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GROWTH OF WATER WAVES ON VISCOUS LIQUIDS

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Abstract

An exact analytical solution to the Navier-Stokes solutions is constructed for an unsteady viscous incompressible flow governing the motion of a monochromatic surface wave on deep water which satisfies surface boundary condition on $z = \eta$, where η is an unknown *a priori* surface. It is shown that the wave-induced motion on a viscous liquid considered here falls in the class of irrotational flow in which the viscous effects are important.

1. Introduction

Lamb [7, Section 349] considers the effect of viscosity on water waves. In that article, he constructs an analytical solution for a viscous flow over a monochromatic surface wave

$$\eta = a\cos(kx \pm \sigma t), \qquad ak \ll 1 \tag{1.1}$$

where a is the wave amplitude, k is the wavenumber and σ is the angular frequency.

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By neglecting the inertia terms in the Navier-Stokes equations and assuming the elevation of the free surface is given by

$$\eta = -\frac{k}{n}(A - iC)e^{ikx+nt}, \tag{1.2}$$

where A and C are constants, and n is an integer he shows that since the ratio

$$\frac{C}{A} = \mp\frac{2\nu k^2}{\sigma} \ll 1 \tag{1.3}$$

the motion is *approximately* irrotational with a velocity potential

$$\varphi = Ae^{-2\nu k^2 t + kz + i(kx \pm \sigma t)}, \tag{1.4}$$

where ν is the kinematic viscosity of the fluid.

In the footnote of the same article (p. 627) he shows the corresponding vorticity is given by

$$\omega = \mp 2\sigma k a e^{-2\nu k^2 t + \beta z}\cos\{kx \pm (\sigma t + \beta z)\}, \tag{1.5}$$

where $\beta = (\sigma/2\nu)^{1/2}$ and $a = kA/\sigma$. Lamb argues that the vorticity diminishes rapidly from the surface downwards and 'owing to the oscillatory character of the motion, the sign of the vorticity which is being diffused inwards from the surface is continually being reversed, so that beyond a stratum of thickness comparable with $2\pi/\beta$ the effect is insensible, ...'.

The presence of a non-zero vorticity implies that the motion is *rotational*. Thus, the question arises, what is the source of this vorticity? To answer this question it is necessary to examine the manner in which the viscosity of a viscous liquid affects the potential flow away from the boundary-layer [12].

In this paper, we will construct an exact analytical solution to the viscous, unsteady, two-dimensional, incompressible Navier-Stokes equations governing the wave-induced motion over a monochromatic surface water wave. The solution obtained here is another example of viscous potential flow as was first pointed out by Joseph and Wang [5].

The main purpose of the present study is to construct a complimentary model to that constructed originally by Miles [8] and use the above mentioned solution to determine the growth of monochromatic surface wave. In a pioneering paper, Miles constructed a model for an inviscid laminar parallel shear flow with a velocity profile $U(z)$ over a two-dimensional surface wave with wavenumber k and wave speed c

$$z = a\cos k(x - ct) \equiv \eta(x,t). \tag{1.6}$$

In this model, he neglects non-linear effects (of second order in ka) and, in addition, first order (in ka) perturbations in the Turbulent Reynolds stresses. With these idealizations imposed, he derives an average momentum flux, from shear flow to the surface wave

$$F = \pi\rho(-U''\overline{\mathscr{W}^2}/kU')_c. \tag{1.7}$$

In (1.7), ρ is density of the air; primes indicate differentiation with respect to z; the overbar is an average taken over an integral number of wavelengths; and the subscript c implies evaluation at the *critical layer*, $z = z_c$, where

$$U(z_c) = c. \tag{1.8}$$

Miles determines the z-component of the wave-induced velocity, $\mathscr{W}(x,z)$ from the following boundary-value problem

$$L\mathscr{W} = (U - c)\nabla^2\mathscr{W} - U''\mathscr{W} = 0, \tag{1.9}$$

$$\mathscr{W}(x,0) = (U - c)(\partial\eta/\partial x), \qquad \mathscr{W}(x,\infty) = 0. \tag{1.10}$$

He argued that the corresponding average energy flux Fc (being of second order in the amplitude) implies an exponential wave growth. In his inviscid laminar model the energy transfer is concentrated in the critical layer, having infinitesimal thickness, and embeds the singularity of (1.9) due to the neglect of both non-linear and diffusive effects.

The method to be described here, for the solution of unsteady Navier-Stokes equations, is very ad hoc and fundamentally simple – although not always simple to execute. The method relies on disregarding the inviolate nature of the equation(s) to be solved. The equations are then decomposed into parts, which are equated to a common factor, in such a manner that a general solution (containing the appropriate number of arbitrary functions) can be constructed for at least one part. The form of the arbitrary function(s) is then obtained by means of the requirement that the other part be satisfied.

Using this solution we will show that the wave-induced motion has exactly zero vorticity. We will also use our analytical solution to construct an alternative model for the transfer of energy from the wind to water waves. The model constructed here is based on Miles' critical-layer theory in which the effect of turbulence is implicitly included through the prescribed wind velocity profile.

2. Exact Solution to the Navier-Stokes Equations

We consider the unsteady viscous incompressible flow of a monochromatic surface wave propagating in positive x-direction with speed $c = \sigma/k$. The wave elevation, η, is given by

$$\eta(x,t) = a\cos(kx - \sigma t), \qquad ak \ll 1 \tag{2.1}$$

where $k = 2\pi/\lambda$ is the wave number, λ is the wavelength and σ is the angular frequency.

The equations of motion are governed by the Navier-Stokes equations, which in two dimensions (taking the z-axis vertically upwards), may be expressed in dimensionless form as, (see Goldstein [3, p.121])

$$\mathscr{U}_t + \mathscr{U}\mathscr{U}_x + \mathscr{W}\mathscr{U}_z = -\mathscr{P}_x + R^{-1}\mathscr{U}_{xx} + \mathscr{U}_{zz}, \tag{2.2a}$$

$$R^{-1}(\mathscr{W}_t + \mathscr{U}\mathscr{W}_x + \mathscr{W}\mathscr{W}_z) = -\mathscr{P}_z + R^{-2}\mathscr{W}_{xx} + R^{-1}\mathscr{W}_{zz}, \tag{2.2b}$$

$$\mathscr{U}_x + \mathscr{W}_z = 0, \tag{2.2c}$$

where R is the Reynolds number.

As was stated in the introduction, the method of solution employed here is very ad hoc in nature. The solution is obtained in the following manner: we first construct a solution to the steady state counterpart of equations (2.2) (reduced equations) by neglecting the x-variation in the pressure. Having obtained such solution, $\mathscr{U}$ and $\mathscr{W}$ velocity components are modified, through their arbitrary constants, so as to take into account the unsteady nature of the full equations; the expression for the pressure obtained for the reduced equations is not physical and hence discarded. This is because this expression for the pressure is only a function of z and the pressure for the full solution must vary with both x and z. Finally, the modified solutions for $\mathscr{U}$ and $\mathscr{W}$, obtained for the reduced equations, are substituted into (2.2a) and (2.2b) to obtain expressions for the pressure gradients x and z, respectively. Upon integrating these expressions and matching the arbitrary functions of integration we obtain a full solution to the unsteady Navier-Stokes equations (2.2).

Introducing the stream function Ψ defined by $\mathscr{U} = \Psi_z, \mathscr{W} = -\Psi_x$, and neglecting the unsteady terms, equations (2.2) become

$$\Psi_z\Psi_{xz} - \Psi_x\Psi_{zz} = -\mathscr{P}_x + \Psi_{zzz} + R^{-1}\Psi_{zxx}, \tag{2.3a}$$

$$\mathscr{P}_z + R^{-1}(\Psi_x\Psi_{xz} - \Psi_z\Psi_{xx} + \Psi_{xzz}) = -R^{-2}\Psi_{xxx}. \tag{2.3b}$$

We now set $\mathscr{P}_x = 0$ and split equation (2.3a) into two parts, namely

$$\Psi_z \Psi_{xz} - \Psi_x \Psi_{zz} = \mathscr{F}(x, z, \Psi, \Psi_x, \Psi_z, \Psi_{zz}, \Psi_{xz}, \Psi_{xx}), \tag{2.4a}$$

$$\Psi_{zzz} + R^{-1} \Psi_{zxx} = \mathscr{F}(x, z, \Psi, \Psi_x, \Psi_z, \Psi_{zz}, \Psi_{xz}, \Psi_{xx}) \tag{2.4b}$$

in such a way that the general solution of at least one of these can be developed. The form of the arbitrary functions is determined by requiring that the other equation is also satisfied. There will then remain certain arbitrary constants which we select in such a way that equation (2.3b) is satisfied. Finally, we substitute the solutions in (2.2a) and (2.2b) and find $\mathscr{P}(x, z, t)$ such that solutions satisfy (2.2a)-(2.2c).

It is clear that the choice made for $\mathscr{F}$ strongly influences the general solution obtained, the labour involved and the final result.[1] For our problem here the simplest choice namely $\mathscr{F} = 0$ will suffice. With this choice our system becomes

$$\mathscr{P}_x = 0, \tag{2.5}$$

$$\Psi_z \Psi_{xz} - \Psi_x \Psi_{zz} = 0, \tag{2.6}$$

$$\Psi_{zzz} + R^{-1} \Psi_{zxx} = 0 \tag{2.7}$$

and $\mathscr{P}_z$ is defined by

$$\mathscr{P}_z = -R^{-2} \Psi_{xxx} - R^{-1}(\Psi_x \Psi_{zx} - \Psi_z \Psi_{xx} + \Psi_{xzz}). \tag{2.8}$$

The general solution of equation (2.6) has the explicit form, with arbitrary functions ϖ and χ,

$$\Psi = \varpi[z + \chi(x)] \tag{2.9}$$

although it is quite usual that the general solution is implicit. In such a case the details of the computation are more complicated. Substituting equation (2.9) into equation (2.7) we find, with $\xi = z + \chi(x)$ that

$$\varpi'''(\xi)[1 + \alpha^2(\chi')^2] + \alpha^2\varpi''(\xi)\chi''(x) = 0, \tag{2.10}$$

where $\alpha^2 = R^{-1}$ and prime indicates differentiation with respect to x. To eliminate the x dependence and thus determine χ, we set

$$1 + \alpha^2(\chi')^2 = A\alpha^2\chi'', \tag{2.11}$$

where A is an arbitrary constant. Equation (2.11) has the solution

$$\chi(x) = -A \ln \cos(R^{1/2}x/A + C) + c_1. \tag{2.12}$$

[1] Other choices of $\mathscr{F}$ have also been employed. However, it is found that the only suitable choice, leading to the desired form of solution which satisfies the boundary conditions, is $\mathscr{F} = 0$.

Then ϖ satisfies

$$\varpi''' + A^{-1}\varpi'' = 0$$

or

$$\varpi(\xi) = \Gamma + \gamma A\xi + \varepsilon \exp(-\xi/A). \tag{2.13}$$

Finally we see that

$$\begin{aligned} \Psi(x, z) = \Gamma + \gamma Az - \gamma A^2 \ln\cos(R^{1/2}x/A + C) \\ + \varepsilon \mathrm{e}^{-z/A} \cos(R^{1/2}x/A + C), \end{aligned} \tag{2.14}$$

where Γ, γ, A, C and ε are arbitrary constants.

Now equations (2.5) and (2.8) are satisfied if we set $\gamma = 0$. Hence we obtain

$$\Psi = \Gamma + \varepsilon \exp(-z/A)\cos(R^{1/2}x/A + C), \tag{2.15a}$$

$$\mathscr{U} = -\varepsilon A^{-1} \exp(-z/A)\cos(R^{1/2}x/A + C), \tag{2.15b}$$

$$\mathscr{W} = \varepsilon A^{-1} R^{1/2} \exp(-z/A)\sin(R^{1/2}x/A + C), \tag{2.15c}$$

$$\mathscr{P} = -\tfrac{1}{2}(\varepsilon A)^{-2} \exp(-2z/A) + D. \tag{2.15d}$$

3. Wave-induced Motion

Appealing to the restriction posed on the size of the wave ($ka \ll 1$) we adopt the usual practice and linearize boundary conditions on $z = \eta$ and apply them instead on the free surface $z = 0$. Thus, the boundary conditions to be satisfied are (c.f. Miles [9])[2]

$$\mathscr{W} = -\eta_x, \qquad \Psi = \eta, \qquad \text{on } z = 0 \tag{3.1a,b}$$

$$\mathscr{W} \to 0, \qquad \text{as } z \to \infty. \tag{3.1c}$$

To determine the wave-induced velocities and pressure, we let $\varepsilon = a, A = R^{1/2}k^{-1}, C = -\sigma t$ and $\Gamma = 0$ in (2.15), to obtain

$$\Psi = a \exp(-kzR^{-1/2})\cos(kx - \sigma t), \tag{3.2a}$$

$$\mathscr{U} = -akR^{-1/2} \exp(-kzR^{-1/2})\cos(kx - \sigma t), \tag{3.2b}$$

$$\mathscr{W} = ak \exp(-kzR^{-1/2})\sin(kx - \sigma t), \tag{3.2c}$$

$$\mathscr{P} = -\tfrac{1}{2}a^{-2}k^2R^{-1} \exp(-2kzR^{-1/2}) + D. \tag{3.2d}$$

Clearly (3.2b) and (3.2c) satisfy the continuity equation (2.2c).

[2] In what follows all variables are made non-dimensional.

Substituting (3.2b) and (3.2c) into (2.2a) we have

$$\mathscr{P}_x = ak\sigma R^{-1/2} \exp(-kzR^{-1/2}) \sin(kx - \sigma t)$$

or

$$\mathscr{P} = -a\sigma R^{-1/2} \exp(-kzR^{-1/2}) \cos(kx - \sigma t) + f(z). \tag{3.3}$$

Substituting (3.2b) and (3.2c) into (2.2b), we get

$$\begin{aligned} \mathscr{P}_z = {} & ak\sigma R^{-1} \exp(-kzR^{-1/2}) \cos(kx - \sigma t) \\ & + a^2k^3R^{-3/2} \exp(-2kzR^{-1/2}) \end{aligned}$$

or

$$\begin{aligned} \mathscr{P} = & - a\sigma R^{-1/2} \exp(-kzR^{-1/2}) \cos(kx - \sigma t) \\ & - \tfrac{1}{2}a^2k^2R^{-1} \exp(-2kzR^{-1/2}). \end{aligned} \tag{3.4}$$

Comparing (3.3) with (3.4) we see they are the same if

$$f(z) = -\tfrac{1}{2}a^2k^2R^{-1} \exp(-2kzR^{-1/2}).$$

Hence the solution of the Navier-Stokes equations (2.2) is:

$$\Psi = a \exp(-kzR^{-1/2}) \cos(kx - \sigma t), \tag{3.5a}$$

$$\mathscr{U} = -akR^{-1/2} \exp(-kzR^{-1/2}) \cos(kx - \sigma t), \tag{3.5b}$$

$$\mathscr{W} = ak \exp(-kzR^{-1/2}) \sin(kx - \sigma t), \tag{3.5c}$$

$$\begin{aligned} \mathscr{P} = {} & -a\sigma R^{-1/2} \exp(-kzR^{-1/2}) \cos(kx - \sigma t) \\ & -\tfrac{1}{2}a^2k^2R^{-1} \exp(-2kzR^{-1/2}). \end{aligned} \tag{3.5d}$$

Note that on $z = 0$

$$\Psi = a \cos(kx - \sigma t)$$

and $\mathscr{W} = -\eta_x$, also $\mathscr{W} \to 0$ as $z \to \infty$. Thus the above solution satisfies the boundary condition (3.1a-c).

Finally the vorticity is given by

$$\omega = -\left(\frac{\partial^2}{\partial z^2} + R^{-1} \frac{\partial^2}{\partial x^2} \right) \Psi. \tag{3.6}$$

Substituting (3.5a) into (3.6) we see that $\omega = 0$. This implies that the wave-induced motion is irrotational without any approximation.

It is now instructive to identify the source of non-zero vorticity inferred by Lamb's solution. Lamb [7] essentially considers the same problem but linearizes the Navier-Stokes equations by neglecting the inertia terms. Thus, he seeks a solution to the Stokes equations (see Lamb [7, p.625]) which yields a non-zero vorticity (eqn 1.5 of Section 1). The velocity components used by Lamb to arrive at (1.5) are those given by (1.4) in terms of a velocity potential φ ($\mathscr{U} = -\varphi_x$ and $\mathscr{W} = -\varphi_z$). This velocity components, being a solution of the Stokes equations, do not satisfy the Navier-Stokes equations.[3] Consequently we conclude that the source of non-zero vorticity in Lamb's solution is principally due to construction of a solution to an approximate equations of motion. It is to be noted that although the magnitude of the inertia terms are small compared to the diffusion terms, and thus can be neglected, nevertheless their exclusion leads to a solution which yields an approximate expression for the vorticity whose magnitude may be negligibly small but not identically zero and hence, by definition, such a motion is not irrotational.

4. The Growth of Surface Waves

If we consider the growth of monochromatic surface waves

$$\eta = a(t)\cos(kx - \sigma t), \qquad ak \ll 1 \tag{4.1}$$

with $a(0) = a_0$. Adopting the exact analytical solution for the Navier-Stokes equation (derived in Sections 2 and 3), we may write

$$\mathscr{U} = -a(t)kR^{-1/2}\exp\left(-kzR^{-1/2}\right)\cos(kx - \sigma t), \tag{4.2}$$

$$\mathscr{W} = a(t)k\exp\left(-kzR^{-1/2}\right)\sin(kx - \sigma t). \tag{4.3}$$

Note that (4.2) and (4.3) are identical to (3.2b,c), except in the present case the amplitude a is a function of time t, and still satisfy the continuity equation (2.2c).

Substituting (4.2) and (4.3) in (2.2a) yields an expression for $\mathscr{P}_x$, whilst substituting in (2.2b) gives an expression for $\mathscr{P}_z$. Integrating the expression for $\mathscr{P}_x$ with respect to x, integrating the expression for $\mathscr{P}_z$ with respect to z, and comparing constant of integrations between the two, we find the following expression for the pressure

$$\begin{aligned}\mathscr{P} = R^{-1/2}\left[\dot{a}(t)\sin(kx - \sigma t) - a(t)\sigma\cos(kx - \sigma t)\right]\exp\left(-kzR^{-1/2}\right) \\ - \tfrac{1}{2}a^2(t)k^2R^{-1}\exp\left(-2kzR^{-1/2}\right),\end{aligned} \tag{4.4}$$

[3] In general not every solution of the Navier-Stokes equations satisfy the Stokes equations, and *vice versa*.

where a dot over a symbol indicates differentiation with respect to time.

Following Miles [8] we take the pressure on the undisturbed mean free surface, $z = 0$, to be

$$\mathscr{P}_a = (\alpha + i\beta)a(t)\cos(kx - \sigma t), \tag{4.5}$$

where $\alpha + i\beta$ is a dimensionless pressure coefficient[4] which are in general functions of both the wave speed c and the wavenumber k.

Substituting (4.5) into (4.4) we thus find the amplitude $a(t)$ satisfies the following initial value problem

$$-\tfrac{1}{2}R^{-1}k^2a^2(t) = \left\{R^{-1/2}\left[i\dot{a}(t) + \sigma a(t)\right] + (\alpha + i\beta)a(t)\right\}e^{i(kx-\sigma t)}. \tag{4.6}$$

Invoking the restriction $ak \ll 1$, posed on (4.1), we may neglect the left-hand side of (4.6). Thus, we obtain

$$\dot{a}(t) = i\left[R^{1/2}(\alpha + i\beta) + \sigma\right]a(t). \tag{4.7}$$

Solving (4.7) subject to the initial condition $a(0) = a_0$ we find the following expression for the amplitude of the surface wave

$$a(t) = a_0 \exp\left\{i\left[R^{1/2}(\alpha + i\beta) + \sigma\right]t\right\}. \tag{4.8}$$

The expression (4.8) indicates that the amplitude of the surface wave grows with time and the growth is a function of the energy-transfer parameters α and β, the Reynolds number R and the angular frequency σ.

5. Evaluation of β

The energy-transfer parameter β is related to the rate of growth of surface waves via (Miles [10])

$$\frac{\dot{E}}{E} = \beta\sigma\frac{\rho_a}{\rho_w}\left(\frac{U_*}{c}\right)^2 \tag{5.1}$$

where E is the wave energy, $\dot{E}$ represents its time derivative, ρ_a and ρ_w denote the density of the air and the water respectively, U_* is the friction velocity and c is the wave phase speed.

According to Miles [8] critical layer mechanism

$$\beta = -\pi\left(\frac{w_c''}{kw_c'}\right)\left(\frac{\overline{\mathscr{W}_c^2}}{U_1^2\overline{\eta_x^2}}\right), \tag{5.2}$$

[4] Not to be confused with β in Section 1 and with α in Section 2.

where $w = U - c$, prime and the suffix x denote differentiation with respect to z and x respectively, the subscript c denotes evaluation at the critical point $z = z_c$ where $U = c$, and overbar signifies average over x. Following Miles [8] we take

$$U = U_1 \log(z/z_0) \tag{5.3}$$

which for turbulent flow over water has the support of both theory and experiment (Coles [1]; Hay [4]). In (5.3), z_0 is a non-dimensional effective roughness parameter, $U_1 = U_*/\kappa$ and κ is the von-Karman's constant.

Substituting (4.3) and (5.3) into (5.2) we may express the result as

$$\beta = \pi \xi_c^{-1} \exp\left(-2R^{-1/2}\xi_c\right), \tag{5.4}$$

where

$$\xi_c = \Omega \left(\frac{U_1}{c}\right)^2 e^{c/U_1} \tag{5.5}$$

is the dimensionless critical height and Ω is the Charnock's constant, being typically $O(10^{-3} - 10^{-2})$. The Reynolds number in (5.4) is the wave *Reynolds number* taken as $R = c\lambda/\nu_w$ where ν_w is the viscosity of water.

The end result for β, obtained from (5.4) and (5.5) with $\kappa = 0.4$ and $\Omega = gz_0/U_1^2 = 2.3 \times 10^{-3}$ is plotted in Figure 1. Also plotted for comparison are: the corresponding approximation for Miles' [10] eddy-viscosity and viscoelastic approximations; Miles' [8] critical-layer approximation; Sajjadi's [11] rapid-distortion approximation for $\beta \equiv \beta_c + \beta_v$, where

$$\beta_c = \pi g y_0 c^{-2} e^{c/U_1} L_1^4 [1 - (4 - \pi^2/3) L_1^{-2}],$$

$$\beta_v = 2\kappa^2 L_1; \quad L_1 = [U(\eta_1) - c]/U_1; \quad k\eta_1 = e^{-\gamma}/2 = 0.281$$

($\gamma = 0.5772$ is the Euler's number) and Sajjadi's [11] quasi-laminar approximation β_c. Also plotted in Figure 1 are Townsend's [13] result for $T \equiv -\ln kz_0 = 8$ and Gent and Taylor's [2] results for $ka = 0.01$ and $T = 8$. As can be seen from this figure, the present result agrees well with that of Miles' [10] eddy-viscosity approximation and Sajjadi's [11] rapid-distortion theory.

Based on the present theory, (5.4) can serve as a simple formula in operational wave models, such as WAM (Komen et al. [6]), for the evaluation of the energy-transfer parameter from wind to surface waves.[5] Note,

[5] WAM generation 4 and 5 use a parameterized form of Miles' [7] critical-layer approximation.

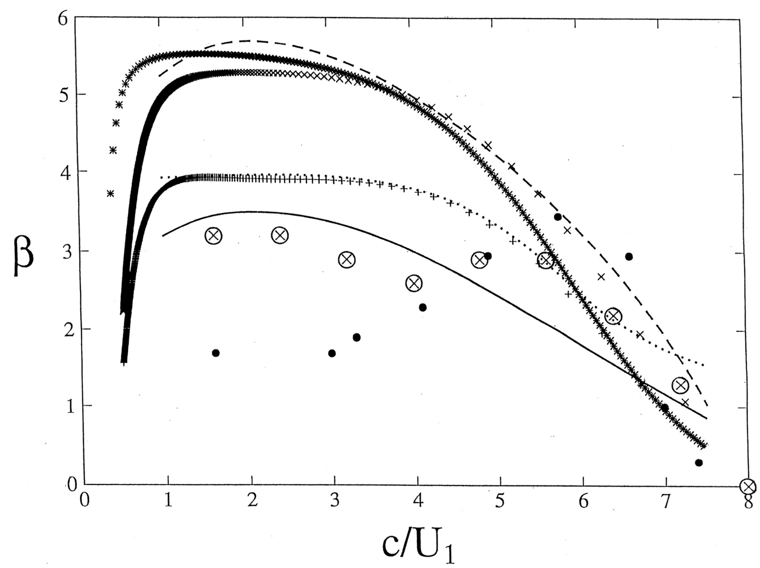

Figure 1: The energy-transfer parameter β, as calculated from: the present formulation with $\Omega = gz_0/U_1^2 = 2.3 \times 10^{-3}$ (***); the rapid-distortion approximation, Sajjadi [11] ($\times\ \times\ \times$); the quasi-laminar approximation, Sajjadi [11] (+++); the eddy-viscosity approximation, Miles [10] ($-\ -\ -$); the critical-layer approximation Miles [8](...); the viscoelastic approximation Miles [10](—); Townsend's [13] numerical integration for $T \equiv \ln(1/ky_0) = 8$ ($\bullet$); Gent and Taylor's [2] numerical integration for $T = 8$ and $ka = 0.01$ ($\otimes$).

although the present theory is based on the exact solution of the Navier-Stokes equations, it is however essentially a quasi-laminar model in which turbulence is implicitly included through the prescribed wind velocity profile. However, it is interesting to note that the present model yields larger values of β compared to Miles' [8] critical-layer approximation and is more in line with Miles' [10] eddy-viscosity model and Sajjadi's [11] rapid-distortion theory.

6. Conclusions

An exact analytical solution to the Navier-Stokes solutions is constructed for an unsteady viscous incompressible flow governing the motion of a monochromatic surface water. The solution obtained here yields a zero vorticity and therefore is irrotational. This solution is another example of viscous potential flow as was discussed by Joseph and Wang [5].

The method adopted for the solution of unsteady Navier-Stokes equations is based on an ad hoc technique for solving non-linear partial differential equations. The method is based on disregarding the inviolate nature of the equations. The equations are then decomposed into parts, which are equated to a common factor, in such a manner that a general solution (containing the appropriate number of arbitrary functions) was constructed for one part. The form of the arbitrary functions are then obtained by means of the requirement that the other part be satisfied.

The analytical solution obtained here is also used to construct an alternative expression for the energy-transfer parameter from the wind to water waves within the classical critical-layer approximation of Miles' [8] quasi-laminar model which can be used in operational wave models.

References

[1] D. Coles, The law of the wake in the turbulent boundary layer, J. Fluid Mech. 1 (1956), 191.

[2] P. R. Gent and P. A. Taylor, A numerical model of the flow above water waves, J. Fluid Mech. 77 (1976), 105.

[3] S. Goldstein, Modern Developments in Fluid Dynamics, Vol. 1, Oxford University Press, 1938.

[4] J. S. Hay, Some observations of air flow over the sea, Quart. J. Roy. Met. Soc. 81 (1955), 307.

[5] D. D. Joseph and J. Wang, The dissipation approximation and viscous potential flow, J. Fluid Mech. 505 (2004), 365.

[6] G. J. Koman, L. Cavaleri, M. Donelan et al., Dynamics and Modelling of Ocean Waves, Cambridge University Press, 1994.

[7] H. Lamb, Hydrodynamics, 6th ed., Cambridge University Press, 1932.

[8] J. W. Miles, On the generation of surface waves by shear flows, J. Fluid Mech. 3 (1957), 185.

[9] J. W. Miles, On the generation of surface waves by shear flows, Part 5, J. Fluid Mech. 30 (1967), 163.

[10] J. W. Miles, Surface wave generation: a viscoelastic model, J. Fluid Mech. 322 (1996), 131.

[11] S. G. Sajjadi, On the growth of a fully non-linear Stokes wave by turbulent shear flow. Part 2: rapid distortion theory, Math. Engng. Ind. 6 (1998), 247.

[12] S. G. Sajjadi, Viscous potential flow for water waves, Adv. Appl. Fluid Mech. 7 (2010), 13-30.

[13] A. A. Townsend, Flow in a deep turbulent boundary layer over a surface distorted by water waves, J. Fluid Mech. 55 (1972), 719.