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HOW MANY LATIN RECTANGLES ARE THERE?

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Abstract

Until now the problem of counting Latin rectangles $m \times n$ has been solved with an explicit formula for $m = 2, 3$ and 4 only. In the present paper, an explicit formula is provided for the calculation of the number of Latin rectangles for any order m. The results attained up to now become particular cases of this new formula. Furthermore, putting $m = n$, the number of Latin squares of order n can also be obtained in an explicit form.

0. Introduction

A Latin rectangle $m \times n$ is a matrix with n rows and n columns the elements of which are chosen in $[n] = \{1, ..., n\}$ so that two elements are never the same, neither on the same row nor on the same column. It is said that such a Latin rectangle has order m. From the definition it follows that $m \leq n$ and that each row of a Latin rectangle is a permutation of $[n]$.

Furthermore, it is clear that it is always possible to standardize the first row making it the same as the permutation $1 2 \cdots n.$ In such a case, we say that we are dealing with a "reduced Latin rectangle".

If we call the number of Latin rectangles $m \times n$ with $L(m, n)$ and the number

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of reduced Latin rectangles with the same dimension with $K(m, n)$, then it is clear that $L(m, n) = n! K(m, n)$.

The problem of counting Latin rectangles has engaged several generations of mathematicians, but the results reached up to now, as we will see later, are limited to certain special cases.

With this paper, we intend to finally supply an explicit formula for the calculation of $K(m, n)$ for any value of the order m.

The partial results attained up to now will result special cases of such a formula.

The result obviously also allows the calculation of the number of Latin squares of order n, which represent the special case of $m \times n$ Latin rectangles in which m takes on its maximum admissible value n.

1. A Brief Survey of Results

A Latin rectangle consists of m permutations of $[n]$ which, taken two by two, do not have fixed points. It is from this point of view that the problem was initially studied by Montmort, Euler and Lucas.

It seems that the solution in the simplest case $m = 2$, known as "derangement problem", can be found going back to Montmort [6]. It consists of the number D_n of the permutations of $[n]$ without fixed points given by:

$$D_n = \sum_{0}^{n} {}_k (-1)^k \frac{n!}{k!} \tag{1.1}$$

and equivalent to $K(2, n)$.

In 1891, Lucas expounded the famous "ménage problem" which consists of counting the ways of arranging n couples at a round table so that men and women alternate and no husband and wife are adjacent to one another. The problem, examined since 1878 by Tate, was also studied by Cayley and Muir, however no satisfactory results were reached.

The solution to ménage problem is equivalent to the enumeration of all the permutations of $[n]$ which are discordant with both the permutations $12 \cdots n$ and $23 \cdots n1$. The generalisation of the above mentioned problem – known as "the cyclical Touchard problem of index m" or "problem of m-discordant permutations" –

proposes the counting of all the permutations σ of $[n]$ so that: $\forall i \in [n],\ \sigma(i) - i \not\equiv 0, 1, \ldots, m - 1 \pmod n$.

It is also said that this problem is equivalent to the enumeration of the "very reduced Latin rectangles" of order $m + 1$, the number of which is indicated with $V(m, n)$, that have the first m permutations σ_j, with $j \in [m]$, in the canonical form: $\sigma_j(i) = i + j - 1 \pmod n$.

The solution in the simplest case $m = 2$ was found by Touchard [15] and consists of Touchard's famous numbers U_n, equivalent to $V(2, n)$, expressed by:

$$U_n = \sum_{0}^{n}{}_{k} (-1)^k \frac{2n}{2n-k}\binom{2n-k}{k}(n-k)!. \tag{1.2}$$

For the subsequent case $V(3, n)$ recursive algorithms have been obtained by Riordan [12] and by Yamamoto [18]. However, an explicit formula was only provided in 1967 by Moser [7]. In the case $V(4, n)$, the biggest yet dealt with, there is only one recursive result by Whitehead [16] and possibly an explicit formula by Nechvatal [8] also in 1979.

However, let us return to the more general and more complex problem of the calculation of $K(m, n)$, which represents the aim of this paper.

The first attempts at the calculation of $K(3, n)$ go back to Jacob and to Kerawala [5], who found a recursive formula. The following tidy explicit formula for $K(3, n)$ is, on the other hand, attributed to Yamamoto (see [1] and [10]):

$$K(3, n) = n! \sum_{a+b+c=n} (-1)^b 2^c \frac{a!}{c!}\binom{3a+b+2}{b}. \tag{1.3}$$

Furthermore, in 1944, Riordan obtained an expression of $K(3, n)$ in terms of Touchard's numbers U_n and subsequently, in 1946 [11], the well-known formula:

$$K(3, n) = \sum_{0}^{\left[\frac{n}{2}\right]}{}_{k}\binom{n}{k} D_k D_{n-k} U_{n-2k}, \tag{1.4}$$

(with $U_0 = 1$) which expresses $K(3, n)$ in terms of D_k and U_k.

It is necessary to say that until now, in this line of research, no other progress has been achieved since, for $m > 3$, it has not been possible to obtain $K(m, n)$ in terms of $K(i, n)$ and $V(j, n)$ with $i, j < m$.

The case of $K(4, n)$, which is the most complex yet to be dealt with successfully, was only solved with an explicit formula in 1979, this was achieved independently by Nechvatal [8] and by Athreya et al. [1].

Subsequently, in 1980, Pranesachar [10] and Nechvatal [9], by different means, found a way to express $K(m, n)$ for any value of m by means of the Möbius function of the lattice of partitions of a set. The limit of these research works is that they take the calculation of $K(m, n)$ back to the enumeration of other combinatorial objects, such as the partitions of an integer, for which no explicit formulas are known, and thus do not allow an explicit formula for $K(m, n)$ to be obtained. A further tidy result of this type was achieved by Gessel in [3].

Until now, then, no explicit formula is known which permits the calculation of $K(m, n)$ whatever the value of m.

We would like to conclude this section remembering that another interesting line of research tried to get asymptotic expressions of $K(m, n)$. The first significant paper of this kind is attributed to Erdös and Kaplansky [2] in 1946, subsequent results were obtained by Yamamoto in [17] and Stein in [14].

Finally, in recent years, Godsil and McKay [4] achieved an asymptotic valuation of $V(m, n)$.

2. Notation and Preliminaries

Very useful concepts in the study of permutations without fixed points are those of board and of rook polynomial.

A board is a nonempty subset of $\mathbb{P} \times \mathbb{P}$ $(\mathbb{P} =$ set of positive integers$)$, the elements of board are called squares. Considering a board C it is usually indicated with $r_k(C)$ the number of different ways of placing k non-attacking rooks on it. The rook polynomial of C in the symbolic variable x, that we will call $R(C)$, is given by $\sum_k r_k(C)x^k$.

If we write $\mathfrak{S}_n$ for the set of all permutations of $[n]$, then every $\sigma \in \mathfrak{S}_n$ can be thought of as a board, called the "graph" of σ, the squares of which are the couples $(i, \sigma(i))\ \forall i \in [n]$. It is furthermore obvious that *m* permutations, two by two without fixed points, make up a board of $m \cdot n$ squares, if in such a board the first permutation is the identical one $\sigma(i) = 1$, it will hereon be indicated with $C^{(m)}$.

We furthermore state, to have a greater number of symbols, that a number put up to the right of a symbol does not denote a raising to a power of the same, but it acts as a new symbol (therefore a^2, it is not the square of *a*). When we want to indicate *a* raised to *m*, we write $(a)^m$.

We will call C^g the board, included in $C^{(m)}$, formed by the *g*th permutation of $C^{(m)}$ $(g \in [m])$. Since, as has been mentioned previously, to speak about $C^{(m)}$, it is the same as to speak about a reduced Latin rectangle of order *m*, we will refer often to C^g as the *g*-th line of $C^{(m)}$ (which is not to be confused with the *g*th row or column of $C^{(m)}$ like a board that are different things). We can even say that a subset of C^g has "grade" *g*.

At this point let us remember a classic result which joins the rook polynomials to the permutations without fixed points. We consider the permutations as boards and, taking the board $B \subseteq [n] \times [n]$, we indicate with $N_s(B)$ the number of permutations of [*n*] which have exactly *s* squares in common with *B*. So giving us the following tidy relation:

$$N_s(B) = \sum_{s}^{n} {}_k (-1)^{k-s} \binom{k}{s} (n-k)!\, r_k(B) \tag{2.1}$$

for the proof of this see [13, Chapter 2.3].

Now, let us introduce some conventions in the use of symbols. The sets are always indicated with capital letters and the number of the elements which make them up with the corresponding lower case letter (therefore $a = |A|$). $C(A)$ will be the complementary of the set *A* in the universe set.

$T^{(m)}$ will represent a generic system of independent rooks – that is which do not attack each other – put on $C^{(m)}$ and T^g, with $g \in [m]$, will be the part of the system contained on the gth line of $C^{(m)}$ $(T^g = T^{(m)} \cap C^g)$ and thus it will be: $T^{(m)} = T^1 \cup T^2 \cup \cdots \cup T^m$.

Furthermore, if $A \subseteq C^{(m)}$, then we will say that $\mathcal{R}(A)$ is the *projection* for rows of A in $C^{(m)}$, consisting of all the squares of $C^{(m)}$ which have any square of A in their own row (obviously, $A \subseteq \mathcal{R}(A)$). We will also say that $\mathcal{R}_g(A)$ is the *projection* for rows of A on the g-th line of $C^{(m)}$ and we will put $\mathcal{R}_g(A) = \mathcal{R}(A) \cap C^g$, with the consequence that: $\mathcal{R}(A) = \mathcal{R}_1(A) \cup \cdots \cup \mathcal{R}_m(A)$. Similarly, speaking of columns instead of rows, we can define $\mathcal{C}(A)$ and $\mathcal{C}_g(A)$.

Finally, we define the set $\mathcal{I}(A) = \mathcal{R}(A) \cap \mathcal{C}(A)$ as the "impression" of A and the set $\mathcal{O}(A) = \mathcal{R}(A) \cup \mathcal{C}(A)$ as the "shadow" of A. Thus $\mathcal{O}(T^{(m)})$ will be the set of all the squares of $C^{(m)}$ subject to the attack of any rook of $T^{(m)}$ and which therefore cannot contain other independent rooks from those of $T^{(m)}$.

To indicate the number of ways in which the set A can generally be arranged, considering the restrictions which have been imposed on it, we will write $\pi(A)$. So, we shall obtain that $K(m, n) = \pi(C^{(m)})$.

As it is known, a partition in k blocks of a set A is formed by a collection of nonempty sets A_i, with $i \in [k]$, two by two disjoint and such that $\bigcup_1^k {}_i A_i = A$. We will indicate with $\prod(A)$ the set of the partitions of A; if $\pi \in \prod(A)$ and π has k blocks, we say that $|\pi| = k$; finally, we put $\prod_n = \prod([n])$.

Let us also remember that the refinement of two partitions π_1 and π_2, with π_1, $\pi_2 \in \prod(A)$, is the partition of A consisting of all the nonempty intersections of some block of π_1 with some block of π_2.

If $X = \{x_1, ..., x_s\}$ is a set of variables, we put, for economy of space:

$$\binom{n}{X} = \binom{n}{x_1, ..., x_s}$$

and furthermore:

$$\sum X = \sum_{x_i \in X} x_i; \quad \prod X = \prod_{x_i \in X} x_i; \quad \prod X! = \prod_{x_i \in X} x_i!$$

and, ranging each x_i within its own domain:

$$\sum_X = \sum_{x_1} \cdots \sum_{x_s} .$$

We will indicate with $(n)_k = n(n-1)\cdots(n-k+1)$ the falling factorial of n and with $\langle n \rangle_k = n(n+1)\cdots(n+k-1)$ the raising factorial of n. Now, given that $(n)_n = n!$ we intend to put $\langle n \rangle_n = n¡$ even if in other literature this symbolism has been used to indicate the subfactorial of n.

Let us conclude this section with some recalls relative to the permutations of $[n]$.

If $\sigma \in \mathfrak{S}_n$ and $\sigma(i) = a_i, \forall i \in [n],$ then we can also say that σ corresponds to the word $a_1 \cdots a_n$. It is well known, see [13, p. 17] and following, that σ can be shared in an unambiguous way in the product of disjoint cycles on the elements of $[n]$ and that it can have a “standard representation”, which we will indicate with $s_1 \cdots s_n$, writing *a*) the elements of each cycle with the largest element first and *b*) arranging the cycles in increasing order of their largest element. In such a case each cycle will start with a left-to-right maximum, i.e., with an element s_i so that $s_i > s_j$ for each $j < i$.

Another way to describe σ can be achieved by indicating with b_i the number of elements j of its standard representation on the left of i with $j > i$ and defining $I(\sigma) = (b_1, ..., b_n)$ the “inversion table” of σ. In fact, it can be easily proved, see [13, Proposition 1.3.9], that between the $\sigma \in \mathfrak{S}_n$ and the $I(\sigma)$ there is a bijection and furthermore that: $0 \le b_i \le n - i, \ \forall i \in [n]$.

3. The Associated Partitions

In this section, we want to show how the computation of $K(m, n)$ can be taken back to the enumeration of a double system of partitions of $[n]$.

The first step in the argument is to reiterate the method of calculation by means of systems of independent rooks expressed by the formula (2.1).

Let us suppose that we want to determine the number $K(m, n)$ of all the possible $C^{(m)}$ and that we have already counted all the possible arrangements of the first $m-1$ lines $C^{(m-1)}$, the number of ways in which $C^{(m)}$ can be arranged can be obtained easily, by means of the formula (2.1), once $r_k(C^{(m-1)})$, for $k = 0, \ldots, m-1$, are known.

In fact, if $T^{(m-1)}_{1/m-1} = T^{1}_{1/m-1} \cup \cdots \cup T^{m-1}_{1/m-1}$ is the independent generic system of rooks on $C^{(m-1)}$ and therefore $t^{(m-1)}_{1/m-1} = t^{1}_{1/m-1} + \cdots + t^{m-1}_{1/m-1} = k$, we will have that:

$$\pi(C^m) = \sum_{0}^{n} {}_{t^{(m-1)}_{1/m-1}} (-1)^{t^{(m-1)}_{1/m-1}} (n - t^{(m-1)}_{1/m-1})!\, \pi(T^{(m-1)}_{1/m-1}). \tag{3.1}$$

Now we are trying to calculate $\pi(C^{m-1})$ with the same assumptions. Now $\pi(C^{m-1}) = \pi(T^{(m-1)}_{1/m-1})\pi(C^{m-1} - T^{m-1}_{1/m-1})$ and, if we consider the generic $T^{(m-2)}_{1/m-2} \subseteq \mathcal{I}(T^{(m-1)}_{1/m-1})$ and $T^{(m-2)}_{2/m-2} \subseteq \mathcal{I}(C^{m-1} - T^{m-1}_{1/m-1})$, we will have that:

$$\pi(C^{m-1}) = \sum_{t^{(m-2)}_{1/m-2}} \sum_{t^{(m-2)}_{2/m-2}} (-1)^{t^{(m-2)}_{1/m-2}} (t^{(m-1)}_{1/m-1} - t^{(m-2)}_{1/m-2})!$$

$$\cdot\, \pi(T^{(m-2)}_{1/m-2})(-1)^{t^{(m-2)}_{2/m-2}} (n - t^{(m-1)}_{1/m-1} - t^{(m-2)}_{2/m-2})!\, \pi(t^{(m-2)}_{2/m-2}). \tag{3.2}$$

Repeating the argument for the subsequent line C^{m-2}, we see that:

$$C^{m-2} = T^{m-2}_{1/m-2} \cup (T^{m-2}_{1/m-1} - T^{m-2}_{2/m-2}) \cup (T^{m-2}_{1/m-1} \cap T^{m-2}_{2/m-2})$$

$$\cup\, (T^{m-2}_{2/m-2} - T^{m-2}_{1/m-1}) \cup (C^{m-2} - (T^{m-2}_{1/m-2} \cup T^{m-2}_{1/m-1} \cup T^{m-2}_{2/m-2})) \tag{3.3}$$

and, calling: $T^{(m-3)}_{1/m-3}$, $T^{(m-3)}_{2/m-3}$, $T^{(m-3)}_{3/m-3}$, $T^{(m-3)}_{4/m-3}$, $T^{(m-3)}_{5/m-3}$ the generic $T^{(m-3)}$ included in the impression of each of the five components of the union of which at (3.3), we will be able to say, referring always to (2.1), that:

$$\pi(C^{m-2}) = \pi(T^{m-2}_{1/m-2})\pi(T^{m-2}_{1/m-1} - T^{m-2}_{2/m-2})$$

$$\cdot\, \pi(T^{m-2}_{1/m-1} \cap T^{m-2}_{2/m-2})\pi(T^{m-2}_{2/m-2} - T^{m-2}_{1/m-1})\pi(C^{m-2}$$

$$- T^{m-2}_{1/m-1} \cup T^{m-2}_{1/m-2} \cup T^{m-2}_{2/m-2})$$

$$= \sum_{t^{(m-3)}_{i/m-3}} (-1)^{\sum_1^5 {}_i t^{(m-3)}_{i/m-3}} (t^{m-2}_{1/m-2} - t^{(m-3)}_{1/m-3})!$$

$$\cdot\, (|\, T^{m-2}_{1/m-1} - T^{m-2}_{2/m-2}\,| - t^{(m-3)}_{2/m-3})!(|\, T^{m-2}_{1/m-1} \cap T^{m-2}_{2/m-2}\,| - t^{(m-3)}_{3/m-3})!$$

$$\cdot\, (|\, T^{m-2}_{2/m-2} - T^{m-2}_{1/m-1}\,| - t^{(m-3)}_{4/m-3})!(|\, C^{m-2} - (T^{m-2}_{1/m-1} \cup T^{m-2}_{1/m-2} \cup T^{m-2}_{2/m-2})\,|$$

$$- t^{(m-3)}_{5/m-3})!\,\pi(T^{(m-3)}_{1/m-3})\pi(T^{(m-3)}_{2/m-3})\pi(T^{(m-3)}_{3/m-3})\pi(T^{(m-3)}_{4/m-3})\pi(T^{(m-3)}_{5/m-3}). \quad (3.4)$$

Continuing in this way in order to count all the possible arrangements of the line l, we should take all the possible subsets of rooks which are situated on it, considering the partition refinement of C^l consisting of all their nonempty intersections T^l_j, $j = 1, \ldots, p_l$, and of the complementary of their union $T^l_0 = C^l - \bigcup_1^{p_l} {}_j T^l_j$ and finally choosing $p_l + 1$ systems of independent rooks $T^{(l-1)}_{j/l-1}$, $j = 0, 1, \ldots, p_l$, with $T^{(l-1)}_{j/l-1} \subseteq \mathcal{I}(T^l_j)$.

Then we will have:

$$\pi(C^l) = \prod_0^{p_l} {}_j \pi(T^l_j) = \sum_{t^{(l-1)}_{j/l-1}} (-1)^{\sum_0^{p_l} {}_j t^{(l-1)}_{j/l-1}} \prod_0^{p_l} {}_j (t^l_j - t^{(l-1)}_{j/l-1})!\,\pi(T^{(l-1)}_{j/l-1}) \quad (3.5)$$

and we can conclude that:

$$K(m, n) = \prod_2^m {}_l \pi(C^l)$$

$$= \sum_{t^{(l-1)}_{j/l-1}} (-1)^{\sum_2^m {}_l \sum_0^{p_l} {}_j t^{(l-1)}_{j/l-1}} \prod_2^m {}_l \prod_0^{p_l} {}_j (t^l_j - t^{(l-1)}_{j/l-1})!\,\pi(T^{(l-1)}_{j/l-1}). \quad (3.6)$$

The situation, therefore, seems to be somewhat complex, but an idea which allows us to control it is that of identifying any subset of $C^{(m)}$ by means of its two projections, for rows and for columns, on the main diagonal $C^1 = \{(i, i), i \in [n]\}$.

Given a set of grade g A^g, we consider in fact $\mathcal{R}_1(A^g)$ and $\mathcal{C}_1(A^g)$. Now, if $g = 1$, then $\mathcal{R}_1(A^1) = \mathcal{C}_1(A^1) = A^1$ and projections and set coincide. If, on the other hand, $g > 1$, then the set A^g determines $\mathcal{R}_1(A^g)$ and $\mathcal{C}_1(A^g)$ in one way only. Vice versa given $A_1^g, A_2^g \subseteq C^1$ with $|A_1^g| = |A_2^g| = |A^g| = p$, A^g will be one of the p! permutations of the square board $\mathcal{R}(A_1^g) \cap \mathcal{C}(A_2^g)$. Furthermore, if B is a board of forbidden positions for A^g and we set $\overline{B} = B \cap \mathcal{R}(A_1^g) \cap \mathcal{C}(A_2^g)$, we will have that: $\pi(A^g) = \sum_0^p {}_k (-1)^k (p-k)!\, r_k(\overline{B})$.

In the light of this new approach, a system of independent rooks $T^{(l)} = T^1 \cup \cdots \cup T^l$ determines the subsets $R^i = \mathcal{R}(T^i)$ and $C^i = \mathcal{C}(T^i)$, for $i \in [l]$, and $R^0 = \mathcal{R}(C(T^{(l)})) = C\left(\bigcup_1^l {}_i R^i\right)$ and $C^0 = \mathcal{C}(C(T^{(l)})) = C\left(\bigcup_1^l {}_i C^i\right)$. Thus it characterizes two partitions, each one with $l+1$ blocks, the $\{R^i\}$ and the $\{C^i\}$, with $i = 0, \ldots, l$, the first for the set of the rows and the second for that of the columns.

Now, if we return to the computation performed with (3.5) of all the possible arrangements of $C^{(l)}$, we will have that $T_{j/l-1}^{(l-1)}$ characterizes a partition in l blocks $R_{j/l-1}^i = \mathcal{R}(T_{j/l-1}^i)$, with $i = 0, \ldots, l-1$, of the block $\mathcal{R}(T_j^l)$ of the partition $\{\mathcal{R}(T_j^l)\}$, $j = 0, \ldots, p_l$, in $p_l + 1$ blocks of the set of the rows.

Consequently, if we put together all the elements of grade i of the various partitions $R_{j/l-1}^i$ and we set $R_l^i = \bigcup_0^{p_l} {}_j R_{j/l-1}^i$, then we will obtain that $\{R_l^i\}$, with $i = 0, \ldots, l-1$, is a partition in l blocks of the set of the rows of our board. Similarly, we can construct the partition $\{C_l^i\}$, with $i = 0, \ldots, l-1$, of the set of columns.

Repeating this for each line from the mth to the second, we will eventually have $m-1$ couples of partitions of the set of the rows and of that of the columns: $\{R_j^i\}$, and $\{C_j^i\}$, with $j = 2, \ldots, m$ and $i = 0, 1, \ldots, j-1$.

Intersecting each of these partitions with C^1, we obtain as many partitions of $C^1 = \{(i, i)\}$ with $i \in [n]$. In such partitions, the projections for rows and for columns of the sets of grade 1, which belong to C^1, obviously coincide.

Calculating the number of all the possible arrangements of these $2(m-1)$ partitions, respecting the condition that the projections of the sets of grade 1 must coincide, is equivalent, from what has been said, to calculating the product $\prod_2^m {}_l \prod_0^{p_l} {}_j \pi(T_{j/l-1}^{(l-1)})$ which appears in (3.6).

To do this, it is natural to consider the partition refinement of the $m-1$ $\{R_j^i \cap C^1\}$ partitions and that of the $m-1$ $\{C_j^i \cap C^1\}$ partitions. Putting by analogy $R_1^0 = C_1^0 = C^1$, the blocks of the refinement partitions will be given from:

$R_{\alpha_m, \ldots, \alpha_1} = R_m^{\alpha_m} \cap R_{m-1}^{\alpha_{m-1}} \cap \cdots \cap R_1^{\alpha_1}$, with $0 \le \alpha_j \le j-1 \ \forall j \in [m]$, and from $C_{\beta_m, \ldots, \beta_1} = C_m^{\beta_m} \cap C_{m-1}^{\beta_{m-1}} \cap \cdots \cap C_1^{\beta_1}$ with similar limitations on the indices β_j.

We will then have:

$$\prod_2^m {}_l \prod_0^{p_l} {}_j \pi(T_{j/l-1}^{(l-1)}) = \prod_0^{j-1} {}_{\alpha_j} \pi(R_{\alpha_m, \ldots, \alpha_1}) \prod_0^{j-1} {}_{\beta_j} \pi(C_{\beta_m, \ldots, \beta_1}). \qquad (3.7)$$

We will also say that the pair of partitions $\{R_{\alpha_m, \ldots, \alpha_1}\}$ and $\{C_{\beta_m, \ldots, \beta_1}\}$ of $[n]$ is "associated" with the collection of systems of independent rooks $T_{j/l-1}^{(l-1)}$, with $l = 2, \ldots, m$ and $j = 0, \ldots, p_l$.

4. The Blocks of the Associated Partitions

Before being able to develop the calculation of (3.6) using (3.7), we must examine closely the meaning of the indices $\alpha_m, ..., \alpha_1$ and $\beta_m, ..., \beta_1$ which respectively, mark the blocks $R_{\alpha_m, ..., \alpha_1}$ and $C_{\beta_m, ..., \beta_1}$.

In order to do this, we first of all define the concept of "covering". Taking the index α_i, with $\alpha_i > 0$, we say that α_i "covers" α_{α_i} and we write $\alpha_i \vdash \alpha_{\alpha_i}$ or $\kappa(\alpha_i) = \alpha_{\alpha_i}$.

From the definition it immediately follows that:

(a) if $\alpha_i = 0$, then it does not cover any other index, since α_0 does not exist;

(b) if $\alpha_i \vdash \alpha_s$, then $s < i$.

Thus, if $\alpha_s > 0$, applying to it more times the function of covering κ, then we will always arrive at an index of value 0.

On the contrary, taking an index α_l of value 0, we can consider all the indices α_s which have the property $\alpha_s \vdash \alpha_l$ (that is $\kappa^{-1}(\alpha_l)$). Repeating this procedure more times, we get all the indices which, with a finite number of applications of the function κ, finish in α_l.

If we suppose that $\alpha_h = 0$ and we put $\kappa^0(\alpha_s) = \alpha_s$, then we will be able to define $Z_h = \{\alpha_j \mid \alpha_h = 0 \text{ and } \exists k \in \mathbb{N} \text{ so that } \kappa^{\kappa}(\alpha_j) = \alpha_h\}$ which we will call the "component *h*" of the indices $\alpha_m, ..., \alpha_1$. Furthermore, as $\alpha_1 = 0$, we will have that $Z_1 \neq \varnothing$.

In this way, we obtain a partition of the set of indices $\{\alpha_m, ..., \alpha_1\}$ in blocks made up from Z_h, with $h \in [m]$.

We will say that such a subdivision represents the structure of the indices $\alpha_m, ..., \alpha_1$ and we will write that $\sigma(\alpha_m, ..., \alpha_1) = Z_1 \cup Z_{z_2} \cup \cdots \cup Z_{z_a}$ with $1 < z_2 < \cdots < z_a \leq m$. We will also write $\sigma_l(\alpha_m, ..., \alpha_1) = Z_l$ and $\zeta(\alpha_m, ..., \alpha_1) =$ number of the α_j which are equal to zero. It is clear that, if $Z = \{\alpha_{i_1}, ..., \alpha_{i_s}\}$ is a component, then $\alpha_{i_s} = 0$.

Moreover, we will put, to be brief $R_{\alpha_j} = R_{\alpha_m, \ldots, \alpha_1}$, $C_{\beta_j} = C_{\beta_m, \ldots, \beta_1}$, $\alpha_j = \alpha_m, \ldots, \alpha_1$ and $\beta_j = \beta_m, \ldots, \beta_1$.

It is necessary to pay attention to the fact that Z_h is not only a subset $I \subseteq [m]$, but a subset of the indices α_j, for $j \in I$, each with its own value.

The following result allows to count the number of the R_{α_j} at the base of the structure of their indices.

Proposition 4.1. *Let* $\sigma(\alpha_m, \ldots, \alpha_1) = Z_1 \cup Z$ *and* $Z = Z_{z_2} \cup \cdots \cup Z_{z_a}$. *Then*:

(a) *if we suppose Z to be variable*, *then the number of the possible sets of indices will be*: $\binom{m-1}{z} z!$.

(b) *if*, on the other hand, *we keep Z constant, then the possible* α_j *will be* $(m-1-z)!$.

In fact, to determine Z we will have, first of all, have to choose the z places of its indices in the set $\{2, \ldots, m\}$ and this can be done in $\binom{m-1}{z}$ ways. Furthermore, if the selected indices are $\alpha_{j_1}, \ldots, \alpha_{j_z}$ (with $j_1 > \cdots > j_z$), then it can be seen that α_{j_z} must be equal to 0, $\alpha_{j_{z-1}}$ can assume the values 0 and α_{j_z} and so on. Therefore, the last index has only one possible value, the penultimate two values, etc., thus all the possible ways to attribute a value to $\alpha_{j_1}, \ldots, \alpha_{j_z}$ are $1 \cdot 2 \cdot \cdots \cdot z = z!$. This proves (a).

If, on the other hand, Z is fixed, then the places of the indices of Z_1 are also fixed. Now, the last index of Z_1 on the right α_1 can only have the value 0, the penultimate only 1 and thus, for an argument identical to the previous, the number of possible values of the $m - z$ indices is equal to $1 \cdot 1 \cdot 2 \cdot \cdots \cdot (m-1-z) = (m-1-z)!$. Thus (b) too is proved.

It is also possible to calculate the number of possible α_j, in terms of the data of singular components with the following result, which we shall just state.

Proposition 4.2. *If* $\sigma(\alpha_m, ..., \alpha_1) = Z_1 \cup Z_{z_2} \cup \cdots \cup Z_{z_a}$, *then* $|Z_1|$ *and* $|Z_{z_i}|$, *with* $i = 2, ..., a$, *constitute a partition of the integer m in which the number of parts equal to s will be* λ_s. *The number of possible* α_j *with this structure will therefore be the same as*:

$$\frac{m!}{1^{\lambda_1} 2^{\lambda_2} \cdots m^{\lambda_m} \lambda_1! \lambda_2! \cdots \lambda_m!}. \tag{4.1}$$

So, if $\alpha_s \vdash \alpha_l$, then we have that R_{α_j} is a subset of $\mathcal{R}_1\left(\bigcup_x T^{\alpha_l}_{x/l-1}\right)$, with x that ranges in a subset of $\{0, 1, ..., p_l\}$, and thus it lies in the projection for rows on the first line of a set of rooks of grade α_l included in the impression of a set of rooks of grade $\alpha_s = l$. If instead α_s does not cover α_l, then the set of rooks of grade α_l is not included in the impression of the set of rooks of grade α_s and so the number of elements in their intersection varies according to the variation of the projection for rows or for columns.

From this, it follows that if, using the symbolism of Section 3, we take $\mathcal{R}_1(T^l_j) - \mathcal{R}_1\left(\bigcup_1^{l-1} {}_i T^i_{j/l-1}\right)$, we see that it will be composed of the union of all the R_{α_j} with the same Z_l component. Vice versa, if we fix the Z_l component and make the other α_j vary in all possible ways, we obtain a collection of sets R_{α_j} the union of which will be equal to $\mathcal{R}_1(T^l_j) - \mathcal{R}_1(T^{(l-1)}_{j/l-1})$ for some j. Furthermore, if $l = 1$, since $T^{(l-1)}_{j/l-1}$ does not exist, the union of all the R_{α_j} with the same Z_1 will be given by $\mathcal{R}_1(T^1_j)$ for some j.

Naturally, the same argument is true for the sets C_{β_j} and the components of the indices β_j.

5. The Enumeration of Latin Rectangles

Now, let us try, applying the contents of the previous section, to give an explicit form to (3.6) in terms of the data of the two associated partitions $\{R_{\alpha_j}\}$ and $\{C_{\beta_j}\}$, and that is in terms of the sets of variables $R = \{r_{\alpha_j}\}$ and $C = \{c_{\beta_j}\}$.

First of all, we observe that, if $l > 1$, $t_j^l - t_{j/l-1}^{(l-1)} = | T_j^l | - | T_{j/l-1}^{(l-1)} | = | \mathcal{R}_1(T_j^l) | - | \mathcal{R}_1(T_{j/l-1}^{(l-1)}) | = | \mathcal{R}_1(T_j^l) - \mathcal{R}_1(T_{j/l-1}^{(l-1)}) |$ but, following what was said before, $\mathcal{R}_1(T_j^l) - \mathcal{R}_1(T_{j/l-1}^{(l-1)})$ is formed, in such a case, from the union of all the sets R_{α_j} with the same Z_l component and vice versa.

Therefore, if we put, $\forall l \in [m]$, $Q(Z_l) = \{r_{\alpha_j} | \sigma_l(\alpha_j) = Z_l\}$, $\tilde{Q}(Z_l) = \{c_{\beta_j} | r_{\beta_j} \in Q(Z_l)\}$ and $q(Z_l) = \sum Q(Z_l)$, thus we have that, if $l > 1$, $j \in \{0, ..., p_l\}$ exists so that:

$$q(Z_l) = t_j^l - t_{j/l-1}^{(l-1)} \tag{5.1}$$

and vice versa.

Now, we will compute the product $\prod_2^m {}_l \prod_0^{p_l} {}_j (T_{j/l-1}^{(l-1)})$ that, for (3.7), is the same as $\prod_0^{j-1} {}_{\alpha_j} \prod_0^{j-1} {}_{\beta_j} \pi(R_{\alpha_j}) \pi(C_{\beta_j})$.

The first partition $\{R_{\alpha_j}\}$ can be chosen in a completely arbitrary way and thus the number of its possible arrangements is given by the multinomial coefficient $\binom{n}{R}$. The second partition is, on the other hand, subject to some restrictions.

First of all, for $l > 1$, $T_{j/l-1}^{(l-1)} \subseteq \mathcal{I}(T_j^l)$ and so $| \mathcal{C}_1(T_j^l) - \mathcal{C}_1(T_{j/l-1}^{(l-1)}) | = | \mathcal{R}_1(T_j^l) - \mathcal{R}_1(T_{j/l-1}^{(l-1)}) | = q(Z_l)$ and since, following the same reasoning as we have already done, $| \mathcal{C}_1(T_j^l) - \mathcal{C}_1(T_{j/l-1}^{(l-1)}) | = \sum \tilde{Q}(Z_l)$, we have

$$\sum \tilde{Q}(Z_l) = q(Z_l). \tag{5.2}$$

Furthermore, taking a generic set of rooks of grade l it is clear that $|\,\mathcal{C}_1(T^l)\,| = |\,\mathcal{R}_1(T^l)\,|$.

Now, if $l = 1$, then $T^l \subseteq C_1$ and even $\mathcal{C}_1(T^l) = \mathcal{R}_1(T^l)$, but, for the reasons stated in Section 4, $\mathcal{R}_1(T)^1$ is made up of the union of all the R_{α_j} with the same Z_1, and the same argument is valid for $\mathcal{C}_1(T^1)$, therefore:

$$\bigcup_{\sigma_1(\alpha_j)=Z_1} R_{\alpha_j} = \bigcup_{\sigma_1(\beta_j)=Z_1} C_{\beta_j} \tag{5.3}$$

and

$$\sum \tilde{Q}(Z_1) = q(Z_1). \tag{5.4}$$

Furthermore, the restrictions (5.2) and (5.4) imposed on c_{β_j} imply that, for T_j^l, with $j = 0, ..., p_l$, $|\,\mathcal{C}_1(T_j^l)\,| = |\,\mathcal{R}_1(T_j^l)\,|$.

This can be easily proved for complete induction on l considering that, if $l = 1$, then the result has already been expressed by (5.4), while, if $l > 1$ and we suppose that we have already proved this $\forall j \in [l-1]$, it follows from the consideration that:

$$|\,T_j^l\,| = \sum_1^{l-1} i|\,T_{j/l-1}^i\,| + |\,\mathcal{R}_1(T_j^l) - \mathcal{R}_1(T_{j/l-1}^{(l-1)})\,|.$$

Therefore, there are no other restrictions on c_{β_j}, apart from those expressed by (5.2) and (5.4).

If we now group the C_{β_j} sets on the basis of the value of their component Z_1, then (5.3) allows us to state that:

$$\prod_{\beta_j} \pi(C_{\beta_j}) = \prod_{Z_1} \binom{q(Z_1)}{\tilde{Q}(Z_1)} \tag{5.5}$$

on the condition, however, that the C variables also respect the restrictions imposed by (5.2).

Let us finally examine the $t_{j/l-1}^{(l-1)}$ which appears in (3.6) as exponents of -1.

Now, for (5.1), if $l > 1$, $t_{j/l-1}^{(l-1)} = t_j^l - q(Z_l)$ for any Z_l and so $\sum_0^{p_l} j\, t_{j/l-1}^{(l-1)} = \sum_0^{p_l} j\, t_j^l - \sum_{z_l} q(Z_l) = n - \sum_{z_l} q(Z_l)$. Therefore, being $l = 2, ..., m$, the exponent of –1 in (3.6) will be the same as $n(m-1) - \sum_2^m l \sum_{z_l} q(Z_l)$. Furthermore, since, as we have already seen, $n = \sum R = \sum_{Z_1} q(Z_1)$, adding and subtracting n it can be expressed by: $nm - \sum_1^m l \sum_{z_l} q(Z_l)$.

Finally, set $W = \{r_{\alpha_j} \mid \zeta(\alpha_j) \text{ odd}\}$ and considering that every r_{α_j} variable compares $\zeta(\alpha_j)$ times in $\sum_1^m l \sum_{z_l} q(Z_l)$, we will have that the exponent of –1 can be substituted by $nm - \sum W$, since the even multiples of r_{α_j} can obviously be omitted.

Using all these results in (3.6), we obtain the following remarkable result:

Theorem 5.1.

$$K(m, n) = \sum_R \sum_{\sum \tilde{Q}(Z_l) = q(Z_l)} {}_C (-1)^{nm + \sum_1^m l \sum_{Z_l} q(Z_l)} \binom{n}{R} \prod_2^m {}_l \prod_{Z_l} q(Z_l)!$$

$$\cdot \prod_{Z_1} \binom{q(Z_1)}{\tilde{Q}(Z_1)} = \sum_{\sum R = n} {}_R \sum_{\sum \tilde{Q}(Z_l) = q(Z_l)} {}_C (-1)^{nm + \sum W} \frac{\prod_0^m l \prod_{Z_l} q(Z_l)!}{\prod R! \prod C!}$$

$$= (-1)^{n(m-1)} \sum_{\sum R = n} {}_R \sum_{\sum \tilde{Q}(Z_l) = q(Z_l)} {}_C \frac{\prod_0^m l \prod {}_{Z_l} (-q(Z_l))!}{\prod R! \prod C!}$$

$$= (-1)^{nm} \sum {}_R \sum {}_C \binom{n}{R} \prod_1^m {}_l \prod {}_{Z_l} (-1)^{q(Z_l)} \binom{q(Z_l)}{\tilde{Q}(Z_l)} \cdot \prod_C c_{\beta_j}^{\zeta(\beta_j) - 1}. \quad (5.6)$$

Here, by analogy with the preceding symbolism, we have set $Z_0 = \varnothing$ since α_0 does not exist. Thus $q(Z_0) = \sum R = n$ since the elements of $Q(Z_0)$, not being subject to any restrictions, are all the elements of R.

So, (5.6) is an explicit formula for the computation of $K(m, n)$ in $2m!$ variables R and C, while $q(Z_l)$ with $l \in [m]$ and $\sum W$ are sums of particular subsets of R.

This therefore represents the result which we proposed to achieve with the present paper.

The C variables, in contrast to the R variables, are not, however, between their independent since they must be subject to the restrictions $\sum \tilde{Q}(Z_l) = q(Z_l)$ for $l \in [m]$.

If we want to limit ourselves to considering only independent variables, then we can proceed as follows.

For each Z_l component we indicate with $d(Z_l)$ the c_{β_j} variable with $\sigma_l(\beta_j) = Z_l$ and all the indices β_j which are different from those of Z_l equal to zero, and we put $D = \{d(Z_l)\}$.

Now, $d(Z_l) = q(Z_l) - \sum_{c_{\beta_j} \in \tilde{Q}(Z_l) - D} c_{\beta_j}$ and thus the variables of D can be obtained from those of $C - D$.

Furthermore, if $\sigma_l(\beta_j) = Z_l$, then $c_{\beta_j} \in \tilde{Q}(Z_l)$ and so $c_{\beta_j} \le q(Z_l)$. Thus, if we put $\mu_{\beta_j} = \min_{\sigma_l(\beta_j) \neq \varnothing}(q(\sigma_l(\beta_j)))$, then $\forall c_{\beta_j} \in C - D$, we will have that $c_{\beta_j} \le \mu_{\beta_j}$ and such a restriction guarantees that $d(Z_l) \ge 0$.

Using this new symbolism (5.6) can be rewritten like this:

$$K(m, n) = \sum_{0}^{n} {}_{r_{\alpha_j}} \sum_{0}^{\mu_{\beta_j}} {}_{c_{\beta_j}} (-1)^{nm + \sum W} \binom{n}{R} \frac{\prod_1^m {}_l \prod_{Z_l} q(Z_l)!}{\prod (C - D)! \prod D!} \tag{5.7}$$

with $r_{\alpha_j} \in R$ and $c_{\beta_j} \in C - D$.

Now, if $|Z_l| = s$, then for Proposition 4.1, the possible $d(Z_l)$ is $\binom{m}{s}(s-1)!$. Furthermore, if $s = 1$, then all the $d(Z_l)$ coincide with the c_{β_j} which has all the indices at 0 and so, in such a case, instead of $\binom{m}{1}0! = m$ we only have one distinct element and $|D| = \sum_1^m s\binom{m}{s}(s-1)! - (m-1)$.

Thus, in (5.7), other than the $m!$ independent variables R, there are the $m! + m - 1 - \sum_1^m s\binom{m}{s}(s-1)!$ independent variables $C - D$.

6. Simplifications of the Formula

We have seen that (5.7) needs $2m! + m - 1 - \sum_1^m s\binom{m}{s}(s-1)!$ independent variables for the computation of $K(m, n)$.

It is possible, though, to effect two types of elimination among these parameters which allow us to reduce their number considerably, even though this fact makes (5.7) lose its symmetry. This is obviously important when we would like to calculate concretely $K(m, n)$ for m and n prefixed.

Let us therefore examine the two possible reductions of the independent variables R and $C - D$.

(A) We consider r_{α_j} and c_{α_j} with $\zeta(\alpha_j) = 1$ and so with $\sigma(\alpha_j) = Z_1$. In such an assumption $q(\sigma(\alpha_j))$ contains a unique element and so, for (5.4), $c_{\alpha_j} = r_{\alpha_j}$ and, in (5.7), $c_{\alpha_j}!$ is simplified with $q(\sigma(\alpha_j))! = r_{\alpha_j}!$. As far as r_{α_j} is concerned instead, if we put: $F_0 = \{r_{\alpha_j} \mid \zeta(\alpha_j) = 1\}$, $f_0 = \sum F_0$, $\overline{Q_0} = R - F_0$ and $\tilde{F}_0 = \{c_{\alpha_j} \mid r_{\alpha_j} \in F_0\}$, then we will have that the variables of F_0 do not appear in any set $Q(Z_l)$ with $l > 1$ and that:

$$\sum_R \binom{n}{R} = \sum_{F_0} \sum_{\overline{Q_0}} \binom{n}{F_0, \overline{Q_0}} = \sum_{F_0} \sum_{\overline{Q_0}} \binom{n}{f_0} \binom{f_0}{F_0} \binom{n - f_0}{\overline{Q_0}}$$

$$= \sum_{\overline{Q}_0} \sum_{f_0} ((m-1)!)^{f_0} \frac{n!}{f_0! \prod \overline{Q}_0!} \tag{6.1}$$

since, for Proposition 4.1, $|F_0| = (m-1)!$. Furthermore the F_0 appears among the exponents of –1 with their total f_0. The $2(m-1)!$ variables of F_0 and of $\tilde{F}_0$ can therefore be substituted by f_0.

(B) Let us now consider the r_{α_j} and c_{α_j} with $\sigma(\alpha_j) = Z_h \cup Z_1$ (and so $\zeta(\alpha_j) = 2$) and $\min(z_1, z_h) = 1$ and put, $\forall s \in [m]$: $F_s = \{r_{\alpha_j} | \zeta(\alpha_j) = 2$ and $|\sigma_s(\alpha_j)| = 1\}$, $f_s = \sum F_s$, $\overline{Q}(Z_s) = Q(Z_s) - F_s$, $\tilde{F}_s = \{c_{\alpha_j} | r_{\alpha_j} \in F_s\}$ and $\overline{q}(Z_s) = q(Z_s) - f_s$. First, we observe that, if $r_{\alpha_j} \in F_s$, then it does not appear among the exponents of –1 since $\zeta(\alpha_j)$ is even. Now, if $\sigma(\alpha_j) = Z_s \cup Z_v$, $|Z_v| = m-1$ and so $q(Z_v)$ has only one element and, for (5.2) and (5.4), $c_{\alpha_j} = r_{\alpha_j}$. Therefore, in (5.7), if $c_{\alpha_j} \in \tilde{F}_s$, then $c_{\alpha_j}!$ is simplified with $q(Z_v)! = r_{\alpha_j}!$. Moreover, $\forall s \in [m]$:

$$\sum_{F_s} \frac{q(Z_s)!}{\prod F_s!} = \sum_{F_s} \frac{(f_s + \overline{q}(Z_s))!}{\prod F_s!}$$

$$= \sum_{F_s} \frac{(f_s + \overline{q}(Z_s))!}{f_s!} \binom{f_s}{F_s} = ((m-2)!)^{f_s} \frac{(f_s + \overline{q}(Z_s))!}{f_s!} \tag{6.2}$$

since $|F_s| = (m-2)!$, and so also the F_s and the $\tilde{F}_s$ are eliminated and substituted by f_s. We must, however, by careful because, if $m = 2$, then F_1 and F_2 are equal.

In conclusion, putting $\overline{R} = R - \bigcup_0^m {}_i F_i$, $\overline{C} = \{c_{\alpha_j} | r_{\alpha_j} \in \overline{R}\}$ and $\overline{D} = \{d(\sigma_l(\alpha_j))| c_{\alpha_j} \in \overline{C}\}$, we have that (5.7) transforms itself into:

$$K(m, n) = \sum_{f_0} \sum_{f_s} \sum_{0}^{n} {}_{\overline{R}} \sum_{0}^{\mu\beta_j} {}_{\overline{C}-\overline{D}} (-1)^{nm+\sum W} ((m-1)!)^{f_0} ((m-2)!)^{\sum_1^m s f_s}$$

$$\cdot \frac{1}{\prod(\overline{C}-\overline{D})!\prod\overline{D}!}\binom{n}{f_0,\ ...,\ f_m,\ \overline{R}}$$

$$\cdot \prod_1^m {}_s(f_s+\overline{q}(Z_s))!\prod_1^m {}_l\prod_2^{m-2}{}_{|Z_l|}q(Z_l)!. \tag{6.3}$$

It is, however, possible to accomplish a further step to simplify (6.3). In fact, putting $f = \sum_1^m {}_s f_s$, we have that:

$$\sum_{f_1}\cdots\sum_{f_m}\prod_1^m {}_s\frac{(f_s+\overline{q}(Z_s))!}{f_s!} = \sum_{f_1}\cdots\sum_{f_m}\prod_1^m {}_s\overline{q}(Z_s)!\binom{f_s+\overline{q}(Z_s)}{\overline{q}(Z_s)}$$

$$= \prod_1^m {}_s\overline{q}(Z_s)!\binom{f+\sum_1^m {}_s\overline{q}(Z_s)+m-1}{f} \tag{6.4}$$

and so (6.3) becomes:

$$K(m,\ n) = \sum_0^n {}_{f_0}\sum_0^n {}_f\sum_0^n {}_{\overline{R}}\sum_0^{\mu\beta_j} {}_{\overline{C}-\overline{D}}(-1)^{nm+\sum W}((m-1)!)^{f_0}$$

$$\cdot ((m-2)!)^f\binom{n}{f_0,\ f,\ \overline{R}}\frac{1}{\prod(\overline{C}-\overline{D})!\prod\overline{D}!}$$

$$\cdot\frac{\left(f+\sum_1^m {}_s\overline{q}(Z_s)+m-1\right)!}{\left(\sum_1^m {}_s\overline{q}(Z_s)+m-1\right)!}\prod_1^m {}_l\prod_2^{m-2}{}_{|Z_l|}q(Z_l)!\prod_1^m {}_s\overline{q}(Z_s)!. \tag{6.5}$$

From the independent variables R we have so eliminated the $(m-1)!$ of F_0 and the $(m-2)!$ of each F_s, with $s \in [m]$, and therefore, $|\ \overline{R}\ | = m! - (m-1)! - m(m-2)!$ $= m! - (2m-1)(m-2)!$.

The C variables have undergone the same reduction. However, it is necessary to add f_0 and f and subtract the $\overline{D}$, which are as many as the components Z_l with

$1 < |Z_l| < m-1$, and so equal to $\sum_2^{m-2} h\binom{m}{h}(h-1)!$ plus the c_{α_j} with all the α_j indices equal to zero (which is determined by the m equivalent restrictions $\overline{q}(Z_s) = \sum(\tilde{Q}(Z_s) - \tilde{F}(Z_s))$ with $s \in [m]$), and thus:

$$|\,\overline{C} - \overline{D}\,| = m! - (m-1)! - m(m-2)! - \left(\sum_2^{m-2} h\binom{m}{h}(h-1)! + 1\right)$$

$$= m! - (2m-1)(m-2)! - 1 - \sum_2^{m-2} h\binom{m}{h}(h-1)!$$

$$= m! - 1 - \sum_2^{m} h\binom{m}{h}(h-1)!. \tag{6.6}$$

The independent parameters of (6.5) are therefore all together: $2m! - (2m-1)(m-2)! + m + 1 - \sum_1^m h\binom{m}{h}(h-1)!$.

7. The Simplest Cases

Let us see what in concrete terms happens calculating the formulas obtained in Sections 5 and 6 for the first values of $m = 2, 3, 4$.

(A) $m = 2$. r_{α_j} are of $r_{\alpha_2\alpha_1}$ which can therefore assume the values r_{10} and r_{00}. Furthermore, $C - D = \varnothing$, $c_{00} = r_{00}$ and $c_{10} = r_{10}$ and so, applying (5.7), we have:

$$K(2, n) = \sum_0^n {}_{r_{10}} \sum_0^n {}_{r_{00}} (-1)^{2n+r_{10}} r_{00}! \binom{n}{r_{10}, r_{00}} \frac{r_{00}!\,r_{10}!}{c_{00}!\,c_{10}!} = \sum_0^n {}_{r_{10}} (-1)^{r_{10}} \frac{n!}{r_{10}!} \tag{7.1}$$

which, for (1.1), is equivalent to D_n.

(B) $m = 3$. r_{α_j} are of $r_{\alpha_3\alpha_2\alpha_1}$. As seen in Section 6, the $F_0 = \{r_{210}, r_{110}\}$, $F_1 = \{r_{200}\}$, $F_2 = \{r_{100}\}$, $F_3 = \{r_{010}\}$ and the homologous c_{α_j} are eliminated. Furthermore, $\overline{q}(Z_1) = \overline{q}(Z_2) = \overline{q}(Z_3) = r_{000}$ and $c_{000} = r_{000}$ and so, applying

(6.5), we have that:

$$K(3,\ n) = \sum_{0}^{n} {}_{f_0} \sum_{0}^{n} {}_{f} \sum_{0}^{n} {}_{r_{000}} (-1)^{3n-f_0-r_{000}} 2^{f_0} 1^{f} \binom{n}{f_0,\ f,\ r_{000}}$$

$$\cdot \frac{(r_{000}!)^3 (f + 3r_{000} + 2)!}{c_{000}!(3r_{000} + 2)!}$$

$$= \sum_{f_0 + f + r_{000} = n} (-1)^{f} 2^{f_0} \frac{n!\, r_{000}!}{f_0!} \binom{3r_{000} + f + 2}{f} \tag{7.2}$$

and we find (1.3) again.

(C) $m = 4$. r_{α_j} are of $r_{\alpha_4\alpha_3\alpha_2\alpha_1}$. The $F_0 = \{r_{3210}, r_{2210}, r_{1210}, r_{1110}, r_{2110}, r_{3110}\}$, $F_1 = \{r_{3200}, r_{2200}\}$, $F_2 = \{r_{3100}, r_{1100}\}$, $F_3 = \{r_{1010}, r_{2010}\}$, $F_4 = \{r_{0110}, r_{0210}\}$ and the homologous c_{α_j} are eliminated. Furthermore, $q(Z_1') = r_{1000} + r_{1200}$, $q(Z_1'') = r_{2100} + r_{0100}$, $q(Z_1''') = r_{0010} + r_{3010}$, $q(Z_2') = r_{0200} + r_{1200}$, $q(Z_2'') = r_{2000} + r_{2100}$, $q(Z_3) = r_{3000} + r_{3010}$, and $\bar{q}(Z_1) = r_{0000} + r_{3000} + r_{2000} + r_{0200}$, $\bar{q}(Z_2) = r_{0000} + r_{3000} + r_{0100} + r_{1000}$, $\bar{q}(Z_3) = r_{0000} + r_{2000} + r_{1000} + r_{0010}$, $\bar{q}(Z_4) = r_{0000} + r_{0100} + r_{0010} + r_{0200}$. We besides have that: $c_{1000} = r_{1000} + r_{1200} - c_{1200}$, $c_{0100} = r_{0100} + r_{2100} - c_{2100}$, $c_{0010} = r_{0010} + r_{3010} - c_{3010}$, $c_{0200} = r_{0200} + r_{1200} - c_{1200}$, $c_{2000} = r_{2000} + r_{2100} - c_{2100}$, $c_{3000} = r_{3000} + r_{3010} - c_{3010}$, $c_{0000} = r_{0000} + r_{3000} + r_{2000} + r_{0200} - c_{3000} - c_{2000} - c_{0200}$, that: $\bar{R} = \{r_{1000}, r_{1200}, r_{2100}, r_{0100}, r_{0010}, r_{3010}, r_{0200}, r_{3000}, r_{2000}, r_{0000}\}$ and that: $\bar{C} - \bar{D} = \{c_{1200}, c_{2100}, c_{3010}\}$ and, applying (6.3), we obtain:

$$K(4,\ n) = \sum_{f_0} \sum_{f_s} \sum_{\bar{R}} \sum_{\bar{C}-\bar{D}} (-1)^{4n-f_0-r_{1000}-r_{0100}-r_{0010}-r_{3000}-r_{2000}-r_{0200}}$$

$$6^{f_0} 2^{f_1+f_2+f_3+f_4} \binom{n}{f_0,\ f_1,\ f_2,\ f_3,\ f_4,\ \bar{R}} (f_2 + r_{0000} + r_{3000} + r_{0100} + r_{1000})!$$

$$\cdot (f_3 + r_{0000} + r_{2000} + r_{1000} + r_{0010})!(f_4 + r_{0000} + r_{0100} + r_{0010} + r_{0200})!$$

$$\cdot \frac{(f_1 + r_{0000} + r_{3000} + r_{2000} + r_{0200})!}{c_{0000}!c_{3000}!c_{2000}!c_{0200}!} (r_{0200} + r_{1200})!(r_{2000} + r_{2100})!$$

$$\cdot (r_{3000} + r_{3010})! \binom{r_{1000} + r_{1200}}{c_{1000}, c_{1200}} \binom{r_{2100} + r_{0100}}{c_{2100}, c_{0100}} \binom{r_{0010} + r_{3010}}{c_{0010}, c_{3010}}$$

$$= \sum_{f_0 + \sum f_s + \sum \overline{R} = n} \sum_{\overline{C} - \overline{D}} (-1)^{f_0 + r_{1000} + r_{0100} + r_{0010} + r_{3000} + r_{2000} + r_{0200}}$$

$$6^{f_0} 2^{\sum_1^4 s f_s} \frac{n!}{f_0! \prod_1^4 s f_s! \prod \overline{R}! c_{0000}!} c_{1200}!$$

$$\cdot \binom{r_{2000} + r_{2100}}{c_{2000}, c_{2100}} c_{2100}! \binom{r_{3000} + r_{3010}}{c_{3000}, c_{3010}} c_{3010}!$$

$$\cdot \binom{r_{1000} + r_{1200}}{c_{1000}, c_{1200}} \binom{r_{2100} + r_{0100}}{c_{2100}, c_{0100}} \binom{r_{0010} + r_{3010}}{c_{0010}, c_{3010}} \binom{r_{0200} + r_{1200}}{c_{0200}, c_{1200}}$$

$$\cdot (f_1 + r_{0000} + r_{3000} + r_{2000} + r_{0200})! (f_2 + r_{0000} + r_{3000} + r_{0100} + r_{1000})!$$

$$\cdot (f_3 + r_{0000} + r_{2000} + r_{1000} + r_{0010})! (f_4 + r_{0000} + r_{0100} + r_{0010} + r_{0200})!, \quad (7.3)$$

that is the result already obtained by Pranesachar and others in [1]. If instead we apply (6.5), then we obtain:

$$K(4, n) = \sum_{f_0 + f + \sum \overline{R} = n} \sum_{\overline{C} - \overline{D}} (-1)^{f_0 + r_{1000} + r_{0100} + r_{0010} + r_{3000} + r_{2000} + r_{0200}}$$

$$6^{f_0} 2^{f} \frac{n!}{f_0! \prod \overline{R}! \prod (\overline{C} - \overline{D})!} (r_{0000} + r_{3000} + r_{2000} + r_{0200})!$$

$$\cdot (r_{0000} + r_{3000} + r_{0100} + r_{1000})! (r_{0000} + r_{2000} + r_{1000} + r_{0010})!$$

$$\cdot (r_{0000} + r_{0100} + r_{0010} + r_{0200})! (r_{1000} + r_{1200})! (r_{2100} + r_{0100})!$$

$$\cdot (r_{0010} + r_{3010})! (r_{0200} + r_{1200})! (r_{2000} + r_{2100})! (r_{3000} + r_{3010})!$$

$$\cdot \binom{f + 4r_{0000} + 2(r_{3000} + r_{2000} + r_{0100} + r_{1000} + r_{0200} + r_{0010}) + 3}{f} \quad (7.4)$$

which is an improvement on the results known up to now, since it needs only 15 independent variables (the ten of $\overline{R}$, the three of $\overline{C} - \overline{D}$ and the two f, f_0) as compared with the 18 of the formula of Pranesachar, Athreya and Singhi.

8. Another Point of View

In conclusion, we want to show how Theorem 5.1 can have another interpretation which sheds light on its combinatory nature in a more profound way.

The circumstance — which would not have escaped a careful reader — that the elements of R and of C are as many as those of $\mathfrak{S}_m$, and that is $m!$, is not casual. In fact, if we interpret the indices $\alpha_m, ..., \alpha_1$ and $\beta_m, ..., \beta_1$ as the inversion tables of one of the permutations of $[m]$, putting $b_i = \alpha_{m+1-i}$ (or $b_i = \beta_{m+1-i}$), we will have two bijective maps between C and R and $\mathfrak{S}_m$, since $0 \le \alpha_{m+1-i} \le m - i$.

Furthermore, $\zeta(\alpha_j)$ will be the same as the number of cycles of $\sigma \in \mathfrak{S}_m$, which corresponds in this way to r_{α_j}. However it is not true – as could be thought – that the components Z_l of α_j correspond, in some way, to the cycles of the permutation σ corresponding to r_{α_j}.

To achieve this result we must introduce a new concept. Let us take a $\sigma \in \mathfrak{S}_m$, written in its standard representation and put, $\forall i \in [m] k_i$ equal to $n + 1 - t$, where t is the element furthest on the right among those to the left of i satisfying $t > i$ (or if $i = s_h$, $k_i = m + 1 - s_t$ with $s_t > s_h$ and t maximum); moreover we set $k_i = 0$ if there are no elements greater than i on the left of i. We say that $K(\sigma) = (k_1, ..., k_m)$ is the "covering table" of σ. It can be proved that the function $K(\sigma)$ is a bijection. Furthermore, it is clear that $0 \le k_i \le m - i$, $\forall i \in [m]$, and that, if $k_i = 0$, i is a left-to-right maximum of the standard representation of σ.

Now, if we put $k_i = \alpha_{m+1-j}$, we have that, $\forall i \in [m]$, $0 \le k_i \le m - i$ and therefore that $(k_1, ..., k_m)$ can be interpreted as the covering table of a $S(r_{\alpha_j}) \in \mathfrak{S}_m$. It can easily be proved that $S(r_{\alpha_j})$ is a bijection between R and $\mathfrak{S}_m$ and that, in this case too, $\zeta(\alpha_j)$ is the number of the cycles of $S(r_{\alpha_j})$. Here however, if $s_h s_{h+1} \cdots s_{h+p}$ are the elements of a cycle of σ written in its standard representation and if we take $k_{s_h}, k_{s_{h+1}}, \ldots, k_{s_{h+p}}$, then we have that $\{\alpha_{m+1-s_h}, \alpha_{m+1-s_{h+1}}, \ldots, \alpha_{m+1-s_{h+p}}\}$ constitute a component Z_{m+1-s_h} of α_j.

In the light of this new bijective map, the results obtained previously can be expressed in a new combinatory language. In fact, we can now consider the new variables r_σ and c_ϑ, the indices of which consist of elements of $\mathfrak{S}_m (\sigma, \vartheta \in \mathfrak{S}_m)$ and again indicate their sets with R and C. Furthermore, writing $\gamma | \sigma \in \mathfrak{S}_m$ to say that γ is a cycle of σ, we can put $Q(\gamma) = \{r_\sigma | \gamma | \sigma\}$ and $q(\gamma) = \sum Q(\gamma)$; corresponding meaning, going from r_σ to c_ϑ, will have $\tilde{Q}(\gamma)$ and $\tilde{q}(\gamma)$. In this way, (5.6) can be reformulated like this:

$$K(m, n) = \sum_{\sum R = n} {}_R \sum_{\sum \tilde{Q}(\gamma) = q(\gamma)} {}_C (-1)^{nm + \sum W'} \frac{\prod_{\gamma|\sigma} q(\gamma)!}{\prod R! \prod C!}$$

$$= (-1)^{n(m-1)} \sum_{\sum R = n} {}_R \sum_{\sum \tilde{Q}(\gamma) = q(\gamma)} {}_C \frac{\prod_{\gamma|\sigma} (-q(\gamma))_{¡}}{\prod R! \prod C!}, \qquad (8.1)$$

where W' indicates the set of all the r_σ in which σ has an odd number of cycles.

The simplifications of Section 6, which conduct us to (6.5), can also be read more clearly now. In fact, F_0 consists of all the r_σ in which σ is made up of only one cycle of order m, while F_s, with $s \in [m]$, is formed of those r_σ in which σ has a fixed point, made up of the element $m + 1 - s$, and a cycle of order $m - 1$ which permutes the other elements of $[m]$.

Furthermore, (8.1) reminds somehow the result attained by Gessel [3].

9. The Latin Squares

When $m = n$, we find ourselves facing the Latin squares, much more famous than the Latin rectangles for their applications in various branches of mathematics.

The number of $n \times n$ Latin squares is usually indicated by $L(n)$.

If we put $m = n$ in (5.6) and in (8.1), and, abandoning the condition that the first row is in standard form, we multiply everything by $n!$, we obtain the following elegant result which allows us to count of the number of Latin squares of any order.

Theorem 9.1.

$$L(n) = n! \sum_{\sum R = n} {}_R \sum_{\tilde{Q}(Z_l) = q(Z_l)} {}_C \frac{\prod_0^n {}_l \prod_{Z_l} (-q(Z_l))\text{¡}}{\prod R! \prod C!}$$

$$= n! \sum_{\sum R = n} {}_R \sum_{\tilde{Q}(\gamma) = q(\gamma)} {}_C \frac{\prod_{\gamma \mid \sigma} (-q(\gamma))\text{¡}}{\prod R! \prod C!}. \qquad (9.1)$$

In which the $2n!$ parameters R and C, and their totals $q(Z_l)$ and $q(\gamma)$ previously defined, appear.

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