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FRACTIONAL-STEP METHOD FOR SPECTRAL ACTION BALANCE EQUATION

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Abstract

Spectral action balance equation is an equation that used to simulate short-crested wind-generated waves in shallow water areas such as coastal regions and inland waters. This equation consists of two spatial dimensions, wave direction and wave frequency. In this paper, we introduce the splitting scheme and proved that it is consistent to the central spaces scheme with same accuracy. This numerical scheme was adopted to split the wave spectral action balance equation into four one-dimensional problems, which for each small problem obtains the independently tridiagonal linear systems. Therefore, we can solve these systems by direct or iterative methods at the same time which is very fast when performed by a multi-core computer. The numerical results of these two methods are very close as shown in the numerical experiments.

1. Introduction

A third-generation model is a number of advanced spectral wind-wave models. It has been developed such as WAM model of the WAMDI Group [5], in which all

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processes of wave generation, dissipation and nonlinear wave-wave interactions are accounted for explicitly. WAM model considers problems on oceanic scales, and make use of explicit propagation schemes in geographical and spectral spaces. Tolman [6] developed model based on spectral action balance equation, WAVEWATCH model incorporates all relevant wave-current interaction mechanism, including changes of absolute frequencies due to unsteadiness of depth and currents. The model explicitly accounts for growth and decay of wave energy and for nonlinear resonant wave-wave interactions. Booij et al. [1] and Ris et al. [4] summarized the research attainment in the wave energy, dissipation and nonlinear wave-wave interactions, and developed the third-generation for coastal region in shallow water, SWAN (Simulating WAve Nearshore) model, which can be applied in coastal zones, lakes and estuaries. The model uses the spectral action balance equation to represent the process of wave shoaling, refraction, bottom friction, depth-induced wave breaking, whitecapping, wind input and nonlinear wave-wave interactions reasonably.

For the numerical treatment of the spectral action balance equation, we use the finite difference method with backward time central spaces. The discretization yields a system of linear equation banded-9 that needs to be solved. This system can be solved with a Gauss-Seidel iterative method.

In recent years, computers' evolution is going dramatically fast. Computers have been improved a lot and become much more powerful. One of the new types of computers is a multi-processing computer. So, we should develop algorithms that support and could be suitable for this evolution. In this paper, we introduce the method of fractional steps (see Yanenko [8]) for solving spectral action balance equation. This method splits the original four-dimensional space problem into a set of one-dimensional space problems. At each step, one has to solve m independent one-dimensional space systems of linear equations where m is the number of one-dimensional space problems in appropriate direction. Therefore, we can solve each of these systems of linear equations at every step by m independent parallel processors. This method is preferable for multi-processing computers.

2. Finite Difference Approximation for Spectral Action Balance Equation

We consider the wave spectral action balance equation which described the wave characteristic:

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial x}(c_x N) + \frac{\partial}{\partial y}(c_y N) + \frac{\partial}{\partial \sigma}(c_\sigma N) + \frac{\partial}{\partial \theta}(c_\theta N) = \frac{S}{\sigma},$$

$$\forall\ (x,\ y,\ \sigma,\ \theta) \in \Omega \times \Gamma, \quad t \in [0,\ T],$$

$$\left.\frac{\partial N}{\partial n}\right|_{\partial\Omega \times \partial\Gamma} = 0, \quad t \in [0,\ T],$$

$$N\,|_{t=0} = N_0(x,\ y,\ \sigma,\ \theta), \quad \forall(x,\ y,\ \sigma,\ \theta) \in \Omega \times \Gamma, \tag{1}$$

where Ω and Γ are domains in geographical and spectral $\mathbb{R}^2$, $\partial\Omega$ is boundary of Ω, $\partial\Gamma$ is boundary of Γ, $N_0(x,\ y,\ \sigma,\ \theta)$ is an initial value, and n is a normal direction of each variable, which $N(x,\ y,\ \sigma,\ \theta,\ t)$ is the action density as a function of relative frequency σ, direction θ, horizontal coordinates x and y, and time t. c_x, c_y, c_σ and c_θ are propagation velocities in $-x$, $-y$, $-\sigma$ and $-\theta$ directions, respectively. The first term of the left-hand side of equation (1) represents the local rate of change of action density in time, the second and the third term represent propagation of action density in geographical space, with propagation velocities c_x and c_y in x and y spaces, respectively. The fourth term represents shifting of relative frequency due to variations in depths and currents with propagation velocity c_σ in σ space. The fifth term represents depth-induced and current-induced refractions with propagation velocity c_θ in θ space. And the right hand side the term S is the source term. More details are given in Booij et al. [1] and Ris et al. [4].

We choose a rectangular grid with constant mesh sizes Δx and Δy in x- and y-directions, respectively. The spectral space is divided into elementary bins with a constant directional resolution $\Delta\theta$ and a constant relative frequency resolution $\Delta\sigma/\sigma$. We denote the grid counters as $1 \le i \le N_x$, $1 \le j \le N_y$, $1 \le l \le N_\sigma$ and $1 \le m \le N_\theta$ in x-, y-, σ- and θ- spaces, respectively. All variables are located at points $(i,\ j,\ l,\ m)$. By the finite difference approximation using the backward time central geographical and spectral spaces, we obtain the following approximation of equation (1):

$$\left.\frac{N^n - N^{n-1}}{\Delta t}\right|_{i,\,j,\,l,\,m} + \left.\frac{[c_x N]_{i+\frac{1}{2}} - [c_x N]_{i-\frac{1}{2}}}{\Delta x}\right|_{j,\,l,\,m}^{n}$$

$$+ \left. \frac{[c_y N]_{j+\frac{1}{2}} - [c_y N]_{j-\frac{1}{2}}}{\Delta y} \right|_{i,l,m}^{n}$$

$$+ \left. \frac{[c_\sigma N]_{l+\frac{1}{2}} - [c_\sigma N]_{l-\frac{1}{2}}}{\Delta \sigma} \right|_{i,j,m}^{n}$$

$$+ \left. \frac{[c_\theta N]_{m+\frac{1}{2}} - [c_\theta N]_{m-\frac{1}{2}}}{\Delta \theta} \right|_{i,j,l}^{n} = \left. \frac{S}{\sigma_l} \right|_{i,j,l,m}^{n-1}, \tag{2}$$

where n is a time-level with Δt a time step. We can approximate $[c_x N]_{i+\frac{1}{2}}$, $[c_x N]_{i-\frac{1}{2}}$, $[c_y N]_{j+\frac{1}{2}}$, $[c_y N]_{j-\frac{1}{2}}$, $[c_\sigma N]_{l+\frac{1}{2}}$, $[c_\sigma N]_{l-\frac{1}{2}}$, $[c_\theta N]_{m+\frac{1}{2}}$ and $[c_\theta N]_{m-\frac{1}{2}}$ by using

$$N_{k+\frac{1}{2}} = \frac{N_k + N_{k+1}}{2}, \quad N_{k-\frac{1}{2}} = \frac{N_k + N_{k-1}}{2}, \quad \text{where} \quad k = i,\ j,\ l,\ m. \tag{3}$$

Substituting this equation into (2) and rearranging them, then we have the following equation:

$$N_{i,j,l,m}^{n} + \left. \left(\frac{c_x \Delta t}{2\Delta x}\right) \right|_{i+1,j,l,m} (N_{i+1,j,l,m}^{n}) - \left. \left(\frac{c_x \Delta t}{2\Delta x}\right) \right|_{i-1,j,l,m} (N_{i-1,j,l,m}^{n})$$

$$+ \left. \left(\frac{c_y \Delta t}{2\Delta y}\right) \right|_{i,j+1,l,m} (N_{i,j+1,l,m}^{n}) - \left. \left(\frac{c_y \Delta t}{2\Delta y}\right) \right|_{i,j-1,l,m} (N_{i,j-1,l,m}^{n})$$

$$+ \left. \left(\frac{c_\sigma \Delta t}{2\Delta \sigma}\right) \right|_{i,j,l+1,m} (N_{i,j,l+1,m}^{n}) - \left. \left(\frac{c_\sigma \Delta t}{2\Delta \sigma}\right) \right|_{i,j,l-1,m} (N_{i,j,l-1,m}^{n})$$

$$+ \left. \left(\frac{c_\theta \Delta t}{2\Delta \theta}\right) \right|_{i,j,l,m+1} (N_{i,j,l,m+1}^{n}) - \left. \left(\frac{c_\theta \Delta t}{2\Delta \theta}\right) \right|_{i,j,l,m-1} (N_{i,j,l,m-1}^{n})$$

$$= \left. \left(\frac{\Delta t}{\sigma_l}\right) \right|_{i,j,l,m} S_{i,j,l,m}^{n-1} + N_{i,j,l,m}^{n-1}, \tag{4}$$

where $i = 1, 2, \ldots, N_x$; $j = 1, 2, \ldots, N_y$; $l = 1, 2, \ldots, N_\sigma$; $m = 1, 2, \ldots, N_\theta$.

From Figure 1, we can see that the structure of the coefficient matrix of the linear system of the spectral action balance equation be in the form of banded-9 diagonal. This linear system can be solved by any direct and iterative method under the diagonal dominant condition, that is, for each row of the coefficient matrix, the sum of absolute off diagonal entry must be less than the absolute main diagonal.

Now, we are analyzing the criteria of $\Delta t, \Delta x, \Delta y, \Delta\sigma$ and $\Delta\theta$ for existent and uniqueness solution of this linear system. Let us consider the diagonal dominant condition

$$\begin{aligned}1 > &\left|\left(\frac{\Delta t}{2\Delta x}\right)(c_x)\Big|_{i+1, j, l, m}\right| + \left|\left(\frac{\Delta t}{2\Delta x}\right)(c_x)\Big|_{i-1, j, l, m}\right| \\ &+ \left|\left(\frac{\Delta t}{2\Delta y}\right)(c_y)\Big|_{i, j+1, l, m}\right| + \left|\left(\frac{\Delta t}{2\Delta y}\right)(c_y)\Big|_{i, j-1, l, m}\right| \\ &+ \left|\left(\frac{\Delta t}{2\Delta \sigma}\right)(c_\sigma)\Big|_{i, j, l+1, m}\right| + \left|\left(\frac{\Delta t}{2\Delta \sigma}\right)(c_\sigma)\Big|_{i, j, l-1, m}\right| \\ &+ \left|\left(\frac{\Delta t}{2\Delta \theta}\right)(c_\theta)\Big|_{i, j, l, m+1}\right| + \left|\left(\frac{\Delta t}{2\Delta \theta}\right)(c_\theta)\Big|_{i, j, l, m-1}\right| \\ &\equiv \text{cond}, \end{aligned} \tag{5}$$

where $i = 1, 2, ..., N_x;\ \ j = 1, 2, ..., N_y;\ l = 1, 2, ..., N_\sigma;\ m = 1, 2, ..., N_\theta$.

Next, we try to simplify these stability criteria, by letting

$$\begin{aligned} M_x &\equiv \max_{\forall i, j, l, m} c_x \,|_{i, j, l, m}, \qquad M_y \equiv \max_{\forall i, j, l, m} c_y \,|_{i, j, l, m}, \\ M_\sigma &\equiv \max_{\forall i, j, l, m} c_\sigma \,|_{i, j, l, m}, \qquad M_\theta \equiv \max_{\forall i, j, l, m} c_\theta \,|_{i, j, l, m}. \end{aligned} \tag{6}$$

Substituting the notation (6) into the equations (5) and (6), yields

$$\frac{(\Delta t)(M_x)}{\Delta x} + \frac{(\Delta t)(M_y)}{\Delta y} + \frac{(\Delta t)(M_\sigma)}{\Delta \sigma} + \frac{(\Delta t)(M_\theta)}{\Delta \theta} > \text{cond}. \tag{7}$$

Since

$$\begin{aligned} &4(\Delta t)\max\{M_x, M_y, M_\sigma, M_\theta\} \\ &> \frac{(\Delta t)(M_x)}{\Delta x} + \frac{(\Delta t)(M_y)}{\Delta y} + \frac{(\Delta t)(M_\sigma)}{\Delta \sigma} + \frac{(\Delta t)(M_\theta)}{\Delta \theta} > \text{cond}, \end{aligned} \tag{8}$$

therefore

$$4(\Delta t)\max\{M_x, M_y, M_\sigma, M_\theta\} < 1. \tag{9}$$

Thus the condition of Δt that satisfies the diagonal dominant condition of the linear system is as following:

$$\Delta t < \frac{1}{4\max\{M_x, M_y, M_\sigma, M_\theta\}}. \tag{10}$$

3. Fractional-step Method

In the previous section, the central difference scheme of the spectral action balance equation was described with a very huge coefficient matrix that needs to be solved by any direct and iterative method that takes a lot of computer capacity and operation count.

In this section, we will design a new numerical scheme that reduces the size of the original problem by splitting the original problem into 4 smaller problems. For each smaller problem can be solved easier than the original problem and takes less computer's resource. This method is called "*fractional-step method*."

Let us consider the spectral action balance equation on a domain $\Omega \times \Gamma$ with boundary $\partial\Omega \times \partial\Gamma$:

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial x}(c_x N) + \frac{\partial}{\partial y}(c_y N) + \frac{\partial}{\partial \sigma}(c_\sigma N) + \frac{\partial}{\partial \theta}(c_\theta N) = \frac{S}{\sigma}. \tag{11}$$

Let $\Lambda_x, \Lambda_y, \Lambda_\sigma$ and Λ_θ be approximation operators of $\frac{\partial}{\partial x} c_x(\cdot)$, $\frac{\partial}{\partial y} c_y(\cdot)$, $\frac{\partial}{\partial \sigma} c_\sigma(\cdot)$ and $\frac{\partial}{\partial \theta} c_\theta(\cdot)$, respectively. For each point $(x_i, y_j, \sigma_l, \theta_m, t_k)$, the approximate operators are represented as following:

$$\left.\frac{\partial}{\partial x}(c_x N)\right|_{ijlm}^{k} \approx \Lambda_x N_{ijlm}^k,$$

$$\left.\frac{\partial}{\partial y}(c_y N)\right|_{ijlm}^{k} \approx \Lambda_y N_{ijlm}^k,$$

$$\frac{\partial}{\partial \sigma}(c_\sigma N)\Big|_{ijlm}^{k} \approx \Lambda_\sigma N_{ijlm}^{k},$$

$$\frac{\partial}{\partial \theta}(c_\theta N)\Big|_{ijlm}^{k} \approx \Lambda_\theta N_{ijlm}^{k}.$$

Therefore, the equation (11) can be approximated for each point $(x_i,\ y_j,\ \sigma_l,\ \theta_m,\ t)$ by the following:

$$\frac{\partial N}{\partial t}\Big|_{ijlm} + \Lambda_x N_{ijlm} + \Lambda_y N_{ijlm} + \Lambda_\sigma N_{ijlm} + \Lambda_\theta N_{ijlm} = \frac{S_{ijlm}}{\sigma_l}.$$

For convenient writing, the indices *i*, *j*, *l*, *m* are neglected, yields

$$\frac{\partial N}{\partial t} + \Lambda_x N + \Lambda_y N + \Lambda_\sigma N + \Lambda_\theta N = \frac{S}{\sigma}.$$

Let us consider the backward time of spectral action balance equation at point $(x_i,\ y_j,\ \sigma_l,\ \theta_m)$,

$$\frac{N^{k+1} - N^{k}}{\tau} + \Lambda_x N^{k+1} + \Lambda_y N^{k+1} + \Lambda_\sigma N^{k+1} + \Lambda_\theta N^{k+1} = \frac{S^{k+1}}{\sigma}, \tag{12}$$

and we introduce the splitting scheme

$$\frac{N^{k+\frac{1}{4}} - N^{k}}{\tau} + \Lambda_x N^{k+\frac{1}{4}} = 0, \tag{13}$$

$$\frac{N^{k+\frac{2}{4}} - N^{k+\frac{1}{4}}}{\tau} + \Lambda_y N^{k+\frac{2}{4}} = 0, \tag{14}$$

$$\frac{N^{k+\frac{3}{4}} - N^{k+\frac{2}{4}}}{\tau} + \Lambda_\sigma N^{k+\frac{3}{4}} = 0, \tag{15}$$

$$\frac{N^{k+1} - N^{k+\frac{3}{4}}}{\tau} + \Lambda_\theta N^{k+1} = \frac{S^{k+1}}{\sigma}. \tag{16}$$

Now, we will prove that (13)-(16) are consistent with (12) by rearranging the equations (13)-(16), yields

$$-N^{k}+(I+\tau\Lambda_{x})N^{k+\frac{1}{4}}=0, \tag{17}$$

$$-N^{k+\frac{1}{4}}+(I+\tau\Lambda_{y})N^{k+\frac{2}{4}}=0, \tag{18}$$

$$-N^{k+\frac{2}{4}}+(I+\tau\Lambda_{\sigma})N^{k+\frac{3}{4}}=0, \tag{19}$$

$$-N^{k+\frac{3}{4}}+(I+\tau\Lambda_{\theta})N^{k+1}=\frac{\tau S^{k+1}}{\sigma}, \tag{20}$$

where I is the identity approximation operator. To eliminate $N^{k+\frac{1}{4}}$ in (17) and (18), we multiply the equation (18) by $(I+\tau\Lambda_{x})$ and adding the result to (17), then we obtain

$$-N^{k}+(I+\tau\Lambda_{x})(I+\tau\Lambda_{y})N^{k+\frac{2}{4}}=0. \tag{21}$$

Next, we are going to eliminate $N^{k+\frac{2}{4}}$ in (19) and (21), we multiply (19) by $(I+\tau\Lambda_{x})(I+\tau\Lambda_{y})$ and adding the result to (21), then

$$-N^{k}+(I+\tau\Lambda_{x})(I+\tau\Lambda_{y})(I+\tau\Lambda_{\sigma})N^{k+\frac{3}{4}}=0. \tag{22}$$

Similarly, to eliminate $N^{k+\frac{3}{4}}$ in (20) and (22), we multiply (20) by $(I+\tau\Lambda_{x})(I+\tau\Lambda_{y})(I+\tau\Lambda_{\sigma})$ and adding the result to (22), then

$$\begin{aligned}&-N^{k}+(I+\tau\Lambda_{x})(I+\tau\Lambda_{y})(I+\tau\Lambda_{\sigma})(I+\tau\Lambda_{\theta})N^{k+1}\\&=(I+\tau\Lambda_{x})(I+\tau\Lambda_{y})(I+\tau\Lambda_{\sigma})\left(\frac{\tau S^{k+1}}{\sigma}\right).\end{aligned} \tag{23}$$

Since

$$(I+\tau\Lambda_{x})(I+\tau\Lambda_{y})(I+\tau\Lambda_{\sigma})=I+\tau(\Lambda_{\sigma}+\Lambda_{x}+\Lambda_{y})+O(\tau^{2}) \tag{24}$$

and

$$\begin{aligned}&(I+\tau\Lambda_{x})(I+\tau\Lambda_{y})(I+\tau\Lambda_{\sigma})(I+\tau\Lambda_{\theta})\\&=I+\tau(\Lambda_{x}+\Lambda_{y}+\Lambda_{\sigma}+\Lambda_{\theta})+O(\tau^{2}),\end{aligned} \tag{25}$$

substituting (24) and (25) into (23), yields

$$-N^{k} + [I + \tau(\Lambda_x + \Lambda_y + \Lambda_\sigma + \Lambda_\theta) + O(\tau^2)]N^{k+1}$$

$$= [I + \tau(\Lambda_x + \Lambda_y + \Lambda_\sigma) + O(\tau^2)]\left[\frac{\tau S^{k+1}}{\sigma}\right], \tag{26}$$

then

$$\frac{N^{k+1} - N^{k}}{\tau} + \Lambda_x N^{k+1} + \Lambda_y N^{k+1} + \Lambda_\sigma N^{k+1} + \Lambda_\theta N^{k+1} = \frac{S^{k+1}}{\sigma}, \; O(\tau). \tag{27}$$

Therefore, the scheme (27) and the equivalent scheme (13)-(16) approximate the spectral action balance equation with the same accuracy $O(\tau)$ as the scheme (12).

For the stability criteria for each system, we must choose the type of approximate operators Λ_x, Λ_y, Λ_σ and Λ_θ. Here, we choose these approximate operators as central difference approximation and apply the central difference approximation with the equations (13)-(16), yields

$$\left.\frac{N^{n+\frac{1}{4}} - N^{n}}{\tau}\right|_{i,\,j,\,l,\,m} + \frac{1}{2\Delta x}[(c_x)_{i+1,\,j,\,l,\,m}(N^{n+\frac{1}{4}}_{i+1,\,j,\,l,\,m})$$

$$- (c_x)_{i-1,\,j,\,l,\,m}(N^{n+\frac{1}{4}}_{i-1,\,j,\,l,\,m})] = 0, \tag{28}$$

$$\left.\frac{N^{n+\frac{2}{4}} - N^{n+\frac{1}{4}}}{\tau}\right|_{i,\,j,\,l,\,m} + \frac{1}{2\Delta y}[(c_y)_{i,\,j+1,\,l,\,m}(N^{n+\frac{2}{4}}_{i,\,j+1,\,l,\,m})$$

$$- (c_y)_{i,\,j-1,\,l,\,m}(N^{n+\frac{2}{4}}_{i,\,j-1,\,l,\,m})] = 0, \tag{29}$$

$$\left.\frac{N^{n+\frac{3}{4}} - N^{n+\frac{2}{4}}}{\tau}\right|_{i,\,j,\,l,\,m} + \frac{1}{2\Delta \sigma}[(c_\sigma)_{i,\,j,\,l+1,\,m}(N^{n+\frac{3}{4}}_{i,\,j,\,l+1,\,m})$$

$$- (c_\sigma)_{i,\,j,\,l-1,\,m}(N^{n+\frac{3}{4}}_{i,\,j,\,l-1,\,m})] = 0, \tag{30}$$

$$\left.\frac{N^{n+1} - N^{n+\frac{3}{4}}}{\tau}\right|_{i,\, j,\, l,\, m} + \frac{1}{2\Delta\theta}\left[(c_\theta)_{i,\, j,\, l,\, m+1}(N^{n+1}_{i,\, j,\, l,\, m+1})\right.$$

$$\left. - (c_\theta)_{i,\, j,\, l,\, m-1}(N^{n+1}_{i,\, j,\, l,\, m-1})\right] = \frac{S^{n+1}_{i,\, j,\, l,\, m}}{\sigma}. \tag{31}$$

From the splitting scheme, we get four-tridiagonal systems and solve these systems under the diagonal dominant condition, by considering

$$\tau_x < \frac{2\Delta x}{|(c_x)_{i-1,\, j,\, l,\, m} + (c_x)_{i+1,\, j,\, l,\, m}|},$$

$$\tau_y < \frac{2\Delta y}{|(c_y)_{i,\, j-1,\, l,\, m} + (c_y)_{i,\, j+1,\, l,\, m}|},$$

$$\tau_\sigma < \frac{2\Delta \sigma}{|(c_\sigma)_{i,\, j,\, l-1,\, m} + (c_\sigma)_{i,\, j,\, l+1,\, m}|},$$

$$\tau_\theta < \frac{2\Delta \theta}{|(c_\theta)_{i,\, j,\, l,\, m-1} + (c_\theta)_{i,\, j,\, l,\, m+1}|}, \tag{32}$$

where $i = 1, 2, \ldots, N_x$; $j = 1, 2, \ldots, N_y$; $l = 1, 2, \ldots, N_\sigma$; $m = 1, 2, \ldots, N_\theta$ and $\tau_x, \tau_y, \tau_\sigma, \tau_\theta$ are time splitting of equations (28), (29), (30) and (31), respectively. We let

$$M_x \equiv \max_{\forall i,\, j,\, l,\, m} c_x|_{i,\, j,\, l,\, m}, \quad M_y \equiv \max_{\forall i,\, j,\, l,\, m} c_y|_{i,\, j,\, l,\, m},$$

$$M_\sigma \equiv \max_{\forall i,\, j,\, l,\, m} c_\sigma|_{i,\, j,\, l,\, m}, \quad M_\theta \equiv \max_{\forall i,\, j,\, l,\, m} c_\theta|_{i,\, j,\, l,\, m}. \tag{33}$$

Substituting the notation (33) into the equations (32) and (33), yields

$$\tau_x < \frac{\Delta x}{M_x} < \frac{2\Delta x}{|(c_x)_{i-1,\, j,\, l,\, m} + (c_x)_{i+1,\, j,\, l,\, m}|},$$

$$\tau_y < \frac{\Delta y}{M_y} < \frac{2\Delta y}{|(c_y)_{i,\, j-1,\, l,\, m} + (c_y)_{i,\, j+1,\, l,\, m}|},$$

$$\tau_\sigma < \frac{\Delta \sigma}{M_\sigma} < \frac{2\Delta \sigma}{|(c_\sigma)_{i,\, j,\, l-1,\, m} + (c_\sigma)_{i,\, j,\, l+1,\, m}|},$$

$$\tau_\theta < \frac{\Delta\theta}{M_\theta} < \frac{2\Delta\theta}{|(c_\theta)_{i,j,l,m-1} + (c_\theta)_{i,j,l,m+1}|}. \tag{34}$$

Thus the condition of τ that satisfies the diagonal dominant of the linear system of splitting scheme is the following:

$$\tau < \min\left\{\frac{\Delta x}{M_x}, \frac{\Delta y}{M_y}, \frac{\Delta\sigma}{M_\sigma}, \frac{\Delta\theta}{M_\theta}\right\}. \tag{35}$$

4. Numerical Experiments

In this section, we collect some results calculated using the scheme in the last two sections. We wish to emphasize the diversity of the possible applications.

We begin with spectral action balance equation:

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial x}(c_x N) + \frac{\partial}{\partial y}(c_y N) + \frac{\partial}{\partial \sigma}(c_\sigma N) + \frac{\partial}{\partial \theta}(c_\theta N) = \frac{S}{\sigma},$$

$$\forall (x, y, \sigma, \theta) \in \Omega \times \Gamma, \tag{36}$$

where $t \in [0, T]$, and the initial and boundary conditions are defined as follows:

$$N|_{t=0} = N_0(x, y, \sigma, \theta), \qquad \forall (x, y, \sigma, \theta) \in \Omega \times \Gamma; \tag{37}$$

$$\frac{\partial N}{\partial n} = 0, \qquad \forall (x, y, \sigma, \theta) \in \partial\Omega \times \partial\Gamma, \qquad t \in [0, T]. \tag{38}$$

The specific parameters used in our calculations are as follows:

$$x_l \le x \le x_r, \quad y_l \le y \le y_r, \quad \sigma_l \le \sigma \le \sigma_r, \quad \theta_l \le \theta \le \theta_r,$$

$$x_l = -0.1, \quad x_r = 0.1, \quad y_l = -0.1, \quad y_r = 0.1,$$

$$\sigma_l = 0.04, \quad \sigma_r = 1, \quad \theta_l = 0, \quad \theta_r = 2\pi,$$

$$N_x = 20, \quad N_y = 20, \quad N_\sigma = 20, \quad N_\theta = 20,$$

$$\Delta x = \frac{x_r - x_l}{N_x - 1}, \quad \Delta y = \frac{y_r - yx_l}{N_y - 1}, \quad \Delta\sigma = \frac{\sigma_r - \sigma_l}{N_\sigma - 1}, \quad \Delta\theta = \frac{\theta_r - \theta_l}{N_\theta - 1}, \tag{39}$$

and source term:

$$S(0,\ x_2,\ y_2,\ \sigma_{(N_\sigma+1)/2},\ \theta_{(N_\theta+1)/2}) = 200,$$

$$S(t,\ x,\ y,\ \sigma,\ \theta) = 0, \quad \forall x,\ y,\ \sigma,\ \theta \in \Omega \times \Gamma, \quad t > 0.$$

In this experiment, we simulate a spectral action balance equation in a square domain. The physical configuration consists of a square container filled with wave energy. The central difference and splitting scheme are presented. First, we set the initial values of N as zero for every node in the domain $\Omega \times \Gamma.$ At the initial time, we filled the wave energy into the bottom-left of the domain. At first time step, the energy peaked at that grid point and after that it moves along the direction field of the propagation velocities. The numerical results by these two methods are very similar as shown in Figures 2 and 3.

5. Conclusions and Discussions

We have analyzed the splitting schemes for numerical solution of the spectral action balance equation (1) with time splitting. This method has first-order accuracy in time and second-order accuracy in geographical and spectral spaces. This numerical scheme was adopted to split the wave spectral action balance equation into four one-dimensional problems, which for each small problem obtains the independently tridiagonal linear systems. Therefore, we can solve these systems by direct or iterative methods at the same time which is very fast when performed by a multi-core computer. The numerical results of these two methods are very close as shown in the numerical experiments.

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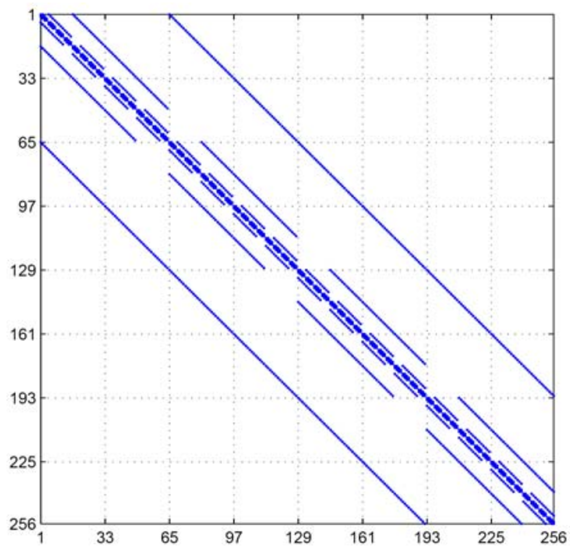

Figure 1. The structural coefficient matrix of the spectral action balance equation by central difference scheme.

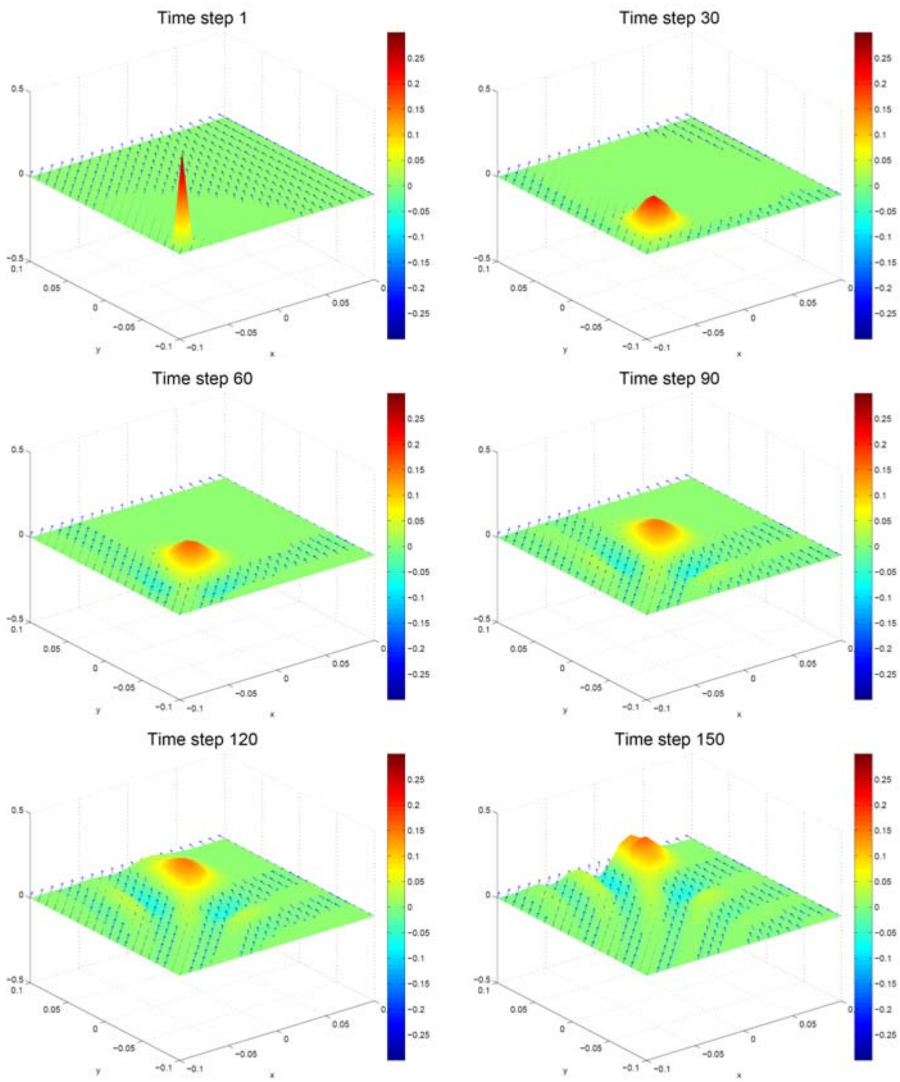

Figure 2. Numerical results for every 30 time step by using central difference scheme.

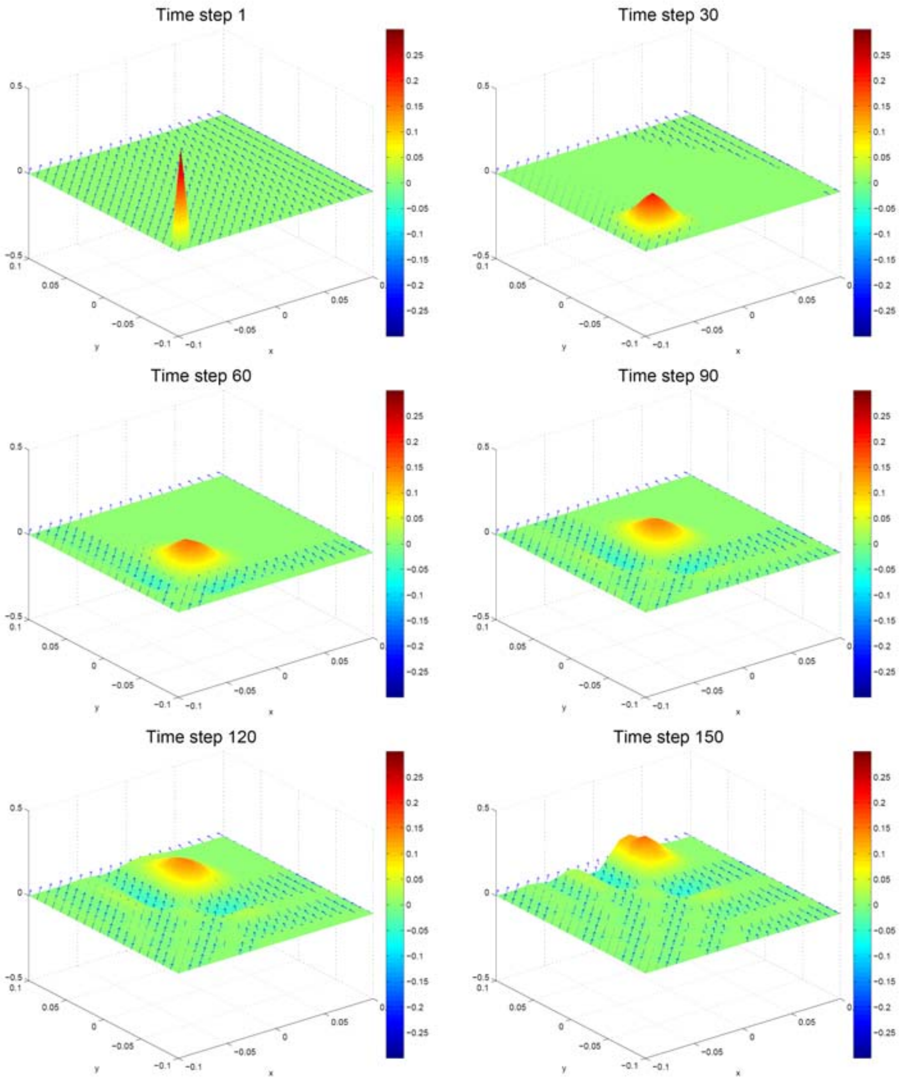

Figure 3. Numerical results for every 30 time step by using splitting scheme.