Advances and Applications in Discrete Mathematics
Volume 5, Number 1, 2010, Pages 69-75
Published Online: March 12, 2010
This paper is available online at http://www.pphmj.com

EXTENSIONS OF MATROIDS TO UNIFORM MATROIDS

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Abstract

Let M be a matroid of rank r on an n-set, which is not direct sum of other matroids. In this paper, we prove that it is possible to determine a sequence of bases $B_1, B_2, ..., B_h$ of M and a sequence of matroids $M_1, M_2, ..., M_h$, where M_i, for $1 \leq i \leq h$, coincides with the base matroid $(M_{i-1})_{B_i}$ and $M_0 = M$, such that last matroid coincides with the uniform matroid $U_{r,n}$. This process is called the extension of M to $U_{r,n}$.

1. Introduction

Let $M = (E, \mathcal{F})$ be a matroid on a set E, having $\mathcal{F}$ as its family of independent sets.

2010 Mathematics Subject Classification: 05B35, 90C27.

Keywords and phrases: matroid, base, base matroid, direct sum of matroids.

This work is partially supported by MIUR (Ministero dell'Istruzione, dell'Università e della Ricerca).

Received November 16, 2009

Let Ξ denote the set of all closed sets of M. Then

$$\mathcal{F} = \{S \subseteq E : |S \cap \theta| \leq r(\theta), \forall \theta \in \Xi\}.$$

The terminology from matroid theory may be obtained from [8].

In [3], the notion of a set saturated with respect to a base has been introduced.

Definition 1. A set $\theta \subseteq E$ is said to be *saturated* with respect to a base B of M, or *B-saturated*, if

$$|\theta \cap B| = r(\theta).$$

Thus, any B-saturated closed set θ satisfies the relation

$$cl(\theta \cap B) = \theta,$$

in other words θ coincides with the closure of its intersection with B.

We simply call θ *saturated* when it is clear from the context whose base is considered. Denoted by Ξ_B the set of all the closed sets of M, saturated with respect to a base B, we consider the family

$$\mathcal{F}_B = \{S \subseteq E : |S \cap \theta| \leq r(\theta), \forall \theta \in \Xi_B\},$$

and the pair

$$M_B = (E, \mathcal{F}_{\mathcal{B}}).$$

Using a theorem of Edmonds and Fulkerson [4], in [3] it is proved that $M_B = (E, \mathcal{F}_{\mathcal{B}})$ is a matroid, named base matroid, which in particular turns out to be a transversal matroid.

An application of these matroids is in the field of inverse combinatorial optimization problems, indeed various inverse problems have been addressed in the recent literature [1, 3].

The main aim of this paper is to prove the following result.

Let M be a matroid of rank r on an n-set, not direct sum of other matroids. Then there exist a sequence of bases $B_1, B_2, ..., B_h$ of M and a sequence of matroids $M_1, M_2, ..., M_h$, where M_i, for $1 \leq i \leq h$, coincides with the base matroid $(M_{i-1})_{B_i}$ and $M_0 = M$, such that last matroid M_h coincides with the uniform matroid $U_{r,n}$.

2. Base Matroids

First recall some properties of the base matroids.

From the definition an independent set S of $\mathcal{F}$ satisfies the condition $|S \cap \theta| \le r(\theta)$, for every $\theta \in \Xi$. In particular, if $\theta \in \Xi_B$, then S is a set of $\mathcal{F}_{\mathcal{B}}$. It implies that

$$\mathcal{F} \subseteq \mathcal{F}_{\mathcal{B}}.$$

The above relation implies that M is isomorphic to M_B, $M \simeq M_B$, if and only if

$$\mathcal{F} = \mathcal{F}_{\mathcal{B}}.$$

The inclusion is proper when a dependent set of M turns out to be independent in M_B and in this case, M is not isomorphic to M_B.

If B is a base of M, then it belongs to $\mathcal{F}_{\mathcal{B}}$ and has maximal cardinality. Then it is also a base of $\mathcal{F}_{\mathcal{B}}$, thus, we have that $r(M) = r(M_B)$. In this context in [6] it is proved that

Lemma 1. *Let M be a matroid and B be one of its bases. Then* $M \simeq M_B$ *if and only if every circuit of M is also circuit of* M_B.

In [6] a circuit C of M is said to be *independent* with respect to a base B or B-independent if

$$|cl(C) \cap B| < |C| - 1.$$

The circuit C is *dependent* with respect to B or B-dependent if it is not independent with respect to B, that is,

$$|cl(C) \cap B| = |C| - 1.$$

Thus, $cl(C)$ is saturated with respect to B.

Lemma 2. *Let C be a circuit of M having cardinality* $r + 1$. *Then C is B-dependent, for every base B of M.*

Proof. If C has cardinality $r + 1$, then $cl(C)$ is isomorphic to M and $|cl(C) \cap B| = |C| - 1$, thus, C is B-dependent, for every base B. □

A consequence is that in the case of a uniform matroid, previous result is satisfied for every circuit. Moreover, in [6] it is proved the following result.

Proposition 1. *Let M be a uniform matroid. Then for every base B of M it is* $M \simeq M_B$.

In [5] the problem of characterizing a rank-*n* matroid *M* isomorphic to M_B for every base *B* of *M* is studied. It is proved that this condition is equivalent to say that for every circuit *C*, there exists a closed *B*-saturated set θ, such that

$$|\theta \cap C| > r(\theta).$$

It implies that $\theta = cl(C)$, and $r(\theta) = |C| - 1$. The condition that $cl(C)$ is *B*-saturated implies

$$|cl(C) \cap B| = |C| - 1,$$

and *C* is *B*-dependent. Note that for every matroid *M* on *E*, the set *E* turns out to be *B*-saturated for every base *B*.

Recall the definition of direct sum of matroids. A matroid *M* on a ground set *E*, whose family of independent sets is $\mathcal{F}$, is direct sum of the matroids $M_1, M_2, \ldots, M_s$ on disjoint sets $E_1, E_2, \ldots, E_s$, respectively, when $E_1, E_2, \ldots, E_s$ is a partition of *E* and $\mathcal{F} = \{I_1 \cup \cdots \cup I_s : I_i \in \mathcal{F}(M_i), 1 \leq i \leq s\}$, where $\mathcal{F}(M_i)$ is the family of independent sets of M_i. Next theorem states the characterization about matroids isomorphic to their base matroids proved in [5].

Theorem 1. *Let M be a matroid on a ground set E. Then M is isomorphic to* M_B, *for every base B, if and only if M is either uniform or direct sum of uniform matroids.*

3. Extension of Matroids

In this section, we consider the problem of studying the possible repetition of the passage from a matroid *M* to one of its base matroids. Let *M* be a matroid of rank *r* and *B* be one of its bases. From Theorem 1, it follows that if *M* is not isomorphic to M_B, then there exists a circuit *C* of *M* which is *B*-independent. It follows that *C* turns out to be an independent set of M_B.

Now we may apply the same previous considerations to M_B. Thus, if M_B is not isomorphic to $(M_B)_{B'}$ in relation to a base B' of M_B, then there is a circuit H of M_B which is B'-independent.

First consider the case of $B' = B$.

Lemma 3. *Let B be a base of M. Then* $M_B \cong (M_B)_B$.

Proof. If a circuit C of M_B is B-independent, then it would have been B-independent in M and then not a circuit of M_B. It follows that the collection of circuits B-dependent of M coincide with the same collection of M_B and the matroids M_B and $(M_B)_B$ are isomorphic. □

Notice that the circuits of M_B are the B-dependent circuits of M; moreover, a base of M_B can be either a base of M, or a subset of E which contains a B-independent circuit of M.

Now we consider the problem of establishing when a circuit C, having cardinality lesser than $r+1$, turns out to be B-dependent for every base B. In other words, our aim is to characterize a circuit C of rank $m < r$, such that

$$|\, cl(C) \cap B \,| = m,$$

for every base B.

This condition implies that every base B may be represented as union of two disjoint independent sets I_1 and I_2 of cardinality m and $r - m$, respectively, and $I_1 = cl(C) \cap B$.

This situation is satisfied when $cl(C)$ is a separator of M [8]. Then also $E - cl(C)$ is a separator of M. Indeed

$$r(cl(C)) + r(E - cl(C)) = r(M)$$

because

$$|\, cl(C) \cap B \,| + |\, (E - cl(C)) \cap B \,| = |\, B \,|.$$

In this case $cl(C)$ and $E - cl(C)$ are components of M [7]. Moreover, denoted $T_1 = cl(C), T_2 = E - cl(C)$, M is direct sum of $M|T_1$ and $M|T_2$, where $M|T_1$ and $M|T_2$ are the restrictions of M to T_1 and T_2.

It follows that if M is not direct sum of other matroids, then a separator cannot exist.

As consequence every circuit C has to be independent with respect to at least one base B.

Then in M_B the elements of C form an independent set.

Now consider a circuit H of M_B. By Lemma 2, H is also a circuit of M, denote by B' a base of M with respect to which H is independent. Then in $(M_B)_{B'}$ the elements of H form an independent set.

We may continue until to exhaust all the circuits of M of rank lesser than r. We obtain a sequence $B_1, B_2, \ldots, B_h$ of bases and in correspondence a sequence of matroids $M_1, M_2, \ldots, M_h$, where $M_i = (M_{i-1})_{B_i}$, $1 \leq i \leq h$ and $M_0 = M$, such that in the last matroid M_h all the possible circuits have rank r, thus, it coincides with the uniform matroid $U_{r,n}$.

We call the process from M to $U_{(r,n)}$ an *extension* of M to $U_{(r,n)}$.

These considerations imply the following result.

Theorem 2. *Let $M = (E, \mathcal{F})$ be a matroid, of rank n on the n-set E, which is not direct sum of other matroids. Then there exist a suitable sequence of bases $B_1, B_2, \ldots, B_h$, $h \geq 1$, of M and a sequence $M_1, M_2, \ldots, M_h$ of matroids, where $M_i = (M_{i-1})_{B_i}$, for $1 \leq i \leq h$ and $M_0 = M$, such that last matroid M_h coincides with the uniform matroid $U_{r,n}$.*

References

[1] M. Cai, Inverse problems of matroid intersection, J. Comb. Optim. 3(4) (1999), 465-474.

[2] R. Cordovil, D. Forge and S. Klein, How is a chordal graph like a supersolvable binary matroid?, Discrete Math. 288(1-3) (2004), 167-172.

[3] M. Dell'Amico, F. Maffioli and F. Malucelli, The base-matroid and inverse combinatorial optimization problems, Discrete Appl. Math. 128(2-3) (2003), 337-353.

[4] J. Edmonds and D. R. Fulkerson, Transversals and matroid partition, J. Res. Nat. Bur. Standards Sect. 69B (1965), 147-153.

[5] F. Maffioli and N. Zagaglia Salvi, On some properties of base-matroids, Discrete Appl. Math. 154(9) (2006), 1401-1407.

[6] F. Maffioli and N. Zagaglia Salvi, A characterization of the base-matroids of a graphic matroid, Contributions to Discrete Mathematics, to appear.

[7] F. Maffioli and N. Zagaglia Salvi, On binary matroids not isomorphic to their base matroids, Ars Combinatoria, to appear.

[8] J. G. Oxley, Matroid Theory, Oxford University Press, New York, 1992.