_{B_{r}}\left|\nabla P_{m}\right|^{2} d x+O\left(r^{2 m+n-1+\alpha}\right)}{\int_{\partial B_{r}}\left|P_{m}\right|^{2} d S_{r}+O\left(r^{2 m+n-1+\alpha}\right)} \\
& =\lim _{r \rightarrow 0} \frac{r \int_{B_{r}}\left|\nabla P_{m}\right|^{2} d x}{\int_{\partial B_{r}}\left|P_{m}\right|^{2} d S_{r}}=m .
\end{aligned}
$$

On the other hand, since

$$
\int_{B_{r}}\left|P_{m}\right|^{2} d x=O\left(r^{2 m+n}\right) \text { and } \int_{B_{r}}|\phi|^{2} d x=O\left(r^{2 m+n+2 \alpha}\right)
$$

taking the hypotheses into consideration, we have

$$
r \int_{B_{r}}\left\{\left|\nabla_{A} \psi\right|^{2}-f\left(|\psi|^{2}\right)|\psi|^{2} d x=r \int_{B_{r}}\left|\nabla P_{m}\right|^{2} d x+O\left(r^{2 m+n-1+\alpha}\right),\right.
$$

and

$$
\int_{\partial B_{r}}|\psi|^{2} d S_{r}=\int_{\partial B_{r}}\left|P_{m}\right|^{2} d S_{r}+O\left(r^{2 m+n-1+\alpha}\right) .
$$

Therefore, we have

$$
\begin{aligned}
\lim _{r \rightarrow 0} M(r) & =\lim _{r \rightarrow 0} \frac{r \int_{B_{r}}\left\{\left|\nabla_{A} \psi\right|^{2}-f\left(|\psi|^{2}\right)|\psi|^{2}\right\} d x}{\int_{\partial B_{r}}|\psi|^{2} d S_{r}} \\
& =\lim _{r \rightarrow 0} \frac{r \int_{B_{r}}\left|\nabla P_{m}\right|^{2} d x}{\int_{\partial B_{r}}\left|P_{m}\right|^{2} d S_{r}}=m .
\end{aligned}
$$

Remark 5.4. By Corollary 5.3, we can define the vanishing order of $\psi$ at $x \in \Omega$ by

$$
\mathcal{O}_{\psi}(x)=\mathcal{O}(x)=\lim _{r \rightarrow 0} M(x, r)=\lim _{r \rightarrow 0} N(x, r) .
$$

Then from Corollary 5.3 and Lemma 4.1, we get

Corollary 5.5. Under the conditions of Theorem 1.1, the vanishing order $\mathcal{O}_{\psi}(x)$ of $\psi$ is a non-negative integer-valued function in $\Omega$ and uniformly bounded from above.

Define the level set $\mathcal{L}_{m}(\psi)$ of $\psi$ for $m=1,2, \ldots$ by

$$
\begin{equation*}
\mathcal{L}_{m}(\psi)=\left\{x \in \Omega ; \mathcal{O}_{\psi}(x)=m\right\} . \tag{5.6}
\end{equation*}
$$

Then the nodal set $\mathcal{N}(\psi)$ and the singular set $\mathcal{S}(\psi)$ have the decompositions as follows:

$$
\begin{aligned}
& \mathcal{N}(\psi)=\bigcup_{m \geq 1} \mathcal{L}_{m}(\psi) \\
& \mathcal{S}(\psi)=\bigcup_{m \geq 2} \mathcal{L}_{m}(\psi)
\end{aligned}
$$

Note that from Corollary 5.3 and Lemma 4.1, the sums are finite.
Now we introduce the blow-up of $\psi$ at any point in $\Omega$. For any $y \in \Omega$ and any $0<R<r_{0}$ with $B_{2 R}(y) \Subset \Omega$, where $r_{0}$ is as in Lemma 4.1. Define

$$
\psi_{y, r}(x)=\frac{\psi(y+r x)}{\left\{r^{-(n-1)} \int_{\partial B_{r}(y)}|\psi|^{2} d S_{r}\right\}^{1 / 2}}
$$

for $0<r<R$ and $x \in B_{2}(0)$.

From Theorem 5.1, there exists an integer $m=m(y) \geq 0$ such that we can write

$$
\psi(y+r x)=r^{m} P_{m}(x)+\phi(y+r x),
$$

where $P_{m}$ is non-zero, complex-valued homogeneous, harmonic polynomial of degree $m$ and $\phi$ satisfies

$$
|\phi(y+r x)| \leq C r^{m+\alpha}, \quad x \in B_{2}(0)
$$

where $\alpha=2-n / q$. Since we can write

$$
r^{-(n-1)} \int_{\partial B_{r}(y)}|\psi|^{2} d S_{r}=r^{2 m} \int_{S^{n-1}}\left|P_{m}(\omega)\right|^{2} d S+O\left(r^{2 m+\alpha}\right) \text { as } \quad r \rightarrow 0
$$

we have

$$
\psi_{y, r}(x)=P_{y}(x)+O\left(r^{\alpha / 2}\right)
$$

where

$$
P_{y}(x)=\frac{P_{m}(x)}{\left\{\int_{S^{n-1}}\left|P_{m}\right|^{2} d S\right\}^{1 / 2}}
$$

so we get

$$
\int_{S^{n-1}}\left|\psi_{y, r}(x)-P_{y}(x)\right|^{2} d x=O\left(r^{\alpha}\right)
$$

as $r \rightarrow 0$. We note that $\left\|P_{y}\right\|_{L^{2}\left(S^{n-1}\right)}=1$. We call this non-zero complex-valued homogeneous, harmonic polynomial $\psi_{y}:=P_{y}$ the homogeneous blow-up of $\psi$ at $y \in \Omega$. The following lemma is a easy result of homogeneity of $\psi_{y}$ which is found in [10].

Lemma 5.6. For every $m \geq 1$, the level set $\mathcal{L}_{m}\left(\psi_{y}\right)$ of the homogeneous blowup $\psi_{y}$ of $\psi$ at $y \in \Omega$ is a real linear subspace of $\mathbb{R}^{n}$, and $\operatorname{dim} \mathcal{L}_{1}\left(\psi_{y}\right) \leq n-1$ and $\operatorname{dim} \mathcal{L}_{m}\left(\psi_{y}\right) \leq n-2$ for $m \geq 2$.

From this lemma, we can define

$$
\mathcal{L}_{m}^{j}(\psi)=\left\{y \in \mathcal{L}_{m}(\psi) ; \operatorname{dim} \mathcal{L}_{m}\left(\psi_{y}\right)=j\right\}
$$

for $j=0,1, \ldots, n-1$ if $m=1$, and $j=0,1, \ldots, n-2$ if $m \geq 2$ and

$$
\begin{align*}
& \mathcal{N}^{j}(\psi)=\bigcup_{m \geq 1} \mathcal{L}_{m}^{j}(\psi) \\
& \mathcal{S}^{j}(\psi)=\bigcup_{m \geq 2} \mathcal{L}_{m}^{j}(\psi) \tag{5.7}
\end{align*}
$$

Then clearly $\mathcal{N}(\psi)=\bigcup_{j=0}^{n-1} \mathcal{N}^{j}(\psi)$ and $\mathcal{S}(\psi)=\bigcup_{j=0}^{n-2} \mathcal{S}^{j}(\psi)$. Here we shall prove that $\mathcal{S}^{j}(\psi)$ is countably $j$-rectifiable (for the notation of $j$-rectifiability, see Federer [7] or Mattila [19]).

Proposition 5.7. (i) For every $j=0,1, \ldots, n-2, \mathcal{S}^{j}(\psi)$ is on a countable union of j-dimensional $C^{1}$ manifolds.
(ii) In particular case where $j=n-2, \mathcal{S}^{n-2}(\psi)$ is on a countable union of ( $n-2$ )-dimensional $C^{1, \alpha}$ manifolds for some $\alpha \in(0,1)$.

Proof. (i) It suffices to show that for every $m \geq 2$ and $j=0,1, \ldots, n-2$, $\mathcal{L}_{m}^{j}(\psi)$ is on a countable union of $j$-dimensional $C^{1}$ manifolds. In fact, we will show that for any $x_{0} \in \mathcal{L}_{m}^{j}(\psi)$, there exists $r=r\left(x_{0}\right)>0$ such that $\mathcal{S}_{m}^{j}(\psi) \bigcap$ $B_{r}\left(x_{0}\right)$ is contained in a (single piece of) $j$-dimensional $C^{1}$ graph.

Choose $0<r \leq r_{0}$, where $r_{0}$ is as Lemma 4.1. Then we can write

$$
\psi(x)=P_{x_{0}}\left(x-x_{0}\right)+\phi(x), \quad x \in B_{r}\left(x_{0}\right) \Subset \Omega,
$$

where $P_{x_{0}}$ is a non-zero homogeneous, harmonic polynomial of degree $m, \phi$ satisfies $|\phi(x)| \leq C\left|x-x_{0}\right|^{m+\alpha}$ and $l_{x_{0}}:=\mathcal{L}_{m}\left(P_{x_{0}}\right)$ is $j$-dimensional real linear subspace of $\mathbb{R}^{n}$. As before, we may assume that $x_{0}=0$. For any sequence $\left\{x_{k}\right\}_{k=1}^{\infty} \subset \mathcal{L}_{m}^{j}(\psi)$ with $x_{k} \rightarrow x_{0}=0 \quad\left(x_{k} \neq 0\right)$, define $\lambda_{k}=\left|x_{k}\right|, y_{k}=x_{k} / \lambda_{k}$. Then $\lambda_{k} \rightarrow 0$ as $k \rightarrow \infty$. After passing to a subsequence, we may assume that $y_{k} \rightarrow y_{0}$ as $k \rightarrow \infty$. Then $\left|y_{0}\right|=1$. If we put

$$
\alpha_{k}=\left\{\lambda_{k}^{-(n-1)} \int_{\partial B_{\lambda_{k}}}|\psi|^{2} d S_{\lambda_{k}}\right\}^{1 / 2}, \quad \psi_{k}(y)=\frac{\psi\left(\lambda_{k} y\right)}{\alpha_{k}}
$$

$\boldsymbol{A}_{k}(y)=\lambda_{k} \boldsymbol{A}\left(\lambda_{k} y\right), \quad f_{k}(y)=\lambda_{k}^{2} f\left(\left|\psi\left(\lambda_{k} y\right)\right|^{2}\right)$ and $B_{k}(y)=\lambda_{k}^{2} B\left(\lambda_{k} y\right)$, then $\psi_{k}$ satisfies

$$
\begin{equation*}
-\nabla_{\boldsymbol{A}_{k}}^{2} \psi_{k}=f_{k}(y) \psi_{k} \tag{5.8}
\end{equation*}
$$

Let $\rho>0$ be fixed. Then we see that as $k \rightarrow \infty$,

$$
\left\|f_{k}\right\|_{L^{\infty}\left(B_{2 \rho}\right)}=\lambda_{k}^{2}\|f\|_{L^{\infty}\left(B_{2 \lambda_{k} \rho}\right)} \rightarrow 0
$$

$$
\begin{aligned}
& \left\|\boldsymbol{A}_{k}\right\|_{L^{\infty}\left(B_{2 \rho}\right)}=\lambda_{k}\|\boldsymbol{A}\|_{L^{\infty}\left(B_{2 \lambda_{k} \rho}\right)} \rightarrow 0, \\
& \left\|B_{k}\right\|_{L^{\infty}\left(B_{2 \rho}\right)}=\lambda_{k}^{2}\|B\|_{L^{\infty}\left(B_{2 \lambda_{k \rho}}\right)} \rightarrow 0, \\
& \left\|\operatorname{div} \boldsymbol{A}_{k}\right\|_{L^{q}\left(B_{2 \rho}\right)}=\lambda_{k}^{2}\|\operatorname{div} \boldsymbol{A}\|_{L^{q}\left(B_{2 \lambda_{k} \rho}\right)} \rightarrow 0 .
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\left\|\psi_{k}\right\|_{L^{2}\left(B_{\rho}\right)}^{2} & =\frac{1}{\alpha_{k}^{2}}\|\psi\|_{L^{2}\left(B_{\lambda_{k} \rho}\right)}^{2} \\
& =\frac{\left(\lambda_{k} \rho\right)^{m+n}\left[\int_{B_{1}}\left|P_{x_{0}}\right|^{2} d x+O\left(\lambda_{k}^{\alpha}\right)\right]}{\lambda_{k}^{m} \int_{\partial B_{1}}\left|P_{x_{0}}\right|^{2} d S+O\left(\lambda_{k}^{m+\alpha}\right)} \leq C .
\end{aligned}
$$

Here the constant $C$ can be chosen uniformly in $\lambda_{k}$. Therefore, applying Lemma 3.1 with $p=2$ to (5.8), there exists a constant $C$ independent of $k$ such that

$$
\begin{aligned}
& \rho^{2}\left\|D^{2} \psi_{k}\right\|_{L^{2}\left(B_{\rho}\right)}+\rho\left\|\nabla \psi_{k}\right\|_{L^{2}\left(B_{\rho}\right)} \\
\leq & C\left\{\left\|\psi_{k}\right\|_{L^{2}\left(B_{2 \rho}\right)}+\rho^{2}\left\|f_{k}(y) \psi_{k}-i\left(\operatorname{div} \boldsymbol{A}_{k}\right) \psi_{k}\right\|_{L^{2}\left(B_{2 \rho}\right)} \leq C .\right.
\end{aligned}
$$

Thus since $\left\{\psi_{k}\right\}$ is uniformly bounded in $W^{2,2}\left(B_{\rho}\right)$ for any fixed $\rho>0$, passing to a subsequence (still denoted by $\left.\left\{\psi_{k}\right\}\right), \psi_{k} \rightarrow \psi^{0}$ weakly in $W_{\text {loc }}^{2,2}\left(\mathbb{R}^{n}\right)$ and strongly in $W_{\text {loc }}^{1,2}\left(\mathbb{R}^{n}\right)$ as $k \rightarrow \infty$. We see that $\psi^{0}$ is harmonic and $\psi^{0}=\psi_{x_{0}}$ $=\psi_{0}$. We note that $y_{k}=x_{k} / \lambda_{k} \in \mathcal{S}_{m}^{j}\left(\psi_{k}\right)$.

Claim. $\mathcal{O}_{\psi_{0}}\left(y_{0}\right) \geq \lim \sup _{k \rightarrow \infty} \mathcal{O}_{\psi_{k}}\left(y_{k}\right)=m$.
In fact, choose $t>0$ small enough, then as $k \rightarrow \infty$,

$$
\begin{aligned}
M_{\psi_{k}}\left(y_{k}, t\right) & =\frac{r I_{\psi_{k}}\left(y_{k}, t\right)}{H_{\Psi_{k}}\left(y_{k}, t\right)} \\
& =\frac{r \int_{B_{t}\left(y_{k}\right)}\left\{\left|\nabla_{A_{k}} \psi_{k}\right|^{2}-f_{k}\left|\psi_{k}\right|^{2}\right\} d x}{\int_{\partial B_{t}\left(y_{k}\right)}\left|\psi_{k}\right|^{2} d S_{t}}
\end{aligned}
$$

$$
\rightarrow \frac{r \int_{B_{t}\left(y_{0}\right)}\left|\nabla \psi_{0}\right|^{2} d x}{\int_{\partial B_{t}\left(y_{0}\right)}\left|\psi_{0}\right|^{2} d S_{t}}=M_{\psi_{0}}\left(y_{0}, t\right)
$$

From the proof of Lemma 4.1, there exists $r_{0}$, $\lambda$ independent of $k$ such that for any $0<r \leq t \leq r_{0}$ and $k$,

$$
e^{\lambda r} M_{\psi_{k}}\left(y_{k}, r\right) \leq e^{\lambda t} \max \left\{1, M_{\psi_{k}}\left(y_{k}, t\right)\right\} .
$$

If we fix $k, t$ and let $r \rightarrow 0$, we have

$$
m=\mathcal{O}_{\psi_{k}}\left(y_{k}\right)=\lim _{r \rightarrow 0} e^{\lambda r} M_{\Psi_{k}}\left(y_{k}, r\right) \leq e^{\lambda t} \max \left\{1, M_{\Psi_{k}}\left(y_{k}, t\right)\right\}
$$

Next, for fixed $t$, letting $k \rightarrow \infty$, we have

$$
m \leq e^{\lambda t} \max \left\{1, M_{\psi_{0}}\left(y_{0}, t\right)\right\}
$$

Finally, letting $t \rightarrow 0$, we have

$$
m \leq \max \left\{1, \mathcal{O}_{\psi_{0}}\left(y_{0}\right)\right\}
$$

Since $m \geq 2$, we see that $\mathcal{O}_{\psi_{0}}\left(y_{0}\right) \leq m$. Thus the claim holds.
From the uniqueness of limit, for original sequence $\left\{y_{k}\right\}$,

$$
m=\limsup _{k \rightarrow \infty} \mathcal{O}_{\psi_{k}}\left(y_{k}\right) \leq \mathcal{O}_{\psi_{0}}\left(y_{0}\right)
$$

On the other hand, since $\psi_{0}$ is non-zero homogeneous polynomial of degree $m$, $\mathcal{O}_{\psi_{0}}\left(y_{0}\right) \leq m$. Thus $\mathcal{O}_{\psi_{0}}\left(y_{0}\right)=m$ and so $y_{0} \in l_{x_{0}}=l_{0}$. From this, we have

$$
\begin{equation*}
\text { Angle }\left\langle\overline{x_{k} x_{0}}, l_{x_{0}}\right\rangle \rightarrow 0 \quad \text { as } \quad k \rightarrow \infty \tag{5.9}
\end{equation*}
$$

where Angle $\left\langle\overline{x_{k} x_{0}}, l_{x_{0}}\right\rangle$ denotes the angle of the vector $\overline{x_{k} x_{0}}$ and the subspace $l_{x_{0}}$. Therefore, for any $x_{0} \in \mathcal{S}_{m}^{j}(\psi)$ and any $\varepsilon \in(0,1)$, there exists $r=r\left(x_{0}, \varepsilon\right)$ $>0$ such that

$$
\begin{equation*}
\mathcal{L}_{m}^{j}(\psi) \cap B_{r}\left(x_{0}\right) \subset B_{r}\left(x_{0}\right) \cap C_{\varepsilon}\left(l_{x_{0}}\right) \tag{5.10}
\end{equation*}
$$

where

$$
C_{\varepsilon}\left(l_{x_{0}}\right)=\left\{z \in \mathbb{R}^{n} ; \operatorname{dist}\left(z-x_{0}, l_{x_{0}}\right) \leq \varepsilon\left|z-x_{0}\right|\right\} .
$$

That is to say,

$$
\mathcal{L}_{m}^{j}(\psi) \cap B_{r}\left(x_{0}\right) \cap C_{\varepsilon}\left(l_{x_{0}}\right)^{C}=\varnothing
$$

Let $P_{k}, P_{0}$ be blow-up polynomials of $\psi$ at $x_{k}, x_{0}$, respectively. Then from [11, Lemma 4.1], $P_{k} \rightarrow P_{0}$ uniformly in $C^{m}\left(B_{1}(0)\right)$. Since

$$
\begin{aligned}
l_{x_{k}}=\mathcal{L}_{m}^{j}\left(P_{x_{k}}\right)= & \left\{x \in \mathbb{R}^{n} ; D^{\alpha} P_{x_{k}}(x)=0 \text { for } 0 \leq|\alpha| \leq m-1\right. \\
& \text { there exists } \left.\beta \text { with }|\beta|=m \text { such that } D^{\beta} P_{x_{k}}(x) \neq 0\right\}
\end{aligned}
$$

$l_{x_{k}} \rightarrow l_{x_{0}}$ as subspaces in $\mathbb{R}^{n}$, i.e., if we denote the orthogonal projections from $\mathbb{R}^{n}$ to $l_{x_{k}}$ and $l_{x_{0}}$ by $\Pi_{l_{x_{k}}}$ and $\Pi_{l_{x_{0}}}$, respectively, then $d\left(l_{x_{k}}, l_{x_{0}}\right):=\| \Pi_{l_{x_{k}}}$ $\Pi_{l_{x_{0}}} \| \rightarrow 0$ as $k \rightarrow \infty$, where $\|\cdot\|$ is the usual operator norm for linear maps. Here we note that $r$ in (5.10) can be chosen uniformly with respect to $x \in \mathcal{L}_{m}^{j}(\psi)$ in a neighbourhood of $x_{0}$ (cf. [11, Step 2 in the proof of Theorem 5.1]). That is to say, for any $x_{0} \in \mathcal{L}_{m}^{j}(\psi)$ and any $\varepsilon \in(0,1)$, there exists $r=r\left(\varepsilon, x_{0}\right)>0$ such that

$$
\mathcal{L}_{m}^{j}(\psi) \cap B_{r}(x) \subset B_{r}(x) \cap C_{\varepsilon}\left(l_{x}\right) \text { for all } x \in \mathcal{L}_{m}^{j}(\psi) \cap B_{r}\left(x_{0}\right)
$$

Therefore, by [19, Lemma 15.13], if $\varepsilon$ is small enough, $\mathcal{L}_{m}^{j}(\psi) \cap B_{r}(x)$ is contained in a $j$-dimensional Lipschitz graph, i.e., there exists a Lipschitz function $f: x_{0}+l_{x_{0}}$
$\rightarrow \mathbb{R}^{n}$ such that

$$
\begin{equation*}
f\left(\left(x_{0}+l_{x_{0}}\right) \cap B_{r}\left(x_{0}\right)\right) \supset \mathcal{L}_{m}^{j}(\psi) \cap B_{r}\left(x_{0}\right) . \tag{5.11}
\end{equation*}
$$

We may assume that $x_{0}=0$ and choose the coordinate system $x=\left(x^{\prime}, x^{\prime \prime}\right)$ such that

$$
\mathbb{R}^{n}=\left(x_{0}+l_{x_{0}}\right) \oplus \mathbb{R}^{n-j}, \quad x^{\prime} \in x_{0}+l_{x_{0}}, \quad x^{\prime \prime} \in \mathbb{R}^{n-j}
$$

Then by (5.9), we see that $x^{\prime \prime}=f\left(x^{\prime}\right)$ is differentiable at $x_{0}^{\prime}$ and $D f\left(x_{0}^{\prime}\right)=0$, where $x_{0}=\left(x_{0}^{\prime}, x_{0}^{\prime \prime}\right)$. Since $x_{0}$ is arbitrary in $B_{r}\left(x_{0}\right) \cap \mathcal{L}_{m}^{j}(u), f\left(x^{\prime}\right)$ is differentiable near $x_{0}^{\prime}$. Again using (5.9), when $x_{k}=\left(x_{k}^{\prime}, x_{k}^{\prime \prime}\right) \rightarrow x_{0}=\left(x_{0}^{\prime}, x_{0}^{\prime \prime}\right)$ in $B_{r}\left(x_{0}\right) \cap \mathcal{L}_{m}^{j}(u), x_{k}+l_{x_{k}} \rightarrow l_{x_{0}}$ as linear subspaces in $\mathbb{R}^{n}$, so we have $\operatorname{Df}\left(x_{k}^{\prime}\right)$ $\rightarrow D f\left(x_{0}^{\prime}\right)=0$ as $k \rightarrow \infty$. Thus $f$ is $C^{1}$ function near $x_{0}^{\prime}$.

Finally, we show that for $m \geq 2, \mathcal{L}_{m}^{n-2}(\psi)$ is on a countable union of $(n-2)$ dimensional $C^{1, \alpha}$ manifolds for some $0<\alpha<1$.

Let $0=x_{0} \in \mathcal{L}_{m}^{n-2}(\psi)$ and $\psi_{x_{0}}=P_{x_{0}}$ be the homogeneous, harmonic blowup of $\psi$ at $x_{0}$ of degree $m$. Then $\operatorname{dim} \mathcal{L}_{m}\left(P_{x_{0}}\right)=n-2$ and $\psi=\psi_{x_{0}}+\phi$, $|\phi(x)| \leq C|x|^{m+\alpha}$ with $\alpha=2-n / q$. Thus we can write $\mathbb{R}^{n}=\mathbb{R}^{2}+\mathcal{L}\left(P_{x_{0}}\right)$ and $\psi_{x_{0}}$ is a harmonic polynomial of degree $m$ on $\mathbb{R}^{2}$. Using the polar coordinate $(r, \theta)$ in $\mathbb{R}^{2}$, we can write

$$
\psi_{x_{0}}=r^{m} \cos m \theta
$$

for some rotation. Denote $x=\left(x^{1}, x^{2}\right)$, where $x^{1} \in \mathbb{R}^{2}, x^{2} \in \mathbb{R}^{n-2}$. For any $x \in$ $\mathcal{S}_{m}^{n-2}(\psi)$ close to $x_{0}$, we have $\nabla \psi(x)=0$ as $m \geq 2$. Thus $\nabla_{x^{1}} \psi_{x_{0}}=-\nabla_{x} \psi(x)$ and $x^{1} \cdot \nabla_{x^{1}} \psi_{x_{0}}\left(x^{1}\right)=m \psi_{x_{0}}\left(x^{1}\right)$ as $\psi_{x_{0}}$ is homogeneous of degree $m$. Here we need the following lemma due to [12].

Lemma 5.8. There exists a constant $C>0$ such that

$$
|\nabla \phi| \leq C\left|x-x_{0}\right|^{m-1+\alpha} \quad \text { for } \quad x \in B_{2 R}\left(x_{0}\right) \Subset \Omega,
$$

where $\alpha=2-n / q$.
Proof. We may assume that $x_{0}=0$. By Lemma 5.2, for any $1<p \leq q$,

$$
\begin{equation*}
\|\Delta \psi\|_{L^{p}\left(B_{r}\right)} \leq C r^{m-2+\alpha+n / p} \tag{5.12}
\end{equation*}
$$

where $\alpha=2-n / q$. Let $\Gamma$ be the fundamental solution of the Laplace operator $\Delta$.

Define

$$
\begin{aligned}
& \tilde{\phi}(x)=\int_{|\xi|<R} \Gamma(x-\xi) \Delta \psi(\xi) d \xi \\
& \Gamma_{k}(x, \xi)=\sum_{|\beta|=k} D^{\beta} \Gamma(-\xi) \frac{x^{\beta}}{\beta!}, \\
& \tilde{P}^{m}(x)=\int_{|\xi|<R} \sum_{k=0}^{m} \Gamma_{k}(x, \xi) \Delta \psi(\xi) d \xi .
\end{aligned}
$$

Then by the construction, we see that

$$
\phi(x)=\tilde{\phi}(x)-\tilde{P}^{m}(x)=\int_{|\xi|<R}\left[\Gamma(x-\xi)-\sum_{k=0}^{m} \Gamma_{k}(x, \xi)\right] \Delta \psi(\xi) d \xi
$$

Here we note that

$$
\begin{equation*}
\left|D^{\beta} \Gamma(x)\right| \leq \frac{C}{|x|^{n+|\beta|-2}} \tag{5.13}
\end{equation*}
$$

where $C=C(n,|\beta|)$. We shall show that $|\nabla \phi(x)| \leq C|x|^{m-1+\alpha}$ in $B_{R}$. We simply write $f=\Delta \psi$. We put

$$
\begin{aligned}
\partial_{j} \phi(x) & =\int_{|\xi|<R}\left[\partial_{x_{j}} \Gamma(x-\xi)-\sum_{k=0}^{m} \partial_{x_{j}} \Gamma_{k}(x, \xi)\right] f(\xi) d \xi \\
& =\sum_{l=1}^{3} J_{l}(x)
\end{aligned}
$$

where

$$
\begin{aligned}
& J_{1}(x)=\int_{|\xi|<2|x|} \partial_{x_{j}} \Gamma(x-\xi) f(\xi) d \xi, \\
& J_{2}(x)=-\int_{|\xi|<2|x|} \sum_{k=1}^{m} \partial_{x_{j}} \Gamma_{k}(x, \xi) f(\xi) d \xi, \\
& J_{3}(x)=\int_{2|x|<|\xi|<R}\left[\partial_{x_{j}} \Gamma(x-\xi)-\sum_{k=0}^{m} \partial_{x_{j}} \Gamma_{k}(x, \xi)\right] f(\xi) d \xi .
\end{aligned}
$$

Put $p^{\prime}=p /(p-1)$. Then by (5.13) and the Hölder inequality, we have

$$
\begin{aligned}
\left|J_{1}(x)\right| & \leq \int_{|\xi| \leq 2|x||x-\xi|^{n-1}} \frac{|f(\xi)|}{} \\
& \leq\left\{\int_{|\xi| \leq 2|x||x-\xi|^{(n-1) p^{\prime}}} d \xi\right\}^{1 / p^{\prime}}\left\{\int_{|\xi| \leq 2|x|}|f(\xi)|^{p} d \xi\right\}^{1 / p} .
\end{aligned}
$$

If $\zeta=x-\xi$, then $|\zeta| \leq 3|x|$ in the integration domain. By (5.12), we have

$$
J_{1}(x) \leq C|x|^{m-1+\alpha}
$$

Next, by (5.13) and the Hölder inequality,

$$
\begin{aligned}
\left|J_{2}(x)\right| \leq & \sum_{k=1}^{m}|x|^{k-1} \int_{|\xi| \leq 2|x||\xi|^{n+k-2}} \frac{|f(\xi)|}{} d \xi \\
= & \sum_{k=1}^{m}|x|^{k-1} \sum_{i=0}^{\infty} \int_{|x| / 2^{i}<|\xi| \leq 2|x| 2^{i}} \frac{|f(\xi)|}{|\xi|^{n+k-2}} d \xi \\
\leq & \sum_{k=1}^{m}|x|^{k-1} \sum_{i=0}^{\infty}\left\{\int_{|x| / 2^{i}<|\xi| \leq 2|x| / 2^{i}} \frac{d \xi}{|\xi|^{(n+k-2) p^{\prime}}}\right\}^{1 / p^{\prime}} \\
& \times\left\{\int_{|\xi| \leq 2|x| / 2^{i}}|f(\xi)|^{p} d \xi\right\}^{1 / p} \\
\leq & \sum_{k=1}^{m}|x|^{k-1} \sum_{i=0}^{\infty}\left(\frac{|x|}{2^{i}}\right)^{-(n+k-2)+n / p^{\prime}}\left(\frac{|x|}{2^{i}}\right)^{m-2+\alpha-n / p} \\
= & |x|^{m-1+\alpha} \sum_{k=1}^{m} \sum_{i=0}^{\infty}\left(\frac{1}{2^{m-k+\alpha}}\right)^{i} \leq C|x|^{m-1+\alpha} .
\end{aligned}
$$

Finally, we estimate $J_{3}(x)$. Since

$$
\begin{aligned}
& \left|\partial_{x_{j}}\left[\Gamma(x-\xi)-\sum_{k=0}^{m} \Gamma_{k}(x, \xi)\right]\right| \\
= & \left|\partial_{x_{j}} \sum_{|\beta|=m+1} D^{\beta} \Gamma(\xi-\theta x) \frac{x^{\beta}}{\beta!}\right|
\end{aligned}
$$

$$
\leq C\left\{\frac{|x|^{m+1}}{|\xi-\theta x|^{n+m}}+\frac{|x|^{m}}{|\xi-\theta x|^{n+m-1}}\right\} \text { for some } 0<\theta<1
$$

Since $|x| \leq|\xi| / 2,|\xi-\theta x| \geq|\xi| / 2$. Therefore, we have

$$
\begin{aligned}
\left|J_{3}(x)\right| \leq & C\left\{|x|^{m+1} \int_{2|x|<|\xi|<R} \frac{|f(\xi)|}{|\xi|^{n+m}} d \xi\right. \\
& \left.+|x|^{m} \int_{2|x|<|\xi|<R} \frac{|f(\xi)|}{|\xi|^{n+m-1}} d \xi\right\} \\
= & C\left\{|x|^{m+1} J_{3,1}(x)+|x|^{m} J_{3,2}(x)\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
& J_{3,1}(x)=\int_{2|x|<|\xi|<R} \frac{|f(\xi)|}{|\xi|^{n+m}} d \xi \\
& J_{3,2}(x)=\int_{2|x|<|\xi|<R} \frac{|f(\xi)|}{|\xi|^{n+m-1}} d \xi
\end{aligned}
$$

As the similar arguments as the estimate of $J_{2}(x)$, we can see that

$$
J_{3,1}(x) \leq C|x|^{-2+\alpha} \text { and } J_{3,2}(x) \leq C|x|^{-1+\alpha}
$$

This completes the proof of Lemma 5.8.
Since $\psi_{x_{0}}$ is a non-zero homogeneous polynomial in $\mathbb{R}^{2}$ and $\Delta_{x^{1}} \psi_{x_{0}}=0$ in $\mathbb{R}^{2}$, it is easily seen that

$$
\left|\nabla_{x^{1}} \psi_{x_{0}}\left(x^{1}\right)\right| \leq C|x|^{m-1}
$$

for some $C>0$ (cf. [12, Lemma 5.2]). From this and Lemma 5.8, we have

$$
\left|x^{1}\right|^{m-1} \leq \frac{1}{C}\left|\nabla_{x^{1}} \psi_{x_{0}}\left(x^{1}\right)\right|=\frac{1}{C}|\nabla \phi(x)| \leq C_{1}|x|^{m-1+\alpha}
$$

for some $C_{1}>0$, i.e., $\left|x^{1}\right| \leq C_{2}|x|^{1+\alpha /(m-1)}$. This implies that $f$ in (5.11) is in fact a $C^{1, \alpha /(m-1)}$ class function. Since $m=m\left(x_{0}\right)$ is bounded from above, this completes the proof of Proposition 5.7.

Corollary 5.9. There exists the following decomposition of the singular set of $\psi$,

$$
\mathcal{S}(\psi)=\mathcal{B}(\psi) \cup \mathcal{G}(\psi)
$$

with $\mathcal{H}^{n-2}(\mathcal{B}(\psi))=0$, where $\mathcal{H}^{n-2}$ is the $(n-2)$-dimensional Hausdorff measure, and $\mathcal{G}(\psi)$ is on a countable union of $C^{1, \alpha}$ class $(n-2)$-dimensional manifolds for some $\alpha \in(0,1)$. Moreover, for any $x \in \mathcal{G}(\psi)$, the $(n-2)$-dimensional density function is equal to one, i.e.,

$$
\Theta^{n-2}(\mathcal{G}(\psi), x)=1
$$

Proof. We put

$$
\begin{aligned}
& \mathcal{B}(\psi)=\bigcup_{j=0}^{n-3} \mathcal{S}^{j}(\psi) \\
& \mathcal{G}(\psi)=\mathcal{S}^{n-2}(\psi)
\end{aligned}
$$

Taking Proposition 5.7 into consideration and applying [7, Theorem 3.2.19], it suffices to check that $\mathcal{S}^{n-2}(\psi)$ is $\mathcal{H}^{n-2}$ measurable. Since $\mathcal{O}_{\psi}(x)$ is an upper semi-continuous in $\Omega$,

$$
\mathcal{S}^{n-2}(\psi)=\left\{x \in \Omega ; \mathcal{O}_{\psi}(x) \geq n-2\right\}
$$

is a closed set, so $\mathcal{S}^{n-2}(\psi)=\mathcal{G}(\psi)$ is a Borel set. Since the Hausdorff measure is a Borel measure, $\mathcal{G}(\psi)$ is $\mathcal{H}^{n-2}$ measurable.

## 6. Structure of the Nodal Set

In this section, we study the nodal set of non-trivial solution $\psi$ of (1.1). We already saw that

$$
\begin{aligned}
\mathcal{N}(\psi) & =\left\{x \in \Omega ; \mathcal{O}_{\psi}(x) \geq 1\right\} \\
& =\bigcup_{m \geq 1} \mathcal{L}_{m}(\psi)
\end{aligned}
$$

Clearly, $\mathcal{L}_{1}(\psi)$ is locally a ( $n-1$ ) -dimensional submanifold if $\psi$ is $C^{1}$ function. As
before, we define

$$
\mathcal{L}_{m}^{j}(\psi)=\left\{y \in \mathcal{L}_{m}(\psi) ; \operatorname{dim} \mathcal{L}_{m}\left(\psi_{y}\right)=j\right\} \text { for } j=0,1, \ldots, n-1
$$

and

$$
\mathcal{N}^{j}(\psi)=\bigcup_{m \geq 1} \mathcal{L}_{m}^{j}(\psi)
$$

Then we see that

$$
\mathcal{N}(\psi)=\bigcup_{j=0}^{n-1} \mathcal{N}^{j}(\psi)
$$

Then we have the following:
Theorem 6.1. (i) $\mathcal{N}^{j}(\psi)$ is on a countable union of $j$-dimensional $C^{1}$ manifolds for $j=0,1, \ldots, n-1$.
(ii) In particular, $\mathcal{N}^{n-1}(\psi)$ is on a countable union of $(n-1)$-dimensional $C^{1, \alpha}$ manifolds with $\alpha=2-n / q$.

Proof. (i) follows from the same arguments as the proof of Proposition 5.7. For the proof of (ii), let $x_{0}=0 \in \mathcal{L}_{1}^{n-1}(\psi)$ and $P$ be the homogeneous blow-up of $\psi$ at $x_{0}=0$. Then $\operatorname{dim} \mathcal{L}_{1}(P)=n-1$. If we write $\mathbb{R}^{n}=\mathbb{R} \times \mathcal{L}_{1}(P), P$ is a monomial of degree one in $\mathbb{R}$. We may assume that $P=x_{1}$. Thus we can write $\psi(x)=x_{1}+$ $\phi(x)$ in $B_{R} \Subset \Omega$ and $|\psi(x)| \leq C|x|^{1+\alpha}$ in $B_{R}$. If we write $x=\left(x_{1}, x^{\prime}\right) \in \mathbb{R} \times$ $\mathcal{L}_{1}(P)$, we see that $\left|x_{1}\right| \leq C|x|^{1+\alpha}$. As in the preceding section, it follows that $\mathcal{L}_{1}^{n-1}(\psi)$ is on a $(n-1)$-dimensional $C^{1, \alpha}$ graph.

We have the decomposition of the nodal set as in Section 5.
Corollary 6.2. (i) We can decompose the nodal set of $\psi$ as follows:

$$
\mathcal{N}(\psi)=\mathcal{D}(\psi) \cup \mathcal{E}(\psi)
$$

where $\mathcal{H}^{n-1}(\mathcal{D}(\psi))=0$ and $\mathcal{E}(\psi)$ is on a union of countable $C^{1, \alpha}$ manifolds.
(ii) For any $x \in \mathcal{E}(\psi)$,

$$
\Theta^{n-1}(\mathcal{E}(\psi), x)=1
$$

where $\Theta^{n-1}(\mathcal{E}(\psi), x)$ is the $(n-1)$-dimensional density function.

## References

[1] J. Aramaki, On an elliptic problem with general nonlinearity associated with superheating field in the theory of superconductivity, Int. J. Pure Appl. Math. 28(1) (2006), 125-148.
[2] J. Aramaki, On an elliptic model with general nonlinearity associated with superconductivity, Int. J. Differ. Equ. Appl. 10(4) (2006), 449-466.
[3] J. Aramaki, A remark on a semi-linear elliptic problem with the de Gennes boundary condition associated with superconductivity, Int. J. Pure Appl. Math. 50(1) (2008), 97-110.
[4] J. Aramaki, A. Nurmuhammad and S. Tomioka, A note on a semi-linear elliptic problem with the de Gennes boundary condition associated with superconductivity, Far East J. Math. Sci. (FJMS) 32(2) (2009), 153-167.
[5] N. Aronszajn, A unique continuation theorem for solutions of elliptic partial differential equations or inequality of second order, J. Math. Pures Appl. 36(9) (1957), 235-249.
[6] C. M. Elliot, H. Matano and Q. Tang, Zeros of a complex Ginzburg-Landau order parameter with applications to superconductivity, European J. Appl. Math. 5 (1994), 431-448.
[7] H. Federer, Geometric Measure Theory, Springer, Berlin, 1969.
[8] N. Garofalo and F.-H. Lin, Monotonicity properties of variational integrals, $A_{p}$ weights and unique continuation, Indiana Univ. Math. J. 35(2) (1986), 245-268.
[9] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, New York, 1983.
[10] Q. Han, Singular sets of solutions to elliptic equations, Indiana Univ. Math. J. 43 (1994), 983-1002.
[11] Q. Han, Schauder estimates for elliptic operators with applications to nodal sets, J. Geom. Anal. 10(3) (2000), 455-480.
[12] Q. Han, R. Hardt and F.-G. Lin, Geometric measure of singular sets of elliptic equations, Comm. Pure Appl. Math. 51 (1998), 1425-1443.
[13] B. Helffer, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and M. Owen, Nodal sets for ground states of Schrödinger operators with zero magnetic field in non-simply connected domains, Comm. Math. Phys. 202 (1999), 629-649.
[14] B. Helffer, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and M. Owen, Nodal sets, multiplicity and superconductivity in non simply connected domains, Lecture Notes in Physics, Vol. 62, J. Berger and K. Rubinstein, eds., 2000, pp. 62-86.
[15] B. Helffer and A. Mohamed, Semiclassical analysis for the ground state energy of a Schrödinger operator with magnetic wells, J. Funct. Anal. 138 (1996), 40-81.
[16] B. Helffer and A. Morame, Magnetic bottles in connection with superconductivity, J. Funct. Anal. 185 (2001), 604-680.
[17] K. Lu and X.-B. Pan, Estimates of upper critical field for the Ginzburg-Landau equations of superconductivity, Physica D 127 (1999), 73-104.
[18] K. Lu and X.-B. Pan, Surface nucleation of superconductivity in 3-dimension, J. Differential Equations 168 (2000), 386-452.
[19] P. Mattila, Geometry of Sets and Measures in Euclidean Spaces, Cambridge Univ. Press, 1995.
[20] X.-B. Pan, Landau-de Gennes model of liquid crystals and critical wave number, Comm. Math. Phys. 239 (2003), 343-382.
[21] X.-B. Pan, Surface superconductivity in 3-dimensions, Trans. Amer. Math. Soc. 356 (2004), 3899-3937.
[22] X.-B. Pan, Nodal sets of solutions of equations involving magnetic Schrödinger operator in three dimension, J. Math. Phys. 48 (2007), 053521.
[23] X.-B. Pan and K. H. Kwek, On a problem related to vortex nucleation of superconductivity, J. Differential Equations 182 (2002), 141-168.
[24] E. Sandier and S. Serfaty, Vortices in the Magnetic Ginzburg-Landau Model, Birkhäuser, Boston, Basel, Berlin, 2007.

