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VISCOUS POTENTIAL FLOW FOR WATER WAVES

SHAHRDAD G. SAJJADI
Division of Geophysics and Planetary Physics
CAMAC, Department of Applied Mathematics
Embry-Riddle Aeronautical University
Florida 32114, U. S. A.

Abstract

A potential flow solution to the Navier-Stokes equations for a viscous incompressible fluid is constructed for gravity waves on deep water. Using this solution, it is shown that the amplitude of the wave decays at one half of the rate originally found by Lamb [6]. The rate of decay deduced from the exact analytical solution derived here is in exact agreement with the direct calculation of Funada and Joseph [2]. The physical consequence of the new rate of decay is discussed.

1. Introduction

Potential flows for which the velocity vector $\boldsymbol{u} = \boldsymbol{\nabla}\phi$ are not, in general, solutions of the Navier-Stokes equations for viscous incompressible fluids unless it can be shown that $\boldsymbol{\nabla} \times \boldsymbol{u} = \mathbf{0}$, as shown by Joseph [5]. For such solutions the viscous term $\mu\nabla^2\boldsymbol{u} = \mu\boldsymbol{\nabla}\nabla^2\phi$ vanishes, but for an incompressible fluid the viscous contribution to the stress does not always vanish [9].

In the present paper, we give an alternative derivation to Joesph [5] and similarly show the manner in which the viscosity of a viscous fluid in

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potential flow away from the boundary layers enters Prandtl's boundary-layer equations. To demonstrate this point, we will derive an analytical solution for a viscous potential flow over a monochromatic wave. Moreover, following Joseph and Wang [4], we revisit Lamb's [6] solution for decaying free gravity waves and give an alternative solution for the viscous potential flow and the viscous correction.

Lamb [6, Section 349] considered the effect of viscosity on water waves. In that article, he constructed an analytical solution for a viscous flow over a monochromatic surface wave

$$y = a\cos(kx \pm \sigma t), \qquad ak \ll 1 \tag{1.1}$$

where a is the wave amplitude, k is the wavenumber and σ is the angular frequency.

By neglecting the inertia terms in the Navier-Stokes equations and assuming the elevation of the free surface

$$\eta = -\frac{k}{n}(A - iC)e^{ikx+nt}, \tag{1.2}$$

where A and C are constants, and n is an integer, he showed that since the ratio

$$\frac{C}{A} = \mp\frac{2\nu k^2}{\sigma} \ll 1 \tag{1.3}$$

the motion is *approximately* irrotational with a velocity potential

$$\phi = Ae^{-2\nu k^2 t + ky + i(kx \pm \sigma t)}, \tag{1.4}$$

where ν is the kinematic viscosity of the fluid.

In the footnote of the same article (p. 627) he showed that the corresponding vorticity is given by

$$\omega = \mp 2\sigma k a e^{-2\nu k^2 t + \beta y}\cos\{kx \pm (\sigma t + \beta y)\}, \tag{1.5}$$

where $\beta = (\sigma/2\nu)^{1/2}$ and $a = kA/\sigma$. Lamb argued that the vorticity diminishes rapidly from the surface downwards and 'owing to the oscillatory character of the motion, the sign of the vorticity which is being diffused inwards from the surface is continually being reversed, so that beyond a stratum of thickness comparable with $2\pi/\beta$ the effect is insensible, ...'.

The presence of a non-zero vorticity implies that the motion is *not* irrotational. Thus, the question arises, what is the source of this vorticity?

Is it solely due to the presence of viscosity? Or is it due to mathematical approximations leading to a solution which yields a non-zero vorticity? To answer these questions, we revisit Lamb's problem and show that, by a slight change in method of the solution, the effect of viscosity on water waves yields a motion with exactly zero vorticity. We will construct an exact analytical solution to the viscous, unsteady, two-dimensional, incompressible Navier-Stokes equations governing the motion of a monochromatic surface wave, of small steepness, on deep water. We satisfy the kinematic and dynamic boundary conditions, as well as the tangential and normal stress conditions, on the free surface whose profile is determined by integrating the kinematic condition.

The method to be described here, for the solution of unsteady Navier-Stokes equations, is very ad hoc and fundamentally simple – although not always simple to execute. The method relies on disregarding the inviolate nature of the equation(s) to be solved. The equations are then decomposed into parts, which are equated to a common factor, in such a manner that a general solution (containing the appropriate number of arbitrary functions) can be constructed for at least one part. The form of the arbitrary function(s) is then obtained by means of the requirement that the other part be satisfied.

2. Boundary-layer Theory

In flows where the fluid speed is large enough the vorticity in the outer layer is zero. In such circumstances, we would like to examine how the viscosity in the viscous potential flow affect the boundary-layer equations.

To begin our analysis it is instructive to consider the Navier-Stokes equations, and for simplicity in two dimensions, which may be expressed in dimensionless form as, (Goldstein [3, p. 121]

$$u_x + v_y = 0, \tag{2.1}$$

$$uu_x + vu_y = -p_x + R^{-1}u_{xx} + u_{yy}, \tag{2.2}$$

$$R^{-1}(uv_x + vv_y) = -p_y + R^{-2}v_{xx} + R^{-1}v_{yy}, \tag{2.3}$$

where R is the Reynolds number.

In describing the Prandtl's boundary-layer theory, Batchelor [1] argues that "... the pressure is approximately uniform across the boundary layer; and if it happens that the variation of p with x just outside the boundary layer is known ... perhaps from the consideration of the inviscid flow equations in the region outside the boundary layer ... the pressure term [in equation (2.2) above] can be regarded as given."

Now assume the flow in the region outside the boundary layer is potential, and let the dynamic viscosity in this region be $\mu \neq 0$ with the velocity $\boldsymbol{U}$ and pressure P being in the free stream. Thus in this region there is a potential flow of a viscous fluid with $\boldsymbol{\nabla} \times \boldsymbol{U} = \mathbf{0}$ (just as in the case of inviscid flow) and consequently there exist a velocity potential Φ such that $\nabla^2\Phi = 0$.

For convenience we re-write equation (2.3) in the following form:

$$R^{-1}(uv_x + vv_y) = -(p - R^{-1}v_y)_y + R^{-2}v_{xx}$$

which indicates that the y derivative of

$$p - R^{-1}v_y \tag{2.4}$$

is $O(\delta)$ across the boundary layer of thickness δ. Hence, in the outer region we may write

$$P - R^{-1}V_y = P - R^{-1}\Phi_y y \tag{2.5}$$

which is the value of (2.4) at the edge of that layer. From the Bernoulli equation, which also holds for a viscous fluid, we have

$$P = -\tfrac{1}{2}U^2 + \text{const.} \tag{2.6}$$

Substituting (2.6) into (2.5) we get

$$-\tfrac{1}{2}U^2 + \text{const} - R^{-1}V_y = p - R^{-1}v_y. \tag{2.7}$$

Since at the wall $v_y = 0$,

$$p_x = -UU_x - R^{-1}V_{xy} \tag{2.8}$$

drives the flow in (2.2). Using the continuity equation for the flow in the outer region we may express (2.8) in the following form:

$$p_x = -UU_x + R^{-1}U_{xx} \tag{2.9}$$

and because in the outer region the flow is potential then in (2.9) $U = \Phi_x$.

Finally, noting that at the wall $u_{xx} = 0$ but $U_{xx} \neq 0$, we obtain the modified Prandtl equation

$$uu_x + vu_y = UU_x - R^{-1}U_{xx} + u_{yy} \tag{2.10}$$

for a steady flow. Compared to Joseph [5], it is much easier to see from (2.10) that when the outer fluid is inviscid, the viscous term $R^{-1}U_{xx}$ vanishes and for high-Reynolds number flows this term is very small.

The derivation of equation (2.10) above, like its counterpart given by Joseph [5], lead to the same conclusions as that arrived by him. However, the present derivation, with the aid of the boundary-layer scaling, makes the conclusions much more clearer.

As was also noted by Joseph [5], there are many cases of irrotational flow in which viscous effects are important. Such examples include the interfacial instability as well as decay of free gravity waves on water which we will examine in detail next.

3. Exact Solution of the Unsteady Flow

We consider the unsteady viscous incompressible flow of a monochromatic surface wave propagating in positive x-direction with speed $c = \sigma/k$. The wave elevation, η, is given by

$$\eta(x,t) = a\cos(kx - \sigma t), \qquad ak \ll 1 \tag{3.1}$$

where $k = 2\pi/\lambda$ is the wave number, λ is the wavelength and σ is the angular frequency.

The equations of motion are governed by the Navier-Stokes equations, which in two dimensions (taking the y-axis vertically upwards), may be expressed in dimensionless form as

$$u_t + uu_x + vu_y = -p_x + R^{-1}u_{xx} + u_{yy}, \tag{3.2a}$$

$$R^{-1}(v_t + uv_x + vv_y) = -p_y + R^{-2}v_{xx} + R^{-1}v_{yy}, \tag{3.2b}$$

$$u_x + v_y = 0, \tag{3.2c}$$

where R is the Reynolds number. The method of solution which will be employed here is very ad hoc in nature. The solution is obtained in the following manner: we first construct a solution to the steady state counterpart of equations (3.2) (reduced equations) by neglecting the x-variation in the pressure. Having obtained such solution, u and v velocity components are modified, through their arbitrary constants, so as to take into account the unsteady nature of the full equations; the expression for the pressure obtained for the reduced equations is not physical and hence discarded. This is because this expression for the pressure is only a function of y and the pressure for the full solution must vary with both x

and y. Finally, the modified solutions for u and v, obtained for the reduced equations, are substituted into (3.2a) and (3.2b) to obtain expressions for the pressure gradients x and y, respectively. Upon integrating these expressions and matching the arbitrary functions of integration we obtain a full solution to the unsteady Navier-Stokes equations (3.2).

Introducing the stream function ψ defined by $u = \psi_y, v = -\psi_x$, and neglecting the unsteady terms, equations (3.2) become

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} = -p_x + \psi_{yyy} + R^{-1}\psi_{yxx}, \tag{3.3a}$$

$$p_y + R^{-1}(\psi_x\psi_{xy} - \psi_y\psi_{xx} + \psi_{xyy}) = -R^{-2}\psi_{xxx}. \tag{3.3b}$$

We now set $p_x = 0$ and split equation (3.3a) into two parts, namely,

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} = F(x, y, \psi, \psi_x, \psi_y, \psi_{yy}, \psi_{xy}, \psi_{xx}), \tag{3.4a}$$

$$\psi_{yyy} + R^{-1}\psi_{yxx} = F(x, y, \psi, \psi_x, \psi_y, \psi_{yy}, \psi_{xy}, \psi_{xx}) \tag{3.4b}$$

in such a way that the general solution of at least one of these can be developed. The form of the arbitrary functions is determined by requiring that the other equation is also satisfied. There will then remain certain arbitrary constants which we select in such a way that equation (3.3b) is satisfied. Finally, we substitute the solutions in (3.2a) and (3.2b) and find $p(x, y, t)$ such that solutions satisfy (3.2a)-(3.2c).

It is clear that the choice made for F strongly influences the general solution obtained, the labour involved and the final result. For our problem here the simplest choice namely $F = 0$ will suffice. With this choice our system becomes

$$p_x = 0, \tag{3.5}$$

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} = 0, \tag{3.6}$$

$$\psi_{yyy} + R^{-1}\psi_{yxx} = 0 \tag{3.7}$$

and p_y is defined by

$$p_y = -R^{-2}\psi_{xxx} - R^{-1}(\psi_x\psi_{yx} - \psi_y\psi_{xx} + \psi_{xyy}). \tag{3.8}$$

The general solution of equation (3.6) has the explicit form, with arbitrary functions ϖ and χ,

$$\psi = \varpi[y + \chi(x)] \tag{3.9}$$

although it is quite usual that the general solution is implicit. In such a case the details of the computation are more complicated. Substituting equation (3.9) into equation (3.7) we find, with $z = y + \chi(x)$ that

$$\varpi'''(z)[1 + \alpha^2(\chi')^2] + \alpha^2\varpi''(z)\chi''(x) = 0, \tag{3.10}$$

where $\alpha^2 = R^{-1}$ and prime indicates differentiation with respect to x. To eliminate the x dependence and thus determine χ, we set

$$1 + \alpha^2(\chi')^2 = A\alpha^2\chi'', \tag{3.11}$$

where A is an arbitrary constant. Equation (3.11) has the solution

$$\chi(x) = -A \ln\cos(R^{1/2}x/A + C) + c_1. \tag{3.12}$$

Then ϖ satisfies

$$\varpi''' + A^{-1}\varpi'' = 0$$

or

$$\varpi(z) = \Gamma + \gamma Az + \varepsilon \exp(-z/A). \tag{3.13}$$

Finally we see that

$$\begin{aligned}\psi(x, y) = \Gamma + \gamma Ay - \gamma A^2 \ln\cos(R^{1/2}x/A + C)\\ + \varepsilon \mathrm{e}^{-y/A} \cos(R^{1/2}x/A + C),\end{aligned} \tag{3.14}$$

where Γ, γ, A, C and ε are arbitrary constants.

Now equations (3.5) and (3.8) are satisfied if we set $\gamma = 0$. Hence, we obtain

$$\psi = \Gamma + \varepsilon \exp(-y/A) \cos(R^{1/2}x/A + C), \tag{3.15a}$$

$$u = -\varepsilon A^{-1} \exp(-y/A) \cos(R^{1/2}x/A + C), \tag{3.15b}$$

$$v = \varepsilon A^{-1}R^{1/2} \exp(-y/A) \sin(R^{1/2}x/A + C), \tag{3.15c}$$

$$p = -\tfrac{1}{2}(\varepsilon A)^{-2} \exp(-2y/A) + D. \tag{3.15d}$$

4. Wave-induced Motion

Appealing to the restriction posed on the size of the wave ($ka \ll 1$), we adopt the usual practice and linearize boundary conditions on $y = \eta$ and apply them instead on the free surface $y = 0$. Thus, the boundary conditions to be satisfied are (c.f. Miles [7] and Sajjadi [8])

$$v = -\eta_x, \quad \psi = \eta, \qquad \text{on } y = 0 \tag{4.1a, b}$$

$$v \to 0, \qquad \text{as } y \to \infty. \tag{4.1c}$$

To determine the wave-induced velocities and pressure, we let $\varepsilon = a, A = R^{1/2}k^{-1}, C = -\sigma t$ and $\Gamma = 0$ in (3.15), to obtain

$$\psi = a \exp(-kyR^{-1/2}) \cos(kx - \sigma t), \tag{4.2a}$$

$$u = -akR^{-1/2} \exp(-kyR^{-1/2}) \cos(kx - \sigma t), \tag{4.2b}$$

$$v = ak\exp(-kyR^{-1/2})\sin(kx-\sigma t), \tag{4.2c}$$
$$p = -\tfrac{1}{2}a^{-2}k^2R^{-1}\exp(-2kyR^{-1/2}) + D. \tag{4.2d}$$

Clearly (4.2b) and (4.2c) satisfy the continuity equation (3.2c).

Substituting (4.2b) and (4.2c) into (3.2a) we have

$$p_x = ak\sigma R^{-1/2}\exp(-kyR^{-1/2})\sin(kx-\sigma t)$$

or

$$p = -a\sigma R^{-1/2}\exp(-kyR^{-1/2})\cos(kx-\sigma t) + f(y). \tag{4.3}$$

Substituting (4.2b) and (4.2c) into (3.2b) we get

$$p_y = ak\sigma R^{-1}\exp(-kyR^{-1/2})\cos(kx-\sigma t)$$
$$+a^2k^3R^{-3/2}\exp(-2kyR^{-1/2})$$

or

$$p = -a\sigma R^{-1/2}\exp(-kyR^{-1/2})\cos(kx-\sigma t)$$
$$-\tfrac{1}{2}a^2k^2R^{-1}\exp(-2kyR^{-1/2}). \tag{4.4}$$

Comparing (4.3) with (4.4) we see they are the same if

$$f(y) = -\tfrac{1}{2}a^2k^2R^{-1}\exp(-2kyR^{-1/2}).$$

Hence the solution of the Navier-Stokes equations (3.2) is:

$$\psi = a\exp(-kyR^{-1/2})\cos(kx-\sigma t), \tag{4.5a}$$
$$u = -akR^{-1/2}\exp(-kyR^{-1/2})\cos(kx-\sigma t), \tag{4.5b}$$
$$v = ak\exp(-kyR^{-1/2})\sin(kx-\sigma t), \tag{4.5c}$$
$$p = -a\sigma R^{-1/2}\exp(-kyR^{-1/2})\cos(kx-\sigma t)$$
$$-\tfrac{1}{2}a^2k^2R^{-1}\exp(-2kyR^{-1/2}). \tag{4.5d}$$

Note that on $y = 0$

$$\psi = a\cos(kx-\sigma t)$$

and $v = -\eta_x$, also $v \to 0$ as $y \to \infty$. Thus the above solution satisfies the boundary condition (3.4a-c).

Finally the vorticity is given by

$$\omega = -\left(\frac{\partial^2}{\partial y^2} + R^{-1}\frac{\partial^2}{\partial x^2}\right)\psi. \tag{4.6}$$

Substituting (4.5a) into (4.6) we see that $\omega = 0$. This implies that the wave-induced motion is irrotational without any approximation.

5. Decay of Free Gravity Waves on Water

It is possible to have a progressive gravity wave of permanent form if the viscosity of the liquid below air is identically zero. Lamb [6, Sections 348 and 349) performed an analysis of the effect of viscosity on these waves. The wave decays and the decay rate may be obtained in two ways: by a dissipation calculation or by a direct (stability) calculation using viscous potential flow. This is analogous to the dissipation and direct calculation of drag. The two decay rates are not the same.

Lamb also constructed an exact solution to this problem: it gives a decay rate different from the two just mentioned; it reduces to the one computed by the dissipation method for long waves ($k \ll (g/\nu^2)^{1/3}$, where k is the wavenumber) and to the one computed directly for short waves ($k \gg (g/\nu^2)^{1/3}$). The dissipation method yields an incorrect result for short waves and the direct method using Viscous Potential Flow (VPF) gives the wrong result for long waves.

Lamb's exact solution also reveals the vorticity near the wave surface, which provides explanations for the aforementioned discrepancies. At the long wave limit, the vorticity is important in a thin boundary layer; thus some kind of pressure correction is needed. We calculate this pressure correction by the method given in Section 2; to arrive at a solution which leads to the same decay rate as Lamb's exact solution for long waves.

6. Dissipation Calculation (Lamb [6, Section 348])

When gravity is important and $\boldsymbol{g} = -\boldsymbol{e}_y g$ where y points upward and $\boldsymbol{e}_y$ is the unit normal in the y direction, the energy equation (2.1) becomes

$$\frac{d}{dt}\int_V \rho(\tfrac{1}{2}u^2 + gy)\, dV = \mathscr{P} - \mathscr{D}, \tag{6.1}$$

where $\mathscr{P}$ is the power of traction and $\mathscr{D}$ is the dissipation (Joseph and Wang [4]). In the present problem we look at functions periodic in x with period λ with $y = \eta(x,t)$ representing the free surface for $-L \leq y \leq \eta$, and in the limit as $L \to \infty$. The gravity term then gives rise to a potential energy

$$\int_V \rho g y\, dV = \int_0^\lambda \frac{\rho g \eta^2}{2}\, dx. \tag{6.2}$$

Lamb notes that when the viscosity is neglected, the progressive wave may be represented by

$$\phi = \alpha c e^{ky}\cos k(x - ct), \qquad \eta = \alpha \sin k(x - ct), \tag{6.3}$$

where $c = \sqrt{g/k}$ for inviscid potential flow is the wave velocity. In fact, this relation between ϕ and η is satisfied only if α is independent of time. Lamb noted that (6.3) will hold and the motion will persist, even with viscosity, provided that the surface stresses calculated on the potential flow are applied. In this case the dissipation in one period is

$$\mathscr{D} = \mathscr{P} = 2\mu k^3 \alpha^2 c^2 \lambda. \tag{6.4}$$

For a free wave, with $\mathscr{P} = 0$,

$$\frac{d}{dt}\left(\int_V \rho \frac{u^2}{2}\, dV + \int_0^\lambda \frac{\rho g \eta^2}{2}\, dx\right) = \frac{d}{dt}\left(\tfrac{1}{2}\rho k \alpha^2 c^2 \lambda\right) = -\mathscr{D} = 2\mu k^3 \alpha^2 c^2 \lambda. \tag{6.5}$$

Equation (6.5) implies that

$$\frac{d\alpha}{dt} = -2\nu k^2 \alpha. \tag{6.6}$$

Thus showing $-2\nu k^2$ is the decay rate from the dissipation calculation.

7. Direct Calculation (Funada and Joseph [2])

The decay of free gravity waves can be treated as a stability problem using the theory of VPF. The stability analysis is a special case of the study of Kevin-Helmholtz stability given by Funada and Joseph [2]. Here, the governing equations are

$$\boldsymbol{u} = \boldsymbol{\nabla}\phi, \qquad \nabla^2\phi = 0, \quad -\infty < y \leq 0 \tag{7.1}$$

$$\left.\begin{array}{ll} \rho\phi_t = -p - \rho g \eta & \text{dynamic condition} \\[2ex] -p + 2\mu\phi_{yy} = 0 & \text{normal stress condition} \\[2ex] \eta_t = \phi_y & \text{kinematic condition} \end{array}\right\} \quad \text{on } y = 0. \tag{7.2}$$

Eliminating p and η from (7.2) and applying the potential

$$\phi = A e^{ky + nt + ikx} \tag{7.3}$$

we obtain

$$n = -\nu k^2 \pm ik\sqrt{g/k - \nu^2 k^2}. \tag{7.4}$$

Hence we see that the amplitude of the wave decays at a rate

$$\frac{d\alpha}{dt} = -\nu k^2 \alpha \tag{7.5}$$

one half of the rate given by (6.6), see Sections 10-11 below. In this case, the wave speed is given by

$$c = \sqrt{g/k - \nu^2 k^2} \tag{7.6}$$

which is slower than the inviscid wave speed $\sqrt{g/k}$ for $\nu k^2 < \sqrt{gk}$. Note that, for very large values of k, $\nu k^2 \gg \sqrt{gk}$ waves do not propagate but simply decay at a rate given by

$$\frac{d\alpha}{dt} = -\frac{g}{2\nu k}. \tag{7.7}$$

8. Exact Solution (Lamb [6, Section 349])

Lamb's exact solution of the problem of decaying free gravity waves differs from the solutions using viscous potential flow, given in Section 7, in that the stress conditions

$$T^{(xy)} = 0, \qquad T^{(yy)} = \gamma\eta_{xx} \tag{8.1}$$

are applied at the free surface ($y = 0$). (Here, the surface tension γ is not relevant in our discussion here and is therefore neglected.)

The condition (8.1) cannot be satisfied by an irrotational flow. To accommodate vorticity, Lamb introduces a stream function ψ and the solution is given by

$$u = \phi_x + \psi_y, \quad v = \phi_y - \psi_x, \quad \rho^{-1}p = -\phi_t - gy \tag{8.2}$$

provided

$$\nabla^2\phi = 0, \qquad \psi_t = \nu\nabla^2\psi. \tag{8.3}$$

It is important to realize that pressure term enters into the stream function equation as the pressure p depends on the viscosity through the velocity potential. Lamb showed that the governing equations can be solved with normal nodes

$$\phi = -Ae^{ky}e^{ikx+nt}, \qquad \psi = -Ce^{my}e^{ikx+nt}, \qquad m^2 = k^2 + n/\nu \tag{8.4}$$

provided

$$(n + 2\nu k^2)^2 + gk + \gamma' k^3 = 4\nu^2 k^3 m, \tag{8.5}$$

where $\gamma' = \gamma/\rho$. When $\nu k^2 \ll \sqrt{gk + \gamma' k^3}$ (long waves) Lamb found that

$$n = -2\nu k^2 \pm i\sqrt{gk + \gamma' k^3}. \tag{8.6}$$

The decay rate $-2\nu k^2$ agrees with the dissipation approximation result (6.6). When $\nu k^2 \gg \sqrt{gk + \gamma' k^3}$ (short waves) and with γ' being neglected,

$$n = -\frac{g}{2\nu k} \tag{8.7}$$

which agrees with the decay rate (7.7) from the direct stability analysis using VPF. This limit is applicable to a very for very viscous fluid and negligible vorticity. Lamb emphasized that this limit "... represents a slow creeping of the fluid towards a state of equilibrium with a horizontal surface."

The decay rate $-\nu k^2$ given by (7.5) is one-half of the exact solution at the long wave limit. This discrepancy is caused by the boundary layer at the free surface, which is not accounted for in the direct stability analysis when using VPF. The vorticity ω is given by

$$\omega = \frac{n}{\nu} C e^{my+ikx+nt}. \tag{8.8}$$

At the long wave limit, however, the vorticity is important in a thin boundary layer. Lamb estimated that the thickness of this boundary layer is $2\pi/\chi$, where $\chi = (\sqrt{gk + \gamma' k^3}/2\nu)^{1/2}$. The situation is different at the short wave limit, where the magnitude of the vorticity is very small and there is no sensible boundary layer. This explains why the decay rate arising from the direct calculation using VPF agrees with the exact solution at the short wave limit.

9. Direct Calculation (Joseph and Wang [4])

At the long wave limit, we require, a pressure correction to the irrotational pressure due to the vorticity layer. In this case, we can solve for the viscous pressure correction from the linearized governing equation and prove that it can be represented by harmonic series. Following Joseph and Wang [4], we first divide the velocity and pressure in the boundary layer near the interface into two parts

$$\boldsymbol{u} = \mathbf{U} + \mathsf{u}, \qquad p = \mathsf{P} + \mathsf{p}, \tag{9.1}$$

where the capital and small symbols denote potential solutions and viscous corrections, respectively. The linearized governing equation for (u, p) is

$$\mathsf{u}_t = -\rho^{-1}\boldsymbol{\nabla}\mathsf{p} + \nu\nabla^2\mathsf{u}. \tag{9.2}$$

Taking the divergence of (9.2) we obtain $\nabla^2 \mathsf{p} = 0$. The solution of p can be expressed as a Fourier series

$$-\mathsf{p} = \sum_{\ell=-\infty}^{\infty} C_\ell e^{nt+\ell y+i\ell x}. \tag{9.3}$$

The zero shear stress condition at the free surface implies that $\mathsf{u}_y \sim O(1)$ and it follows from the continuity equation that $\mathsf{v}_y \sim O(\delta)$, where δ is the boundary layer thickness. The normal stress balance at $y = 0$ is

$$-p + 2\mu\phi_{yy} = 0, \tag{9.4}$$

where the surface tension and the term $2\mu\mathsf{v}_y$ have been neglected. Equation (9.4) can alternatively be written as

$$\rho\phi_t + \rho g\eta - \mathsf{p} + 2\mu\phi_{yy} = 0. \tag{9.5}$$

Substituting the expressions for ϕ from (7.3) and p from (9.3) into (9.5), we have

$$\left(\rho nA + \rho\frac{gk}{n}A + 2\mu k^2 A + C_k\right) e^{nt+ikx} + \sum_{\ell\neq k} C_\ell e^{nt+i\ell x} = 0 \tag{9.6}$$

and by orthogonality, we obtain

$$\rho nA + \rho\frac{gk}{n}A + 2\mu k^2 A + C_k = 0 \quad \text{and} \quad C_\ell = 0 \quad \text{if} \quad \ell \neq k. \tag{9.7}$$

We now list the velocities and stresses at $y = 0$ evaluated on the potential:

$$\left.\begin{aligned} \mathsf{u} &= ikAe^{nt+ikx}, & \mathsf{v} &= kAe^{nt+ikx} \\ \tau^{(yy)} &= 2\mu k^2 Ae^{nt+ikx}, & \tau^{(xy)} &= 2i\mu k^2 Ae^{nt+ikx} \end{aligned}\right\}. \tag{9.8}$$

The power of the pressure correction and power of the shear stress are

$$-\int_0^\lambda \mathsf{v}^*\mathsf{p}\, dx = C_k Ak\lambda, \quad \mathscr{P}_s = \int_0^\lambda \mathsf{u}^*\tau^{(xy)}\, dx = 2\mu A^2 k^3 \lambda, \tag{9.9}$$

where the asterisk denotes conjugate variables. It then follows that

$$C_k = 2\mu k^2 A \quad \text{and} \quad -\mathsf{p} = 2\mu k^2 Ae^{nt+ky+ikx}. \tag{9.10}$$

Inserting the value of C_k into (9.7), we get

$$\rho n + \rho\frac{gk}{n} + 4\mu k^2 = 0 \tag{9.11}$$

and the solution for the potential is

$$\phi = Ae^{ky}e^{-2\nu k^2 t}e^{ik(x\pm\sqrt{g/k-4\nu^2k^2}t)}. \tag{9.12}$$

The amplitude of the wave decays at a rate $-2\nu k^2$ which agrees with the dissipation result and Lamb's exact solution for the long wave waves.

10. Irrotational Waves on Deep Water

Taking the y-axis to be drawn vertically upwards and assuming that the motion is confined to the two-dimensional (x, y)-plane, we consider the unsteady viscous incompressible flow of a monochromatic surface wave

$$y = a\cos(kx - \sigma t), \qquad ak \ll 1$$

on deep water, where $k = 2\pi/\lambda$ is the wave number, λ is the wavelength and σ is the angular frequency, propagating in positive x-direction with speed $c = \sigma/k$.

We construct an analytical solution to the Navier-Stokes equations

$$\mathscr{U}_t + \mathscr{U}\,\mathscr{U}_x + \mathscr{V}\,\mathscr{U}_y = -\rho^{-1}\mathscr{P}_x + \nu(\mathscr{U}_{xx} + \mathscr{U}_{yy}), \tag{10.1a}$$

$$\mathscr{V}_t + \mathscr{U}\,\mathscr{V}_x + \mathscr{V}\,\mathscr{V}_y = -\rho^{-1}\mathscr{P}_y + \nu(\mathscr{V}_{xx} + \mathscr{V}_{yy}), \tag{10.1b}$$

$$\mathscr{U}_x + \mathscr{V}_y = 0, \tag{10.1c}$$

where ρ and ν are the density and the kinematic viscosity of the fluid, respectively, applicable for deep water waves.

Applying the technique described in Section 3 we obtain the solution as (c.f. 2.15a)

$$\psi = \vartheta + \varepsilon\exp(-y/A)\cos(x/A + C), \tag{10.2}$$

where $\mathscr{U} = \psi_y$ and $\mathscr{V} = -\psi_x$.

To determine the 'normal modes' which are periodic with respect to x with a prescribed wavelength $2\pi/k$ we assume a time-factor by letting $\varepsilon = e^{Nt}$, where N to be determined from the boundary conditions, and to preserve the space factor e^{ikx} we let $A = -k^{-1}$ and $\vartheta = 0$. Thus (10.2) becomes

$$\begin{aligned}\psi &= e^{Nt}e^{ky}\cos(-kx + C)\\ &= e^{Nt}e^{ky}[E\cos kx + F\sin kx],\end{aligned} \tag{10.3}$$

where $E = \cos C$ and $F = \sin C$. It is convenient to express (10.3) as

$$\psi = \mathscr{A}e^{ky}e^{ikx+Nt}, \tag{10.4}$$

where $\mathscr{A} = E - iF$ and with the understanding that the real part of (10.4) is implied. Hence, from (10.4) we see at once that

$$\mathscr{U} = \mathscr{A} k e^{ky} e^{ikx+Nt}, \tag{10.5a}$$

$$\mathscr{V} = -\mathscr{A} ik e^{ky} e^{ikx+Nt}. \tag{10.5b}$$

Note that (10.5a) and (10.5b) satisfy the no slip boundary condition at an infinite depth (i.e., as $y \to -\infty$).

Substituting (10.5) into (10.1a), integrating with respect to x, and into (10.1b), integrating with respect to y, we obtain the following expression for the pressure

$$\mathscr{P} = \rho \mathscr{A} iN e^{ky} e^{ikx+Nt}. \tag{10.6}$$

Clearly (10.5) and (10.6) satisfy (10.1a-c), and are therefore the exact solution to the Navier-Stokes equations.

At this stage it is important to note that the vorticity $\omega = \mathscr{V}_x - \mathscr{U}_y$, for this motion, is identically zero. This implies, contrary to Lamb's [6] solution, the motion considered here is irrotational even in the presence of viscosity. Consequently, we can introduce a velocity potential ϕ such that $\mathscr{U} = \boldsymbol{\nabla}\phi$ and from (10.5) we see that

$$\phi = -i\mathscr{A} e^{ky} e^{ikx+Nt}. \tag{10.7}$$

If η denotes the elevation at the free surface, we must have $\eta_t = \mathscr{V}$ on $y = \eta$. Expanding this as a Taylor expansion about $y = 0$, we have to the first order in Taylor expansion

$$\frac{\partial \eta}{\partial t} = \left(\mathscr{V} + \eta \frac{\partial \mathscr{V}}{\partial y} \right)_{y=0}. \tag{10.8}$$

Substituting (10.5b) into (10.8) and integrating with respect to t, we obtain

$$\log(1 + k\eta) = -\frac{\mathscr{A} ik^2}{N} \exp\{ikx + Nt\}.$$

Expanding the logarithm as power series in η and neglecting the quadratic terms in η we get

$$\eta = -\frac{k}{N} \mathscr{A} i e^{ikx+Nt}. \tag{10.9}$$

The dynamic boundary condition at $y = \eta$ is

$$\frac{\mathscr{P}}{\rho} = \frac{\partial \phi}{\partial t} - g\eta, \tag{10.10}$$

where g is the acceleration due to gravity. If γ denote the surface tension, the stress conditions at the surface $y = \eta$ are:

$$T^{(yy)} = \gamma \frac{\partial^2 \eta}{\partial x^2} = -\mathscr{P} + 2\mu \frac{\partial \mathscr{V}}{\partial y}, \tag{10.11a}$$

$$T^{(xy)} = \mu \left(\frac{\partial \mathscr{V}}{\partial x} + \frac{\partial \mathscr{U}}{\partial y} \right) = 0, \tag{10.11b}$$

where μ is the dynamic viscosity of the fluid. Eliminating p between (10.10) and (10.11a), and Taylor expand the resulting equation about $y = 0$, we have

$$g\eta - \left(\frac{\partial \phi}{\partial t} + \frac{\partial \eta}{\partial t}\frac{\partial \phi}{\partial y} + \eta \frac{\partial^2 \phi}{\partial t \partial y} \right) + 2\nu \left(\frac{\partial \mathscr{V}}{\partial y} + \eta \frac{\partial^2 \mathscr{V}}{\partial y^2} \right) = \gamma \frac{\partial^2 \eta}{\partial x^2}. \tag{10.12}$$

Also Taylor expanding (10.11b) about $y = 0$ we get

$$\frac{\partial \mathscr{V}}{\partial x} + \frac{\partial \mathscr{U}}{\partial y} = -\frac{\partial \eta}{\partial x}\frac{\partial \mathscr{V}}{\partial y} - \eta \left(\frac{\partial^2 \mathscr{V}}{\partial x \partial y} + \frac{\partial^2 \mathscr{U}}{\partial y^2} \right). \tag{10.13}$$

Note (10.12) and (10.13) are expanded version of (10.11a) and (10.11b) to first order in Taylor expansion, respectively.

Substituting (10.5), (10.7) and (10.9) into (10.12) and (10.13), we obtain, respectively

$$N - 2\nu k^2 - \frac{\sigma^2}{N} = 2k^2 \left(1 - \frac{\nu k^2}{N} \right) i\mathscr{A} e^{ikx+Nt}, \tag{10.14}$$

$$2N = 3k^2 i\mathscr{A} e^{ikx+Nt}, \tag{10.15}$$

where $\sigma^2 = k(g + \gamma' k^2)$ and $\gamma' = \gamma/\rho$.

Taking the ratio (10.14) and (10.15) and solving for N, we obtain the following quadratic equation, namely

$$N^2 + 2\nu k^2 N + 3\sigma^2 = 0. \tag{10.16}$$

The roots of (10.16) are $N = -\nu k^2 \pm \sqrt{\nu^2 k^4 - 3\sigma^2}$. Now as $\nu^2 k^4 \ll 3\sigma^2$, we see that N may be expressed as $n \pm im$ with

$$n = -\nu k^2 \qquad \text{and} \qquad m = \pm\sqrt{3\sigma^2 - \nu^2 k^4} \approx \pm\sqrt{3}\sigma.$$

Finally letting $\mathscr{A} = \mp a\sigma^*/k$, upon taking the real part, we obtain from (10.9) approximately

$$\eta = a e^{-\nu k^2 t} \cos(kx \pm \sigma^* t), \tag{10.17}$$

where $\sigma^* = \sqrt{3}\sigma$.

Comparing with Lamb's [6, Section 349, eq. 21] we see that the free surface (10.17) is practically identical to that previously obtained by Lamb. Equation (10.17) shows the exponential damping factor is $\frac{1}{2}$ of that of Lamb which is in exact agreement with the direct calculation of Funda and Joseph [2]. Note that, this physically indicates that the free surface does not damps down quite as rapidly as that obtained by Lamb. Note also the velocity potential

$$\phi = -\mathscr{A} i e^{-\nu k^2 t + ky + i(kx \pm \sigma^* t)} \tag{10.18}$$

obtained in the present case has also remarkable similarity with Lamb's [6, eq. 20], except (10.18) refers to a truly irrotational motion. It should be noted that Lamb [6] constructed a solution using the Stokes equation, thereby neglecting the inertia terms on the right-hand side of equations (10.1a) and (10.1b).

It is interesting to note that the velocity potential given by (10.18) satisfies the Laplace equation $\nabla^2\phi = 0$. The gradient of this potential, namely u and v, together with the expression (10.6) for the pressure satisfy the Navier-Stokes equations. This implies that under certain circumstances, such as that presented here, a potential solution can also satisfy the full Navier-Stokes equations.

11. Conclusions

An exact analytical solution to the Navier-Stokes solutions is constructed for an unsteady viscous incompressible flow governing the motion of a monochromatic surface wave on deep water. The method adopted for the solution of unsteady Navier-Stokes equations is based on an ad hoc technique for solving non-linear partial differential equations. The method is based on disregarding the inviolate nature of the equations. The equations are then decomposed into parts, which are equated to a common factor, in such a manner that a general solution (containing the appropriate number of arbitrary functions) was constructed for one part. The form of the arbitrary functions are then obtained by means of the requirement that the other part be satisfied. The results show that, contrary to Lamb [6], the effect of viscosity on water of infinite depth leads to wave-induced motion with identically zero vorticity. Furthermore, since the rate of decay deduced here from the exact analytical solution is one half of the rate derived by Lamb [6], the free surface profile damps down exponentially in time at the rate one half of that originally given by Lamb. Moreover, the new rate of decay in exact agreement with the direct calculation of Funada and Joseph [2].

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