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COCLEFT EXTENSIONS OF MODULE COALGEBRA

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Abstract

Using crossed coproducts, module coalgebra and cocleft extensions, this paper discusses the question about coalgebra cocleft extensions and isomorphism of crossed coproducts coalgebra.

1. Preliminaries

Let H be a Hopf algebra and C be its coalgebra over a field K.

Definition 1. Assume that C is also a weak left H-comodule, and that α is a linear map $\alpha : C \to H \otimes H$, $\alpha(c) = \sum \alpha_1(c) \otimes \alpha_2(c)$, $\forall c \in C$. Then as a vector space $C \times_\alpha H = C \otimes H$ with comultiplication Δ, $\Delta(c \times h) = \sum c_1 \times c_2^1 \alpha_1(c_3) h_1 \otimes c_2^2 \times \alpha_2(c_3) h_2$, $\rho(c) = \sum c^1 \otimes c^2$ is the *left H-comodule structure map*, $\forall c \in C$, $\forall h \in H$. Here we write $c \times h$ for the tensor $c \otimes h$. We say that $C \times_\alpha H$ is a *crossed coproduct* by using ρ and α if $\varepsilon(c \times h) = \varepsilon_C(c) \varepsilon_H(h)$ is its counit and

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coassociativity are satisfied. See also in [6], this is dual of definition of crossed products.

Definition 2. We call the linear map α *normal* if

$$\varepsilon_c(c)1_H = \sum \varepsilon_H(\alpha_1(c))\alpha_2(c) = \sum \alpha_1(c)\varepsilon_H(\alpha_2(c)).$$

See also in [6, Definition 2.1].

Remark 1. The linear map α is normal and S is the antipode of H. Then

$$\varepsilon_c(c)1_H = \sum S(\alpha_1(c))\alpha_2(c) = \sum \alpha_1(c)S(\alpha_2(c)).$$

Definition 3. A coalgebra C is a *right H-module coalgebra*, if

(1) C is a right H-module via: $c \otimes h \mapsto c \cdot h$.

(2) Δ and ε are right H-module maps: $\forall h \in H$, $\forall c \in C$,

$$\Delta(c \cdot h) = \sum c_1 h_1 \otimes c_2 h_2,$$

$$\varepsilon_C(c \cdot h) = \varepsilon_C(c)\varepsilon_H(h).$$

See also in [5].

Definition 4. Let B be a right H-module. Then the invariants of H on B are the set ${}^H B = \{b \in B \mid b \cdot h = b\varepsilon(h), \forall h \in H\}$.

Similar to that in [4, Chapter 1, Definition 1.7.1].

Definition 5. Let $C \subset B$ be a K-coalgebra and H be a Hopf algebra. Then

(1) $C \subset B$ is a *right H-extension* if B is a right H-module coalgebra with ${}^H B = C$,

(2) the right H-extension $C \subset B$ is *H-cocleft* if there exists a coalgebra map $\mu : B \to H$ and $\varepsilon_H \mu = \varepsilon_B$ which is convolution invertible.

Dual of [4, Chapter 7, Definition 7.2.1].

2. Main Result

Theorem. *An H-extension $C \subset B$ is H-cocleft* $\Leftrightarrow B \cong C \times_\alpha H$.

See also in [3, Theorem 3].

Proposition 1. *Let* $C \subset B$ *be a right* H*-extension, which is* H*-cocleft via*: $\mu : B \to H$ *and* $\varepsilon_H \mu = \varepsilon_B$. *Then there is a crossed coproduct action of* C *on* H, *given by*

$$c \cdot h = \sum \mu(c_1) c_2^1 h \mu^{-1}(c_2^2)$$

and a convolution invertible map $\alpha : C \to H \otimes H$ *given by*

$$\alpha(c) = \sum \alpha_1(c) \otimes \alpha_2(c).$$

This action gives B the structure of an H-crossed coproduct over C. Moreover, the coalgebra isomorphism $\phi : C \times_\alpha H \to B$ given by

$$c \times_\alpha h \mapsto \sum c_1 \mu^{-1}(c_2) h$$

is both a left C-comodule and right H-module map, where

$$C \times_\alpha H \text{ is a right } H\text{-module via: } (c \otimes h) \cdot l = \sum c l_1 \otimes h l_2, \quad \forall h, l \in H.$$

To prove this, we need a technical lemma.

Lemma. *Assume that* $C \subset B$ *is a right* H*-extension via*: $W : C \otimes H \to C$, *and that* $C \subset B$ *is* H*-cocleft via* $\mu : B \to H$ *with* $\varepsilon_H \mu = \varepsilon_B$. *Then*

(1) $\mu^{-1} \circ W = M\tau(\mu^{-1} \otimes S)$.

(2) $\forall b \in B$, *there exists a map* $P : B \to C$ *which is both right* H*-module and coalgebra map, then* $P(b) \in C = {}^H B$.

Proof. (1) First observe that since W is a coalgebra map, $\mu^{-1} \circ W$ is the inverse of $\mu \circ W$. Then let $\mu \circ W = M(\mu \otimes I)$

$$[(\mu \circ W) * (\mu^{-1} \circ W)](c \otimes h)$$

$$= M[(\mu \circ W) \otimes (\mu^{-1} \circ W)]\Delta(c \otimes h)$$

$$= M[M(\mu \otimes I) \otimes M\tau(\mu^{-1} \otimes S)]\left(\sum c_1 \times c_2^1 \alpha_1(c_3) h_1 \otimes c_2^2 \times \alpha_2(c_3) h_2\right)$$

$$= \sum \mu(c_1)c_2^1\alpha_1(c_3)h_1 S(\alpha_2(c_3)h_2)\mu^{-1}(c_2^2)$$

$$= \sum \mu(c_1)c_2^1\alpha_1(c_3)h_1 S(h_2)S(\alpha_2(c_3))\mu^{-1}(c_2^2)$$

$$= \sum \mu(c_1)c_2^1\alpha_1(c_3)\varepsilon(h)1_H S(\alpha_2(c_3))\mu^{-1}(c_2^2)$$

$$= \sum \varepsilon(h)\mu(c_1)c_2^1\alpha_1(c_3)S(\alpha_2(c_3))\mu^{-1}(c_2^2)$$

$$= \sum \varepsilon(h)\varepsilon(c_3)\mu(c_1)c_2^1\mu^{-1}(c_2^2)$$

$$= \sum \varepsilon(h)\varepsilon(c_2)c_1 1_H$$

$$= \varepsilon(h)\varepsilon(c)1_H$$

$$= \varepsilon(c \otimes h)1_H.$$

So $\mu^{-1} \circ W$ is the right inverse of $\mu \circ W$, and so $\mu^{-1} \circ W$ by uniqueness of inverses.

(2) $P : B \to C$ is right H-module, $\forall b \in B$, $\forall h \in H$, we have: $b \cdot h \in B$, so we define that: $P(b \cdot h) \triangleq P(b)\varepsilon(h) \in C$.

And $PW(b \otimes h) = W(P \otimes I)(b \otimes h)$.

So $P(b \cdot h) = P(b) \cdot h = P(b)\varepsilon(h)$.

Then $P(b) \in C = {}^H B$. □

Remark 2. Following the condition of this lemma, we also know that:

(1) $\mu(b \cdot h) = \mu(b)h$

$$\mu^{-1}(b \cdot h) = S(h)\mu^{-1}(b).$$

(2) $\varepsilon_B = \varepsilon_C P = \varepsilon_H \mu$.

The lemma enables us to define an inverse to ϕ. Namely, define

$$\psi : B \cong C \times_\alpha H \quad \text{by} \quad b \mapsto P(b_1) \times_\alpha \mu(b_2).$$

Now, let us prove the proposition. First, we show that ϕ and ψ are mutual inverse.

$$\begin{aligned}
&\psi\phi(c \otimes h)\\
&= \psi\left(\sum c_1\mu^{-1}(c_2)h\right)\\
&= (P \otimes \mu)\Delta_c\left(\sum c_1\mu^{-1}(c_2)h\right)\\
&= (P \otimes \mu)\left(\sum c_{11}\mu^{-1}(c_{21})h_1 \otimes c_{12}\mu^{-1}(c_{22})h_2\right)\\
&= \sum P(c_{11}\mu^{-1}(c_{21})h_1) \otimes \mu(c_{12}\mu^{-1}(c_{22})h_2)\\
&= \sum P(c_{11})\varepsilon(\mu^{-1}(c_{21}))\varepsilon(h_1) \otimes \mu(c_{12})\mu^{-1}(c_{22})h_2\\
&= \sum P(c_{11}) \otimes \mu(c_{12})\varepsilon(\mu^{-1}(c_{21}))\ \mu^{-1}(c_{22})\varepsilon(h_1)h_2\\
&= \sum c_1 \otimes \mu(c_2)\mu^{-1}(c_3)h\\
&= \sum c_1 \otimes \varepsilon(c_2)h\\
&= c \otimes h.
\end{aligned}$$

On the other side, we have:

$$\begin{aligned}
&\phi\psi(b)\\
&= \phi(P \otimes \mu)\Delta(b)\\
&= \sum \phi(P(b_1) \otimes \mu(b_2))\\
&= \sum P(b_1)_1\mu^{-1}(P(b_1)_2)\mu(b_2)\\
&= \sum P(b_1)_1\varepsilon(\mu^{-1}(P(b_1)_2)\mu(b_2))\\
&= \sum P(b_1)_1\varepsilon(\mu^{-1}(P(b_1)_2))\varepsilon(\mu(b_2))
\end{aligned}$$

$$= \sum P(b_1)_1 \varepsilon(P(b_1)_2)\varepsilon(\mu(b_2))$$

$$= \sum b_1\varepsilon(\mu(b_2))$$

$$= \sum b_1\varepsilon(b_2) = b.$$

So $\psi\phi = I_{C\times_\alpha H}$, $\phi\psi = I_B$.

Next, we prove that $\phi : c \times_\alpha h \mapsto \sum c_1\mu^{-1}(c_2)h$ is a left C-comodule map. It means we need to check that $\rho_B\phi = (I \otimes \phi)\rho_{C\otimes H}$. Then, we have:

$$\rho_B\phi(c \otimes h)$$

$$= \rho_B\left(\sum c_1\mu^{-1}(c_2)h\right)$$

$$= (P \otimes I)\Delta_B\left(\sum c_1\mu^{-1}(c_2)h\right)$$

$$= \sum (P \otimes I)[(c_1\mu^{-1}(c_2)h)_1 \otimes (c_1\mu^{-1}(c_2)h)_2]$$

$$= \sum P(c_1\mu^{-1}(c_2)h)_1 \otimes (c_1\mu^{-1}(c_2)h)_2$$

$$= \sum P(c_1)_1\varepsilon(\mu^{-1}(c_2))_1\varepsilon(h_1) \otimes (c_1)_2(\mu^{-1}(c_2))_2h_2$$

$$= \sum (c_1)_1 \otimes (c_1)_2\varepsilon(\mu^{-1}(c_2))_1(\mu^{-1}(c_2))_2\varepsilon(h_1)h_2$$

$$= \sum c_1 \otimes c_2\mu^{-1}(c_3)h$$

$$(I \otimes \phi)\rho_{C\otimes H}(c \otimes h)$$

$$= (I \otimes \phi)(\Delta_C \otimes I)(c \otimes h)$$

$$= (I \otimes \phi)(\Delta_C(c) \otimes h)$$

$$= \sum (I \otimes \phi)(c_1 \otimes c_2 \otimes h)$$

$$= \sum c_1 \otimes \phi(c_2 \otimes h)$$

$$= \sum c_1 \otimes c_2 \mu^{-1}(c_3) h.$$

So $\phi : c \times_\alpha h \mapsto \sum c_1 \mu^{-1}(c_2) h$ is a left C-comodule map.

At last, it is clear that ϕ is a right H-module map. This proves Proposition 1. □

Proposition 2. *Let $C \times_\alpha H$ be a crossed coproduct, and define the map*

$$\gamma : C \times_\alpha H \to H \quad by \quad \gamma(c \otimes h) = \varepsilon(c) h.$$

Then γ is convolution invertible, with inverse $\gamma^{-1}(c \otimes h) = S(h)\varepsilon(c)$. In particular $C \hookrightarrow C \times_\alpha H$ is H-cocleft.

Proof. Set $\gamma^{-1}(c \otimes h) = S(h)\varepsilon(c)$. Then it is straightforward to verify that γ^{-1} is a right inverse for γ. For,

$$(\gamma * \gamma^{-1})(c \otimes h)$$

$$= M(\gamma \otimes \gamma^{-1})\Delta(c \otimes h)$$

$$= M(\gamma \otimes \gamma^{-1})\left(\sum c_1 \times c_2^1 \alpha_1(c_3) h_1 \otimes c_2^2 \times \alpha_2(c_3) h_2\right)$$

$$= \sum \varepsilon(c_1) c_2^1 \alpha_1(c_3) h_1 \otimes S(\alpha_2(c_3) h_2) \varepsilon(c_2^2)$$

$$= \sum \varepsilon(c_1) c_2^1 \varepsilon(c_2^2) \alpha_1(c_3) h_1 S(h_2) S(\alpha_2(c_3))$$

$$= \sum \varepsilon(c_1) c_2 \varepsilon(c_3) 1_H \varepsilon(h)$$

$$= \sum c_1 \varepsilon(c_2) 1_H \varepsilon(h) = \varepsilon(c)\varepsilon(h) 1_H = \varepsilon(c \otimes h) 1_H. \qquad \square$$

Similarly, we can check that γ is the right inverse of γ^{-1}. This proves Proposition 2. We have also proved theorem by combining Proposition 1 and Proposition 2.

Corollary 1. *Let $C \times_\alpha H$ be a crossed coproduct, and $C \times_\alpha H \cong C \otimes H$ be left C-comodule. Then $C^{cop} \times_\alpha H^{cop} \to B^{cop}$ as left C^{cop}-comodule map, provided the antipode S of H is bijective.*

Proof. First, from Proposition 2 we know that the map

$$\gamma : C \times_\alpha H \to H \quad \text{by} \quad \gamma(c \otimes h) = \varepsilon(c)h,$$

which is an invertible right H-module map, the module structure maps for $C \times_\alpha H$ and H are given by $f = (I \otimes M)$ and M, respectively.

Let $\overline{S}$ denote the inverse of S and set $\mu = \overline{S}\gamma$. Note that μ is invertible under twist convolution with $\mu^{-1} = \overline{S}\gamma^{-1}$. It follows that if $B \cong C \times_\alpha H$, then B^{cop} is a right H^{cop}-module coalgebra which is cocleft via: $\mu : B^{cop} \to H^{cop}$. Thus, by Proposition 1, $C^{cop} \times_\alpha H^{cop} \to B^{cop}$ as left C^{cop}-comodule, where the map is given by

$$c^{cop} \times_\alpha h^{cop} \mapsto \sum c_2^{cop}\mu^{-1}(c_1^{cop})h^{cop} = \sum c_2^{cop}\overline{S}\gamma^{-1}(c_1^{cop})h^{cop}. \qquad \square$$

Corollary 2. *Let* $C \times_\alpha H$ *be a crossed coproduct. Then* $C \times_\alpha H \to H \otimes C$ *is right* C*-comodule map*.

Proof. According to the theorem we know that $B \cong C \times_\alpha H$. We also know that

$$g : B \to H \otimes C \text{ by } g(b) = \sum \mu(b_1) \otimes P(b_2),$$

which is left C-comodule map, the comodule structure maps for B and $H \otimes C$ are given by $\rho_B = (I \otimes P)\Delta$ and $I \otimes \Delta$, respectively.

So $C \times_\alpha H \to H \otimes C$ is right C-comodule map, where the map is given by

$$g : c \times_\alpha h \mapsto \varepsilon(c_1)h \otimes P(c_2).$$

In other words, we need to check that: $(I \otimes \Delta)g = (g \otimes I)\rho_{C\times_\alpha H}$, then we have $\rho_{C\times_\alpha H}(c \otimes h) = \sum c_1 \otimes h \otimes P(c_2)$,

$$(I \otimes \Delta)g(c \otimes h)$$

$$= \sum (I \otimes \Delta)(\varepsilon(c_1)h \otimes P(c_2))$$

$$= \sum \varepsilon(c_1)h \otimes P(c_2) \otimes P(c_3)$$

and

$$(g \otimes I)\rho_{C \times_\alpha H}(c \otimes h)$$

$$= (g \otimes I)\left(\sum c_1 \otimes h \otimes P(c_2)\right)$$

$$= \sum g(c_1 \otimes h) \otimes P(c_2)$$

$$= \sum \varepsilon(c_1) h \otimes P(c_2) \otimes P(c_3).$$

So $C \times_\alpha H \to H \otimes C$ is right C-comodule map. □

References

[1] Hai Bin Kan, Cohomologous transformations of cocycles for bialgebras (Chinese), Chinese Ann. Math. Ser. A 22(2) (2001), 141-150; translation in Chinese J. Contemp. Math. 22(2) (2001), 125-136

[2] Jin Qi Li, $[C, H]$-Hopf modules and a structure theorem for *H*-module coalgebras, Chinese Ann. Math. Ser. A 16 (3) (1995), 312-319.

[3] Gui-Long Liu, The structure of cocleft *H*-module coalgebra, Chinese Science Bull. 38(20) (1993), 1688-1688.

[4] Susan Montgomery, Hopf algebras and their actions on rings, CBMS Regional Conference Series in Mathematics, 82, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1993.

[5] M. E. Sweedler, Hopf Algebras, W. A. Benjamin, Inc., New York, 1996.

[6] Shuan Hong Wang, *H*-weak comodule coalgebras and crossed coproducts of Hopf algebras, Chinese Ann. Math. Ser. A 16(4) (1995), 471-479.