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EXISTENCE OF WEAK SOLUTIONS FOR A CLASS OF NONUNIFORMLY ELLIPTIC EQUATIONS OF p-LAPLACIAN TYPE IN R^N

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Abstract

Using a variational approach, we study a class of nonlinear elliptic systems derived from a potential and involving the p-laplacian. Under growth and regularity conditions on the nonlinearities f and g, we show the existence of nontrivial solutions by applying a variant of the Mountain Pass theorem.

1. Introduction

In this paper, we deal with the nonlinear elliptic system

$$\begin{cases} -div(h_1(x)|\nabla u|^{p-2}\nabla u) + a(x)|u|^{p-2}u = f(x, u, v) & \text{in} \quad R^N, \\ -div(h_2(x)|\nabla v|^{p-2}\nabla v) + b(x)|v|^{p-2}v = g(x, u, v) & \text{in} \quad R^N, \end{cases} \tag{1.1}$$

where $N \geq 3$, $h_i \in L^1_{loc}(R^N)$, $h_i(x) \geq 1$, $i = 1, 2$, $a, b \in C(R^N)$. We assume that there exist $a_0, b_0 > 0$ such that

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$$a(x) \geq a_0, \quad b(x) \geq b_0, \quad \forall x \in R^N,$$
$$a(x) \to \infty, \quad b(x) \to \infty \quad \text{as} \quad |x| \to \infty. \tag{1.2}$$

We observe that there exists an extensive bibliography on the study of elliptic systems (see [2, 6, 8] and the references therein). In particular, we mention the article [5], where the problem (1.1) was studied with $p = q = 2$ and $h_1 = h_2 = 1$. In the article [4], the authors considered the system (1.1) for $p = q = 2$.

Let $H^1 = H^1(R^N, R^2)$ denote the Sobolev space of pairs $w = (u, v)$ of L^P-functions $u, v : R^N \to R$ with weak derivatives $\frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j}$ $(j = 1, 2, ..., N)$ also in $L^P(R^N)$, endowed with its usual norm

$$\| w \|^p = \int (|\nabla w|^p + |w|^p) dx = \int (|\nabla u|^p + |\nabla v|^p + |u|^p + |v|^p) dx.$$

Throughout this paper, unless specified otherwise, all integrals are understood to be taken over all of R^N. To prove our main results, we introduce the following hypotheses:

(H1) There exists a function $F(x, w) \in C^1(R^N \times R^2, R)$ such that

$$\frac{\partial F}{\partial u} = f(x, w), \frac{\partial F}{\partial v} = g(x, w), \quad \text{for all } x \in R^N, \, w = (u, v) \in R^2.$$

(H2) The nonlinearities $f(x, w), g(x, w) \in C^1(R^N \times R^2, R)$ with $f(x, 0, 0) = g(x, 0, 0) = 0$, for all $x \in R^N$, there exists a positive constant τ_0 such that

$$|\nabla f(x, w)| + |\nabla g(x, w)| \leq \tau_0 |w|^{p-1},$$

for all $x \in R^N$ and $w \in R^2$.

(H3) There exists a constant $\mu > p$ such that

$$0 < \mu F(x, w) \leq w \nabla F(x, w),$$

for all $x \in R^N$, $w \in R^2 \setminus (0, 0)$. Consider the subspace

$$E = \left\{ (u, v) \in H^1(R^N, R^2) : \int_{R^N} (|\nabla u|^p + |\nabla v|^p + a(x)|u|^p + b(x)|v|^p) dx < \infty \right\},$$

then E is a banach space with the norm

$$\| w \|_E^p = \int_{R^N} (|\nabla u|^p + |\nabla v|^p + a(x)|u|^p + b(x)|v|^p)dx.$$

By (1.2), it is clear that

$$\| w \|_E \geq m_0 \| w \|_{H^1(R^N, R^2)} \quad \forall w \in E, \quad m_0 > 0,$$

and the embeddings $E \hookrightarrow H^1(R^N, R^2) \hookrightarrow L^q(R^N, R^2)$, $p \leq q \leq p^*$ are continuous.

Moreover, the embedding $E \hookrightarrow L^q(R^N, R^2)$ is compact (see [4]). We now introduce the space

$$H = \left\{ (u, v) \in E : \int_{R^N} (h_1(x)|\nabla u|^p + h_2(x)|\nabla v|^p + a(x)|u|^p + b(x)|v|^p)dx < \infty \right\}$$

endowed with the norm

$$\| w \|_H^p = \int_{R^N} (h_1(x)|\nabla u|^p + h_2(x)|\nabla v|^p + a(x)|u|^p + b(x)|v|^p)dx.$$

It can easily be shown that H is a banach space with the above norm.

Definition 1.1 (Weak solution)**.** We say that (u, v) is a *weak solution* of (1.1) if

$$\int h_1(x)|\nabla u|^{p-2}\nabla u \nabla \varphi + \int a(x)|u|^{p-2}u\varphi dx = f(x, u, v)\varphi dx,$$

$$\int h_2(x)|\nabla v|^{p-2}\nabla v \nabla \psi + \int b(x)|v|^{p-2}v\psi dx = g(x, u, v)\psi dx,$$

for all $\phi = (\varphi, \psi) \in H$.

2. Main Result

Our main result is stated as follows:

Theorem 2.1. *Assuming that* (1.2) *and* (H1)-(H3) *are satisfied*, *then the system* (1.1) *has at least one nontrivial weak solution in H*.

It is clear that system (1.1) has a variational structure. Let $J : H \to R$ be defined by

$$J(w) = \frac{1}{p}\int (h_1(x)|\nabla u|^p + h_2(x)|\nabla v|^p + a(x)|u|^p + b(x)|v|^p)dx$$

$$-\int F(x, u, v)dx$$

$$= T(w) - p(w) \text{ for } w = (u, v) \in H, \tag{2.1}$$

where

$$T(w) = \frac{1}{p}\int (h_1(x)|\nabla u|^p + h_2(x)|\nabla v|^p + a(x)|u|^p + b(x)|v|^p)dx, \tag{2.2}$$

$$p(w) = \int F(x, u, v)dx. \tag{2.3}$$

Clearly, the critical points of J correspond to the weak solutions of problem (1.1). In general, due to $h(x) \in L^1_{loc}(R^N)$, the functional J may not belong to $C^1(H)$ (in this paper, we do not completely care whether the functional J belongs to $C^1(H)$ or not). This means that we cannot apply directly the Mountain Pass theorem by Ambrosetti-Rabinowitz (see [2, 6]), our approach is based on a weak version of the Mountain Pass theorem by Duc (see [7]).

Proposition 2.2. *Under the assumptions of Theorem* 2.1, *the functional* $J(w)$, $w \in H$ *given by* (2) *is weakly continuously differentiable on H and*

$$\langle J'(w), \phi\rangle = \int_{R^N} (h_1(x)|\nabla u|^{p-2}\nabla u\nabla\phi + h_2(x)|\nabla v|^{p-2}\nabla v\nabla\psi$$

$$+ a(x)|u|^{p-2}u\phi + b(x)|v|^{p-2}v\psi)dx - \int (f(x, u, v)\phi + g(x, u, v)\psi)dx,$$

for all $w = (u, v)$, $\Phi = (\phi, \psi) \in H$.

By conditions (H1)-(H3) and the embedding $H \hookrightarrow E$, it can be shown that the functional P is well defined and of class $C^1(H)$. Moreover, we have

$$\langle P'(w), \phi\rangle = \int_{R^N} (f(x, u, v)\phi + g(x, u, v)\psi)dx,$$

for all $w = (u, v)$, $\Phi = (\phi, \psi) \in H$.

Next, we prove that T is continuous. Let $\{w_n\}$ be a sequence converging to w in $H(\| w_m \|_H \to \| w \|_H)$, where $w_m = (u_m, v_m)$, $m = 1, 2, \ldots$ and $w = (u, v)$. Then

$$| T(w_n) - T(w) | = \left| \frac{1}{p} \left(\int h_1(x)(| \nabla u_m |^p - | \nabla u |^p) + h_2(x)(| \nabla v_m |^p - | \nabla v |^p) \right.\right.$$

$$\left.\left. + \, a(x)(| u_m |^p - | u |^p) + b(x)(| v_m |^p - | v |^p) \right) dx \right|$$

$$= \frac{1}{p} | \, \| w_m \|_H - \| w \|_H \, | \to 0.$$

Thus, T is continuous on H. Next, we prove that for all $w = (u, v)$, $\Phi = (\phi, \psi) \in H$

$$\langle J'(w), \phi \rangle = \int (h_1(x) \nabla u \nabla \phi + h_2(x) \nabla v \nabla \psi + a(x) u \phi + b(x) v \psi) dx.$$

Indeed

$$\langle J'(w), \Phi \rangle = \frac{d}{dt} J(w + t\Phi) \Big|_{t=0}$$

$$= \frac{d}{dt} \left[\int (h_1(x) | \nabla u + t \nabla \phi | + h_2(x) | \nabla v + t \nabla \psi | \right.$$

$$\left. + \, a(x) | u + t\phi |^p + b(x) | v + t\psi |^p) dx \right] \Big|_{t=0}$$

$$= \int (h_1(x) \nabla u \nabla \phi + h_2(x) \nabla v \nabla \psi + a(x) u \phi + b(x) v \psi) dx.$$

Thus, T is weakly differentiable on H. We can conclude that functional T is weakly continuously differentiable on H. Finally, J is weakly continuously differentiable on H.

Proposition 2.3. *The functional* $J(w)$, $w \in H$ *given by* (2.1) *satisfies the Palais-Smale condition*.

Proof. Let $\{w_m = (u_m, v_m)\} \subset H$ be a Palais-Smale sequence, i.e., $| J(w_m) | \leq c$, for all m and $J'(w_m) \to 0$ as $m \to \infty$. For m large enough, we have $\frac{1}{\mu} | \langle J'(w_m), w_m \rangle | \leq \| w_m \|_H$ and by

$$c + \| w_m \|_H \geq J(w_m) - \frac{1}{\mu} \langle J'(w_m), w_m \rangle \geq \left(\frac{1}{p} - \frac{1}{\mu} \right) \| w_m \|_H^p.$$

This shows that $\{w_m\}$ is bounded in H. This implies that there exists $w_0 \in H$ such that at least in sequence; $\{w_m\}$ converges to w_0 weakly in H and strongly in L^p. Using $J'(w_m) \to 0,$ we obtain

$$\langle J'(w_m) - J'(w_0),\, w_m - w_0 \rangle$$

$$= \int h_1(x)(|\nabla u_m|^{p-2}\nabla u_m - |\nabla u_0|^{p-2}\nabla u_0)\nabla(u_m - u_0)dx$$

$$+ \int h_2(x)(|\nabla v_m|^{p-2}\nabla v_m - |\nabla v_0|^{p-2}\nabla v_0)\nabla(v_m - v_0)dx$$

$$+ \int a(x)(|u_m|^{p-2}u_m - |u_0|^{p-2}u_0)\nabla(u_m - u_0)dx$$

$$+ \int b(x)(|v_m|^{p-2}v_m - |u_0|^{p-2}v_0)\nabla(v_m - v_0)dx$$

$$+ \int (f(x, u_m, v_m) - f(x, u, v))dx \to 0.$$

Due to the continuity of the Nemyteskiy operators $u \to |u|^{p-2}u$ and $v \to |v|^{p-2}v$ from $L^p(\Omega)$ into $L^{p/(p-1)}(\Omega)$, and H_2 the last three integrals approach zero.

Observe that for all $x_1, x_2 \in R^N$, and $1 < p < 2$,

$$|x_2|^p \geq |x_1|^p + p|x_1|^{p-2}x_1(x_2 - x_1) + c(p)\frac{|x_1 - x_2|^2}{(|x_1| + |x_2|)^{2-p}}$$

and for $p \geq 2$,

$$|x_2|^p \geq |x_1|^p + p|x_1|^{p-2}x_1(x_2 - x_1) + c(p)\frac{|x_2 - x_1|^p}{2^{p-1} - 1}.$$

Then for $2 \leq p$, we have

$$|h_1^{1/p}\nabla u_0|^p \geq |h_1^{1/p}\nabla u_m|^p + p|h_1^{1/p}\nabla u_m|^{p-2}(h_1^{1/p}\nabla u_m)$$

$$\times (h_1^{1/p}\nabla u_0 - h_1^{1/p}\nabla u_m) + \frac{|h_1^{1/p}\nabla u_m - h_1^{1/p}\nabla u_0|^p}{2^{p-1} - 1}$$

$$\Rightarrow \int |h_1^{1/p}\nabla u_m|^{p-2}(h_1^{1/p}\nabla u_m)(h_1^{1/p}\nabla u_m - h_1^{1/p}\nabla u_0)$$

$$\geq \frac{1}{p}\int |\, h_1^{1/p}\nabla u_m\,|^p - \frac{1}{p}\int |\, h_1^{1/p}\nabla u_0\,|^p$$

$$+\frac{1}{p(2^{p-1}-1)}\int |\, h_1^{1/p}\nabla u_m - h_1^{1/p}\nabla u_0\,|^p$$

$$\Rightarrow \int h_1(x)|\,\nabla u_m\,|^{p-2}\nabla u_m(\nabla u_m - \nabla u_0)$$

$$\geq \frac{1}{p}\int h_1|\,\nabla u_m\,|^p - \frac{1}{p}\int h_1|\,\nabla u_0\,|^p + \frac{1}{p(2^{p-1}-1)}\int h_1|\,\nabla u_m - \nabla u_0\,|^p,$$

similarly

$$-\int h_1(x)|\,\nabla u_0\,|^{p-2}\nabla u_0(\nabla u_m - \nabla u_0)$$

$$\geq -\frac{1}{p}\int h_1|\,\nabla u_m\,|^p + \frac{1}{p}\int h_1|\,\nabla u_0\,|^p + \frac{1}{(p(2^{p-1}-1))}\int h_1|\,\nabla u_0 - \nabla u_m\,|^p.$$

Therefore

$$\int h_1(x)(|\,\nabla u_m\,|^{p-2}\nabla u_m - |\,\nabla u_0\,|^{p-2}\nabla u_0)\nabla(u_m - u_0)$$

$$\geq \frac{2}{p(2^{p-1}-1)}\int h_1|\,\nabla u_m - \nabla u_0\,|^p,$$

similarly we have

$$\int h_2(x)(|\,\nabla v_m\,|^{p-2}\nabla v_m - |\,\nabla v_0\,|^{p-2}\nabla v_0)\nabla(v_m - v_0)$$

$$\geq \frac{2}{p(2^{p-1}-1)}\int h_2|\,\nabla v_m - \nabla v_0\,|^p,$$

then

$$\int h_1|\,\nabla u_m - \nabla u_0\,|^p + \int h_2|\,\nabla v_m - \nabla v_0\,|^p \to 0$$

and

$$\int a(x)|\,u_m - u_0\,|^p + b(x)|\,v_m - v_0\,|^p$$

$$\leq \max_\Omega \sup(a(x),\, b(x))\int |\,u_m - u_0\,|^p + b(x)|\,v_m - v_0\,|^p \to 0,$$

so we have

$$\| w_m - w_0 \|_H = \int h_1 | \nabla u_m - \nabla u_0 |^p + \int h_2 | \nabla v_m - \nabla v_0 |^p$$

$$+ a(x) | u_m - u_0 |^p + b(x) | v_m - v_0 |^p \to 0,$$

therefore, we conclude that $\{w_n\}$ converges strongly to w_0 in *H* and *J* satisfies the Palais-Smale condition on *H*.

To apply the Mountain Pass theorem, we shall prove the following proposition which shows that the functional *J* has the Mountain Pass geometry:

Proposition 2.4. (i) *There exist* $\alpha > 0$ *and* $r > 0$ *such that* $J(w) \geq \alpha$, *for all* $w \in H$ *and* $\| w \|_H = r$.

(ii) *There exists* $w_0 \in H$ *such that* $\| w \|_H > r$ *and* $J(w_0) < 0$.

From (H3), it is easy to see that

$$F(x, z) \geq \min_{| s | = 1} F(x, s) \cdot | z |^\mu > 0, \quad \forall x \in R^N \quad \text{and} \quad | z | \geq 1, \quad z \in R^2, \quad (2.4)$$

$$0 < F(x, z) \leq \max_{| s | = 1} F(x, s) \cdot | z |^\mu, \quad \forall x \in R^N \quad \text{and} \quad 0 < | z | \leq 1, \quad z \in R^2,$$

where $\max_{| s | = 1} F(x, s) \leq C$ in view of (H2). It follows that

$$\lim_{| z | \to 0} \frac{F(x, z)}{| z |^2} = 0 \quad \text{uniformly for } x \in R^N.$$

By using the embeddings $H \hookrightarrow E \hookrightarrow L^p(R^N, R^2)$, with simple calculation, we infer that $\inf_{\| w \|_H = r} J(w) = \alpha > 0$ for $r > 0$ small enough. This implies (i).

(ii) By (2.4), for each compact set $\Omega \subset R^N$, there exists $c = c(\Omega)$ such that

$$F(x, z) > c | z |^\mu \quad \text{for all} \quad x \in \Omega, \quad | z | \geq 1.$$

Let $0 \neq \phi = (\varphi, \psi) \in C^1(R^N, R^2)$ having compact support, for $t > 0$ large enough, we have

$$J(t\phi) = \frac{1}{p} t^p \| \phi \|_H^p - \int F(x, t\phi) dx \leq \frac{1}{p} t^p \| \phi \|_H^p - t^\mu c \int | \phi |^\mu dx,$$

where $c = c(\Omega)$, $\Omega = (\operatorname{supp} \varphi \bigcup \operatorname{supp} \psi)$. Since $\mu > p$, (ii) is proved.

Furthermore, the acceptable set

$$G = \{\gamma \in C([0, 1], H) : \gamma(0) = \gamma(1) = \omega_0\},$$

where w_0 is given in Proposition 2.4, is not empty. So, all the assumptions of the Mountain Pass theorem are satisfied. Therefore, there exists $u \in H$ such that $J(u) \geq \alpha > 0$.

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