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A RENEWAL INPUT ACCESSIBLE AND NON-ACCESSIBLE BATCH SERVICE QUEUE WITH SERVERS' VACATION

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Abstract

This paper considers an infinite buffer single server accessible and non-accessible batch service queue with single and multiple exponential vacations, which has a wide range of applications in several areas including manufacturing and communication systems. The inter-arrival times are general independent and identically distributed random variables and the service times are exponential. We provide a recursive method, using the supplementary variable technique and treating the remaining inter-arrival time as the supplementary variable, to develop the steady-state queue length distributions at pre-arrival and arbitrary epochs. Some numerical results are presented in the form of self explanatory tables and

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graphs. Moreover, some queueing models discussed in the literature are derived as special cases of our model.

1. Introduction

Bulk-service queues have attracted much attention due to practical applicability in the field of communication systems, lift operations, cargo loading and unloading problems, etc. These queues and other variants of general bulk-service queues have been studied by many authors, see Medhi [17, 18], Chaudhry and Templeton [6], Baba [1], Neuts [19], Gold and Tran-Gia [8], Chaudhry and Gupta [5], and Hébuterne and Rosenberg [13]. Further, the customers arriving one at a time must wait in the queue until a sufficient number of customers are accumulated in the queue in order to utilize the server effectively. The server has pre-specified minimum and maximum threshold capacities of service. The concept of accessibility into batches during service has been considered by Gross et al. [11], Kleinrock [15], Sivasamy [21] and Goswami et al. [9]. The infinite buffer queue with accessible and non-accessible batch service rule has been studied by Sivasamy [22], where the arrivals and service times are exponentially distributed. In discrete-time systems, the same type of model has been studied by Goswami et al. [9] with finite and infinite buffers.

In most of the queueing models, on completion of service to the existing customers, the server stays in the empty system awaiting for a new arrival. But there are situations where if the server after completing the service of a customer finds the queue empty, then it goes away for a length of time called *vacation*. This time may be utilized by the server to carry out some additional work. On return from a vacation if it finds one or more customers waiting, it takes them for service on a one-by-one basis until the system empties, after which time it takes another vacation. However, if, on return from a vacation, it finds no customer waiting, then in the case of single vacation, it remains dormant until at least one customer arrives, whereas in the case of multiple vacation it immediately proceeds for another vacation and continues in this manner until it finds at least one waiting customer upon return from a vacation. Queueing models with server vacations are characterized by the fact that the idle time of the server may be utilized for some other jobs.

Vacation models have applications in modelling of computer systems, data communication networks, traffic concentrators and other related areas. The queues with vacation have also attracted many researchers due to their wide importance in the areas of manufacturing, computer communication systems, etc. In the past, several authors have studied queueing systems with vacations, Tian et al. [23], Chaterjee and Mukerjee [4], etc. Chae et al. [3] obtained queue length and waiting time in terms of probability generating function for the continuous-time $GI/M/1$ queueing model with single vacation. Chae and Kim [2] obtained the length of a busy period, the number of customers served during busy period, and the residual inter-arrival time at the instant the busy period ends. The batch service queues with single and multiple exponential vacations have been studied by Sikdar [21], where inter-arrival time of customers and service time of batches are, respectively, exponentially and arbitrarily distributed. The batch service $GI/M^{(a,b)}/1$ queue with multiple vacations has been analyzed by Choi and Han [7]. Further, studies related to batch service queue with servers' vacation are found in Lee et al. [16], Samanta et al. [20], and Gupta and Vijaya [12]. However, the batch service queues with accessible and non-accessible batch service and servers' vacation have not been studied so far. It may be noted that the general uncorrelated arrival process appears to be more appropriate and reasonable than the exponential distribution, as the memoryless property of the exponential arrival process does not always fit in many application areas. Further, the general arrival process includes exponential, deterministic, Erlang distributions etc., as special cases.

The present paper focuses on the study of infinite buffer queue with accessible and non-accessible batch services for both single and multiple exponential vacations. The inter-arrival time of customers and service time of batches are, respectively, arbitrarily and exponentially distributed. We provide a recursive method, using the supplementary variable technique and treating the remaining inter-arrival time as the supplementary variable, to develop the steady-state queue length distributions at pre-arrival and arbitrary epochs. Numerical results have been presented in the form of tables and graphs. Also, we can obtain the result of non-vacation $GI/M^{(a,d,b)}/1/\infty$ queue by taking the vacation parameter sufficiently large, so that the mean vacation time tends to zero. The queueing model presented above has applications in the field of communication systems, polling systems, cinema theaters and many other such related areas.

This paper is organized as follows. Section 2 presents the model description and solution of the model for both single and multiple vacation policies. Various performance measures are presented in Section 3. Some special cases which are matched with existing results in the literature are demonstrated in Section 4. Numerical results in the form of tables and graphs are presented in Section 5. Section 6 concludes the paper.

2. The Model Description and Solution of the Model

Let us consider an infinite buffer single server accessible and non-accessible batch service queue with single and multiple vacations. The inter-arrival times are independent and identically distributed random variables with probability distribution function $A(u)$, probability density function $a(u)$, $u \geq 0$, Laplace-Stieltjes transform (LST) $A^*(\theta)$, $\mathrm{Re}(\theta) \geq 0$. The mean inter-arrival time is $E(A) = -A^{*(1)}(0) = 1/\lambda$ (say), where λ is the mean arrival rate. The customers are served exponentially with parameter μ by a single server in batches of maximum size b with a minimum threshold value a. However, if the server finds $0 \leq n \leq a-1$ customers present in the system at a service completion epoch of a batch, it proceeds to take exponential vacations with parameter ϕ. At the end of the vacation, if the server finds $n \geq a$ customers waiting in the system, it begins to serve them according to batch service rule. Otherwise, if the server sees $0 \leq n \leq a-1$ customers in the system at the end of that vacation, it either goes to another vacation (multiple vacation (MV)) or enters the idle phase (single vacation (SV)). If b or more customers are present in the queue at service initiate epoch/vacation completion, then only b of them are taken into service. It is further assumed that the late entries can join a batch in course of ongoing service as long as the number of customers in that batch is less than $d < b$ (called *maximum accessible limit*). At every departure epoch of service, the server may find the system in any one of the following three cases: (i) $0 \leq n \leq a-1$, (ii) $a \leq n \leq d-1$ and (iii) $n \geq d$. In case (i), the server cannot initiate service, it goes to vacation state. In case (ii), the server takes the entire queue for batch service and admits the subsequent arrivals in the batch while the service is on, till the accessible limit d is reached, and such a batch is called an *accessible batch* (*AB*). In case (iii), it takes $\min(n, b)$ customers for the service and does not allow further arrivals into the batch being served even if the current batch size is not b, that is, when the batch size is greater than or equal to d, the batch becomes non-accessible (NAB) for late arriving customers. The traffic intensity is given by $\rho = \lambda/b\mu < 1$. We analyze both multiple vacation and single vacation models together and for that purpose we introduce an indicator function (δ) as follows: $\delta = 1$ yields the results for the single vacation policy and $\delta = 0$ gives the results for multiple vacation policy.

The state of the system prior to a potential arrival at time t is described by the following random variables, namely,

- $N_s(t) =$ number of customers present in the system including those in service;
- $N_q(t) =$ number of customers present in the queue not counting those in service;
- $U(t) =$ remaining inter-arrival time for the next arrival;
- $\zeta(t) = \begin{cases} 0, & \text{if the server is in dormancy (idle phase)}, \\ 1, & \text{if the server is on vacation}, \\ 2, & \text{if the server is busy with an accessible batch}, \\ 3, & \text{if the server is busy with a non-accessible batch.} \end{cases}$

Let us define the joint probabilities by

$$R_n(u, t)du = Pr(N_s(t) = n, u < U(t) \le u + du, \zeta(t) = 0), u \ge 0, 0 \le n \le a - 1,$$

$$P_{n,0}(u, t)du = Pr(N_s(t) = n, u < U(t) \le u + du, \zeta(t) = 1), u \ge 0, n \ge 0,$$

$$Q_{n,0}(u, t)du = Pr(N_s(t) = n, u < U(t) \le u + du, \zeta(t) = 2), u \ge 0, a \le n \le d - 1,$$

$$Q_{n,1}(u, t)du = Pr(N_q(t) = n, u < U(t) \le u + du, \zeta(t) = 3), u \ge 0, n \ge 0.$$

As we shall discuss the model in limiting case, that is, when $t \to \infty$ the above probabilities will be denoted by $R_n(u) = \lim_{t\to\infty} R_n(u, t)$, $P_{n,0}(u) = \lim_{t\to\infty} P_{n,0}(u, t)$, $Q_{n,0}(u) = \lim_{t\to\infty} Q_{n,0}(u, t)$ and $Q_{n,1}(u) = \lim_{t\to\infty} Q_{n,1}(u, t)$, and their Laplace transforms are $R_n^*(\theta)$, $P_{n,0}^*(\theta)$, $Q_{n,0}^*(\theta)$ and $Q_{n,1}^*(\theta)$, respectively.

To obtain the queue length distribution at arbitrary epochs, we develop relations between distributions of number of customers in the system/queue at pre-arrival and arbitrary epochs. Relating the states of the system at two consecutive time epochs t and $t + dt$, using probabilistic arguments, we have in the steady-state

$$-\frac{d}{du}R_0(u) = \delta\phi P_{0,0}(u), \tag{1}$$

$$-\frac{d}{du}R_n(u) = \delta\phi P_{n,0}(u) + a(u)\delta R_{n-1}(0),\ 1 \le n \le a-1, \tag{2}$$

$$-\frac{d}{du}P_{0,0}(u) = -\delta\phi P_{0,0}(u) + \mu Q_{0,1}(u) + \mu\sum_{i=a}^{d-1} Q_{i,0}(u), \tag{3}$$

$$-\frac{d}{du}P_{n,0}(u) = -\delta\phi P_{n,0}(u) + \mu Q_{n,1}(u) + a(u)P_{n-1,0}(0),\ 1 \le n \le a-1, \tag{4}$$

$$-\frac{d}{du}P_{n,0}(u) = -\phi P_{n,0}(u) + a(u)P_{n-1,0}(0),\quad n \ge a, \tag{5}$$

$$-\frac{d}{du}Q_{a,0}(u) = -\mu Q_{a,0}(u) + \phi P_{a,0}(u) + \mu Q_{a,1}(u) + a(u)\delta R_{a-1}(0), \tag{6}$$

$$\begin{aligned}-\frac{d}{du}Q_{n,0}(u) = {}& -\mu Q_{n,0}(u) + \phi P_{n,0}(u) + \mu Q_{n,1}(u)\\ & + a(u)Q_{n-1,0}(0),\quad a+1 \le n \le d-1,\end{aligned} \tag{7}$$

$$-\frac{d}{du}Q_{0,1}(u) = -\mu Q_{0,1}(u) + \phi\sum_{k=d}^{b} P_{k,0}(u) + \mu\sum_{k=d}^{b} Q_{k,1}(u) + a(u)Q_{d-1,0}(0), \tag{8}$$

$$-\frac{d}{du}Q_{n,1}(u) = -\mu Q_{n,1}(u) + \mu Q_{n+b,1}(u) + a(u)Q_{n-1,1}(0) + \phi P_{n+b,0}(u),\ n \ge 1. \tag{9}$$

Multiplying (1) to (9) by $e^{-\theta u}$ and integrating with respect to u from 0 to ∞, yields

$$-\theta R_0^*(\theta) = \delta\phi P_{0,0}^*(\theta) - R_0(0), \tag{10}$$

$$-\theta R_n^*(\theta) = \delta\phi P_{n,0}^*(\theta) + A^*(\theta)\delta R_{n-1}(0) - R_n(0),\ \ 1 \le n \le a-1, \tag{11}$$

$$(\delta\phi - \theta)P_{0,0}^*(\theta) = \mu Q_{0,1}^*(\theta) + \mu\sum_{i=a}^{d-1} Q_{i,0}^*(\theta) - P_{0,0}(0), \tag{12}$$

$$(\delta\phi - \theta)P_{n,0}^*(\theta) = \mu Q_{n,1}^*(\theta) + A^*(\theta)P_{n-1,0}(0) - P_{n,0}(0),\ \ 1 \le n \le a-1, \tag{13}$$

$$(\phi - \theta)P_{n,0}^*(\theta) = A^*(\theta)P_{n-1,0}(0) - P_{n,0}(0),\ n \ge a, \tag{14}$$

$$(\mu - \theta)Q_{a,0}^*(\theta) = \phi P_{a,0}^*(\theta) + \mu Q_{a,1}^*(\theta) + A^*(\theta)\delta R_{a-1}(0) - Q_{a,0}(0), \tag{15}$$

$$\begin{aligned}(\mu - \theta)Q_{n,0}^*(\theta) = {}& \phi P_{n,0}^*(\theta) + \mu Q_{n,1}^*(\theta)\\ & + A^*(\theta)Q_{n-1,0}(0) - Q_{n,0}(0),\ a+1 \le n \le d-1,\end{aligned} \tag{16}$$

$$(\mu - \theta)Q_{0,1}^{*}(\theta) = \phi\sum_{k=d}^{b} P_{k,0}^{*}(\theta) + \mu\sum_{k=d}^{b} Q_{k,1}^{*}(\theta) + A^{*}(\theta)Q_{d-1,0}(0) - Q_{0,1}(0), \quad (17)$$

$$(\mu - \theta)Q_{n,1}^{*}(\theta) = \mu Q_{n+b,1}^{*}(\theta) + \phi P_{n+b,0}^{*}(\theta) + A^{*}(\theta)Q_{n-1,1}(0) - Q_{n,1}(0),\ n \geq 1. \quad (18)$$

Adding (10) to (18), and taking limit as $\theta \to 0$ and using the normalization condition, we get

$$\sum_{n=0}^{a-1} \delta R_n(0) + \sum_{n=0}^{\infty} P_{n,0}(0) + \sum_{n=a}^{d-1} Q_{n,0}(0) + \sum_{n=0}^{\infty} Q_{n,1}(0) = \lambda. \quad (19)$$

The left hand side of (19) denotes mean number of entrances into the system per unit time and is equal to mean arrival rate λ. We discuss the solution for the model with single vacation policy $(\delta = 1)$ in the subsection below.

2.1. Model with single vacation policy

In this model, we assume that the server takes exactly one vacation each time when the system becomes empty. On the other hand, up on return from a vacation if the server finds $0 \leq n \leq a - 1$ customers in the system, then it becomes idle until the number of customer in the system reaches a or more.

2.1.1. Steady-state distribution at pre-arrival epochs

Let R_n^{-} be the probability that $n\,(0 \leq n \leq a - 1)$ customers waiting in the system at pre-arrival epoch and the server is idle and $P_{n,0}^{-}$ be the probability that $n\,(n \geq 0)$ customers waiting in the system at pre-arrival epoch and the server is on vacation. Further, let $Q_{n,0}^{-}$ denote the probability that the server is busy with an accessible batch of size $n\ (a \leq n \leq d - 1)$, and $Q_{n,1}^{-}$ denote the probability that the server is busy with non-accessible batch of size $n\,(n \geq 0)$ customers waiting in the queue at pre-arrival epoch. These are given by

$$R_n^{-} = \frac{1}{\lambda} R_n(0),\ P_{n,0}^{-} = \frac{1}{\lambda} P_{n,0}(0),\ Q_{n,0}^{-} = \frac{1}{\lambda} Q_{n,0}(0),\ Q_{n,1}^{-} = \frac{1}{\lambda} Q_{n,1}(0), \quad (20)$$

where λ is given by (19).

To obtain R_n^{-}, $P_{n,0}^{-}$ and $Q_{n,i}^{-}$, first we need to evaluate $R_n(0)$, $P_{n,0}(0)$ and $Q_{n,i}(0)$ which is done below.

Setting $\theta = \phi$ in (14), we get after recursive substitution

$$P_{n,0}(0) = B\beta^{n-a+1}, \quad n \geq a-1, \tag{21}$$

where $B = P_{a-1,0}(0)$ and $\beta = A^*(\phi)$. Substituting (21) in (14), we have

$$P^*_{n,0}(\theta) = \frac{B\beta^{n-a}(A^*(\theta) - \beta)}{\phi - \theta}, \quad n \geq a. \tag{22}$$

By using the displacement operator E defined by $Ex_n = x_{n+1}$ for all n, we can write (18) as

$$(\mu - \mu E^b - \theta)Q^*_{n,1}(\theta) = A^*(\theta)Q_{n-1,1}(0) + \phi P^*_{n+b,0}(\theta) - Q_{n,1}(0), \quad n \geq 1. \tag{23}$$

Setting $\theta = \mu - \mu E^b$ in equation (23) and using (22), we obtain

$$(E - A^*(\mu - \mu E^b))Q_{n,1}(0) = \frac{B\phi(\beta - A^*(\mu - \mu E^b))\beta^{n+b+1-a}}{\mu - \mu E^b - \phi}, \quad n \geq 0. \tag{24}$$

The complementary solution of homogeneous difference equation $(E - A^*(\mu - \mu E^b))Q_{n,1}(0) = 0$ of equation (24) is given by

$$Q^{(c)}_{n,1}(0) = Cr^n,$$

where C is an arbitrary constant and r is a real root inside the unit circle of the equation $z = A^*(\mu - \mu z^b)$ for $\rho < 1$.

The particular solution of the difference equation (24) is given by

$$Q^{(p)}_{n,1}(0) = \frac{B\phi\beta^{n+b+1-a}}{\mu - \mu\beta^b - \phi}, \quad n \geq 0.$$

Thus, the general solution of (24) is given by

$$Q_{n,1}(0) = Cr^n + \frac{B\phi\beta^{n+b+1-a}}{\mu - \mu\beta^b - \phi}, \quad n \geq 0. \tag{25}$$

Let $z_j(\theta)$, $1 \leq j \leq b$, be the b roots of $\mu - \mu\beta^b - \theta = 0$ for a fixed θ with $\mathrm{Re}(\theta) \geq 0$. Then the complementary solution of the homogeneous difference equation $(\mu - \mu E^b - \theta)Q^*_{n,1}(\theta) = 0$ of (23) is given by

$$Q^{*(c)}_{n,1}(\theta) = \sum_{j=1}^{b} d_j z_j(\theta),$$

where d_j's are arbitrary constants. Since $\sum_{n=1}^{\infty} Q_{n,1}^*(0) \leq 1,$ so we must have all $d_j = 0.$

Thus, the general solution of (23), using (22) and (25) is

$$Q_{n,1}^*(\theta) = \frac{Cr^{n-1}(A^*(\theta) - r)}{\mu - \mu r^b - \theta} + \frac{B\phi(A^*(\theta) - \beta)\beta^{n+b-a}}{(\phi - \theta)(\mu - \mu\beta^b - \phi)}, \quad n \geq 1. \tag{26}$$

Using (22) and (26) in (15) and (16), after substituting $\theta = \mu,$ we get

$$R_{a-1}(0) = \frac{1}{\omega}\left[K - \frac{B\phi(\omega - \beta)}{\phi - \mu + \mu\beta^b} + C(\omega - r)r^{a-b-1}\right], \tag{27}$$

$$Q_{n,0}(0) = \frac{B\beta\phi(\beta^{n-a} - \omega^{n-a})}{\mu - \mu\beta^b - \phi} + Cr^{a-b}(r^{n-a} - \omega^{n-a}) + K\omega^{n-a},$$

$$a + 1 \leq n \leq d - 1, \tag{28}$$

where $\omega = A^*(\mu)$ and $K = Q_{a,0}(0).$ Thus, the solution of (15) and (16) is

$$\begin{aligned} Q_{n,0}^*(\theta) = {} & \frac{B\phi\beta}{\mu - \mu\beta^b - \phi}\left\{\frac{\beta^{n-a-1}(A^*(\theta) - \beta)}{\phi - \theta} - \frac{\omega^{n-a-1}(A^*(\theta) - \omega)}{\mu - \theta}\right\} \\ & + \frac{K(A^*(\theta) - \omega)\omega^{n-a-1}}{\mu - \theta} \\ & + \frac{C}{r^b}\left\{\frac{r^{n-1}(A^*(\theta) - r)}{\mu - \mu r^b - \theta} - \frac{r^a\omega^{n-a-1}(A^*(\theta) - \omega)}{\mu - \theta}\right\}, \ a \leq n \leq d - 1. \end{aligned} \tag{29}$$

From (17), we get

$$\begin{aligned} Q_{0,1}^*(\theta) = {} & \frac{B\phi}{(\mu - \mu\beta^b - \phi)(\mu - \theta)}\left\{\frac{\beta^{d-a}(A^*(\theta) - \beta)(\mu - \phi)(1 - \beta^{b-d+1})}{(\phi - \theta)(1 - \beta)}\right. \\ & \left. + A^*(\theta)\beta(\beta^{d-a-1} - \omega^{d-a-1}) - \beta^{b-a+1}\right\} + \frac{KA^*(\theta)\omega^{d-a-1}}{\mu - \theta} \\ & + \frac{C}{(\mu - \theta)}\left\{\frac{\mu(A^*(\theta) - r)(r^{d-1} - r^b)}{(\mu - \mu r^b - \theta)(1 - r)} - 1 + A^*(\theta)r^{a-b}(r^{d-a-1} - \omega^{d-a-1})\right\}. \end{aligned} \tag{30}$$

Setting $\theta = \phi$ in (13), we get after recursive substitution,

$$P_{n,0}(0)$$
$$= \beta^{n+1-a}\left[B + \frac{B\phi\mu\psi(a-n-1)\beta^{b-1}}{\mu - \mu\beta^b - \phi} - \frac{C\mu r^n(\beta^{a-n-1} - r^{a-n-1})}{(\mu - \mu r^b - \phi)}\right], \quad 0 \le n \le a-2, \tag{31}$$

where $\psi = A^{*(1)}(\phi)$. Setting $\theta = 0$ in (10) and (11), we get

$$R_n(0) = \phi\sum_{j=0}^{n} P_{j,0}, \quad 0 \le n \le a-1, \tag{32}$$

where $P_{j,0}$, $0 \le j \le a-1$ are obtained by setting $\theta = 0$ in (12) and (13), respectively, and are given as

$$\phi P_{0,0} = \frac{B}{\phi - \mu + \mu\beta^b}\left[\frac{\phi\beta}{\omega} - \mu - \frac{\phi - \mu}{\beta^{a-1}} + \phi\mu(a-1)\psi\beta^{b-a}\right]$$
$$+ C\left[\frac{r^{a-b-1} - 1}{1 - r^b} - \frac{r^{a-b}}{\omega} + \frac{\mu(1 - r^{a-1}\beta^{1-a})}{(\mu - \phi - \mu r^b)}\right] + \frac{K}{\omega}, \tag{33}$$

$$\phi P_{n,0} = \frac{B\beta^{n-a}}{\phi - \mu + \mu\beta^b}\left\{(\phi - \mu)(1-\beta) - \phi\mu\psi\beta^{b-1}\{(a-n)(1-\beta) + \beta\}\right\}$$
$$+ Cr^{n-1}\left\{\frac{\mu\beta^{n-a}\{(r-1)\beta^{a-n} + r^{a-n}(1-\beta)\}}{\mu - \mu r^b - \phi} + \frac{1-r}{1-r^b}\right\}, 1 \le n \le a-1. \tag{34}$$

Setting $\theta = \mu$ in (17) and using (22), (25), (26) and (28), we get

$$K + \frac{B\phi\beta}{\phi - \mu + \mu\beta^b} - Cr^{a-b}$$
$$= (1-\omega)\omega^{a-d}\left(\frac{B\phi(\beta^{d-a+1} - \beta^{b-a+1})}{(\phi - \mu + \mu\beta^b)(1-\beta)} + \frac{C(1 - r^{d-b})}{1-r}\right). \tag{35}$$

Now using (21), (25), (28), (29), (31)-(34) in (19), we obtain

$$\lambda = BT_1 + CT_2, \tag{36}$$

where

$$T_1 = \frac{1}{(\mu - \phi - \mu\beta^b)(1-\beta)}\left[\phi\omega^{a-d-1}\{a(1-\omega) + \omega\}(\beta^{b-a+1} - \beta^{d-a+1})\right.$$
$$\left. + \mu\{a(1-\beta) + \beta(1 - \beta^{b-a})\}\right],$$

$$T_2 = \frac{a(r^{a-b-1}-1)}{(1-r^b)} + \frac{a\mu - \mu r^{a-1}}{\mu - \phi - \mu r^b} + \frac{\phi(1-a+ar-r^a)}{(\mu-\phi-\mu r^b)(1-r)(1-r^b)} - \frac{\mu(1-r^{a-1})}{(\mu-\phi-\mu r^b)(1-r)}$$

$$+ \frac{r^{a-b} - r^{d-b} + 1 + \{a(1-\omega)\omega^{a-d-1} + \omega^{a-d} - 1\}(1-r^{d-b})}{1-r}.$$

Setting $\theta = \phi$ in (12) and using (29)-(31) and (36), we get after simplification

$$C = \frac{BT_3}{T_4}, \tag{37}$$

$$B = \lambda T_4 (T_1 T_4 + T_2 T_3)^{-1}, \tag{38}$$

where

$$T_3 = \frac{\phi\mu}{\mu - \phi - \mu\beta^b}\left[\frac{(\beta-\omega)\omega^{a-d-1}(\beta^{b-a+1} - \beta^{d-a+1})}{(\mu-\phi)(1-\beta)}\right.$$

$$\left. + \frac{\psi\{\beta^{b-a} - 1 - a\beta^{b-a}(1-\beta)\}}{1-\beta} - \frac{\beta^{1-a}(\mu - \mu\beta^b - \phi)}{\phi\mu}\right],$$

$$T_4 = \frac{\mu}{\phi - \mu}\left[\frac{(\mu-\phi)\{(\beta-r)r^{a-b-1} - (1-r)\beta^{1-a}r^{a-1} + (1-\beta)r^{d-b}\} + \mu(r^b - r^d)(1-\beta)}{(1-r)(\mu - \mu r^b - \phi)}\right.$$

$$\left. + \frac{\{(1-\beta) + (\beta-\omega)\omega^{a-d-1}\}(1-r^{d-b})}{1-r}\right].$$

Using (37) in (35), we get

$$K = B\left[\frac{\phi\beta}{\phi - \mu + \mu\beta^b}\left\{\frac{(1-\omega)\omega^{a-d}(\beta^{d-a} - \beta^{b-a})}{1-\beta} - 1\right\}\right.$$

$$\left. + \frac{T_3}{T_4}\left\{r^{a-b} + \frac{(1-\omega)\omega^{a-d}(1-r^{d-b})}{1-r}\right\}\right]. \tag{39}$$

Below we summarize the above results in Theorem 2.1 for pre-arrival epoch probabilities.

Theorem 2.1. *The pre-arrival distributions* R_n^- *that an arrival sees n customers in the system and the server is idle,* $P_{n,0}^-$ *that an arrival sees n customers in the system and the server is on vacation,* $Q_{n,0}^-$ *that the server is busy with an accessible batch and* $Q_{n,1}^-$ *that the server is busy with non-accessible batch are given by*

$$R_n^- = \frac{\phi}{\lambda}\sum_{j=0}^{n} P_{j,0}, \quad 0 \le n \le a-1,$$

$$P_{n,0}^- = \frac{\beta^{n+1-a}}{\lambda}\left[B + \frac{B\phi\mu\psi(a-n-1)\beta^{b-1}}{\mu-\mu\beta^b-\phi} - \frac{C\mu r^n(\beta^{a-n-1}-r^{a-n-1})}{(\mu-\mu r^b-\phi)}\right],$$

$$0 \le n \le a-2,$$

$$P_{n,0}^- = \frac{B\beta^{n-a+1}}{\lambda}, \quad n \ge a-1,$$

$$Q_{n,0}^- = \frac{1}{\lambda}\left[\frac{B\beta\phi(\beta^{n-a}-\omega^{n-a})}{\mu-\mu\beta^b-\phi} + Cr^{a-b}(r^{n-a}-\omega^{n-a}) + K\omega^{n-a}\right], \quad a \le n \le d-1,$$

$$Q_{n,1}^- = \frac{1}{\lambda}\left[Cr^n + \frac{B\phi\beta^{n+b+1-a}}{\mu-\mu\beta^b-\phi}\right], \quad n \ge 0.$$

Proof. To get the desired results, we use (20) in (32), (31), (21), (28) and (25), respectively.

2.1.2. Steady-state distribution at arbitrary epochs

The arbitrary epoch queue length distributions R_n that an arrival sees n customers in the system and the server is idle, $P_{n,0}$ that an arrival sees n customers in the system and the server is on vacation, $Q_{n,0}$ that server is busy with an accessible batch and $Q_{n,1}$ that the server is busy with non-accessible batch are summarized in the following theorem.

Theorem 2.2. *The arbitrary epoch probabilities are given by*

$$P_{0,0} = \frac{B}{\phi(\phi-\mu+\mu\beta^b)}\left[\frac{\phi\beta}{\omega} - \mu - \frac{\phi-\mu}{\beta^{a-1}} + \phi\mu(a-1)\psi\beta^{b-a}\right]$$

$$+ \frac{C}{\phi}\left[\frac{r^{a-b-1}-1}{1-r^b} - \frac{r^{a-b}}{\omega} + \frac{\mu(1-r^{a-1}\beta^{1-a})}{(\mu-\phi-\mu r^b)}\right] + \frac{K}{\phi\omega},$$

$$P_{n,0} = \frac{B\beta^{n-a}}{\phi(\phi-\mu+\mu\beta^b)}\left\{(\phi-\mu)(1-\beta) - \phi\mu\psi\beta^{b-1}\ \{(a-n)(1-\beta)+\beta\}\right\}$$

$$+ \frac{Cr^{n-1}}{\phi}\left\{\frac{\mu\beta^{n-a}\{(r-1)\beta^{a-n} + r^{a-n}(1-\beta)\}}{\mu-\mu r^b-\phi} + \frac{1-r}{1-r^b}\right\}, \quad 1 \le n \le a-1,$$

$$P_{n,0} = \frac{B\beta^{n-a}(1-\beta)}{\phi}, \quad n \geq a,$$

$$Q_{n,0} = \frac{B\phi\beta}{\mu - \mu\beta^b - \phi}\left\{\frac{(1-\beta)\beta^{n-a-1}}{\phi} - \frac{(1-\omega)\omega^{n-a-1}}{\mu}\right\} + \frac{K(1-\omega)\omega^{n-a-1}}{\mu}$$

$$+ C\left\{\frac{(1-r)r^{n-b-1}}{\mu(1-r^b)} - \frac{r^{a-b}(1-\omega)\omega^{n-a-1}}{\mu}\right], \quad a \leq n \leq d-1,$$

$$Q_{0,1} = \frac{1}{\mu}\left[\frac{B\phi\beta}{\phi - \mu + \mu\beta^b}\left\{\frac{(\phi-\mu)\beta^{d-a-1}(1-\beta^{b-d+1})}{\phi} + (\omega^{d-a-1} - \beta^{d-a-1}) + \beta^{b-a}\right\}\right.$$

$$\left. + C\left\{r^{a-b}(r^{d-a-1} - \omega^{d-a-1}) - \frac{1-r^{d-1}}{1-r^b}\right\} + K\omega^{d-a-1}\right],$$

$$Q_{n,1} = \frac{Cr^{n-1}(1-r)}{\mu(1-r^b)} + \frac{B(1-\beta)\beta^{n+b-a}}{\mu - \mu\beta^b - \phi}, \quad n \geq 1.$$

Proof. The results $P_{n,0}$, $0 \leq n \leq a-1$ are from (33) and (34). The other results of the theorem are obtained by setting $\theta = 0$ in (22), (26), (28) and (30), respectively.

Theorem 2.3. *The arbitrary epoch probabilities* $\{R_n\}_0^{a-1}$ *are given by*

$$R_n = -P_{n,0} + P^-_{n-1,0} + R^-_{n-1} - \frac{Cr^{n-1}(\lambda(1-r) - \mu(1-r^b))}{\lambda\mu(1-r^b)^2}$$

$$+ \frac{B\mu(\lambda(1-\beta) - \phi)\beta^{n+b-a}}{\phi\lambda(\phi - \mu + \mu\beta^b)}, \quad 1 \leq n \leq a-1.$$

Finally, the only unknown quantity R_0 *is obtained by using the normalization condition*

$$R_0 = 1 - \left(\sum_{n=1}^{a-1} R_n + \sum_{n=0}^{\infty} P_{n,0} + \sum_{n=a}^{d-1} Q_{n,0} + \sum_{n=0}^{\infty} Q_{n,1}\right).$$

Proof. Differentiating (11) with respect to θ, and setting $\theta = 0$, we obtain

$$R_n = -\phi P^{*(1)}_{n,0}(0) + R^-_{n-1}, \quad 1 \leq n \leq a-1,$$

where $P_{n,0}^{*(1)}(0),\ 1 \leq n \leq a-1,$ can be obtained by differentiating (13) with respect to θ and setting $\theta = 0,$ we obtain

$$P_{n,0}^{*(1)}(0) = \frac{1}{\phi}(P_{n,0} + \mu Q_{n,1}^{*(1)}(0) - P_{n-1,0}^{-}), \quad 1 \leq n \leq a-1.$$

Finally, to know the quantity $Q_{n,1}^{*(1)}(0),$ we differentiate (26) with respect to θ and setting $\theta = 0,$ we get

$$Q_{n,1}^{*(1)}(0) = \frac{Cr^{n-1}(\lambda(1-r) - \mu(1-r^b))}{\lambda\mu^2(1-r^b)^2} - \frac{B\beta^{n+b-a}(\lambda(1-\beta) - \phi)}{\lambda\phi(\phi - \mu(1-\beta^b))}, \quad n \geq 1.$$

2.2. Model with multiple exponential vacation policy

We assume that the server takes vacation each time the number in the system drops below a. If the server returns from a vacation and finds a or more than a customers in the system, then it starts service immediately and continues until the number is less than a. If the server returns from a vacation and finds less than a number of customers in the system, then it begins another vacation immediately, and continues so until it finds a or more than a customers in the system upon returning from a vacation.

2.2.1. Steady-state distribution at pre-arrival epochs

It can be seen that equations (14) and (16)-(18) are independent of δ. Therefore, the pre-arrival probabilities $Q_{n,1}^{-}(0),\ n \geq 0$ and $P_{n,0}^{-}(0),\ n \geq a-1,$ will be same in both single- and multiple-vacation policies. Following the procedure discussed for single vacation policy, we can obtain pre-arrival epoch probabilities for multiple vacation policy.

Setting $\theta = \mu$ in (15) and (16), we get

$$Q_{n,0}(0) = \frac{B\phi(\beta^{n-a+1} - \omega^{n-a+1})}{\mu - \mu\beta^b - \phi} + Cr^{a-b-1}(r^{n-a+1} - \omega^{n-a+1}), \quad a \leq n \leq d-1.$$

Thus, the solution of equations (15) and (16) is

$$Q_{n,0}^{*}(\theta) = \frac{B\phi}{\mu - \mu\beta^b - \phi}\left\{\frac{\beta^{n-a}(A^*(\theta) - \beta)}{\phi - \theta} - \frac{\omega^{n-a}(A^*(\theta) - \omega)}{\mu - \theta}\right\}$$
$$+ \frac{C}{r^{b+1}}\left\{\frac{r^n(A^*(\theta) - r)}{\mu - \mu r^b - \theta} - \frac{r^a\omega^{n-a}(A^*(\theta) - \omega)}{\mu - \theta}\right\}, \quad a \leq n \leq d-1. \quad (40)$$

Setting $\theta = 0$ in (13), we obtain the recursive relation

$$P_{n,0}(0) - P_{n-1,0}(0) = \frac{C(1-r)r^{n-1}}{1-r^b} + \frac{B\mu(1-\beta)\beta^{n+b-a}}{\mu - \mu\beta^b - \phi}, \quad 1 \le n \le a-1.$$

Based on this recursion, we have

$$P_{n,0}(0) = \frac{B(\mu - \mu\beta^{n+b-a+1} - \phi)}{\mu - \mu\beta^b - \phi} + \frac{C(r^{a-1} - r^n)}{1-r^b}, \quad 0 \le n \le a-2.$$

From (17), we obtain

$$Q_{0,1}^*(\theta) = \frac{B\phi}{(\mu - \mu\beta^b - \phi)(\mu - \theta)}\left\{\frac{\beta^{d-a}(A^*(\theta) - \beta)(\mu - \phi)(1 - \beta^{b-d+1})}{(\phi - \theta)(1-\beta)}\right.$$
$$\left. + A^*(\theta)(\beta^{d-a} - \omega^{d-a}) - \beta^{b-a+1}\right\}$$
$$+ \frac{C}{(\mu - \theta)}\left\{\frac{\mu(A^*(\theta) - r)(r^{d-1} - r^b)}{(\mu - \mu r^b - \theta)(1-r)} + A^*(\theta)r^{a-b-1}(r^{d-a} - \omega^{d-a}) - 1\right\}. \quad (41)$$

Setting $\theta = \mu$ in (17) and simplifying, we get

$$\frac{B\phi h(\beta)}{\phi - \mu + \mu\beta^b} - Cr^{a-b-1}h(r) = 0, \quad (42)$$

where $h(x) = \omega^{d-a+1} + \dfrac{x^{b-a+1}(1-\omega)(1-x^{d-b})}{1-x}$.

Now using (19), we obtain

$$\lambda = \frac{B}{\mu - \mu\beta^b - \phi}\left[(\mu - \phi)\left((a-1) + \frac{1-\beta^{b-a+1}}{1-\beta}\right) + \phi\left(\frac{\beta(1-\beta^{d-a})}{1-\beta} - \frac{\omega(1-\omega^{d-a})}{1-\omega}\right)\right]$$
$$+ Cr^{a-b-1}\left[\left(\frac{r^b}{1-r^b}\right)\left((a-1) + \frac{1-r^{b-a+1}}{1-r}\right) + \frac{r(1-r^{d-a})}{1-r} - \frac{\omega(1-\omega^{d-a})}{1-\omega}\right]. \quad (43)$$

Solving the linear equations (42) and (43) for B and C, we have

$$B = \frac{\lambda(1-r^b)(\mu - \mu\beta^b - \phi)h(r)}{(1-r^b)\{(\mu - \phi)f(\beta) + \phi g(\beta)\}h(r) - \phi h(\beta)\{[f(r) - g(r)]r^b + g(r)\}},$$

$$C = \frac{B\phi h(\beta)}{r^{a-b-1}h(r)(\phi - \mu + \mu\beta^b)},$$

where $f(x) = (a-1) + \dfrac{(1-x^{b-a+1})}{1-x}$ and $g(x) = \dfrac{x(1-x^{d-a})}{1-x} - \dfrac{\omega(1-\omega^{d-a})}{1-\omega}$.

Theorem 2.4. *The pre-arrival distributions* $P^-_{n,0}$ *that an arrival sees n customers in the system and the server is on vacation*, $Q^-_{n,0}$ *that server is busy with an accessible batch and* $Q^-_{n,1}$ *that the server is busy with non-accessible batch are given by*

$$P^-_{n,0} = \frac{1}{\lambda}\left\{\frac{B(\mu - \mu\beta^{n+b-a+1} - \phi)}{\mu - \mu\beta^b - \phi} + \frac{C(r^{a-1} - r^n)}{1 - r^b}\right\}, \quad 0 \le n \le a-2,$$

$$P^-_{n,0} = \frac{B\beta^{n-a+1}}{\lambda}, \; n \ge a-1,$$

$$Q^-_{n,0} = \frac{1}{\lambda}\left\{\frac{B\phi(\beta^{n-a+1} - \omega^{n-a+1})}{\mu - \mu\beta^b - \phi} + Cr^{a-b-1}(r^{n-a+1} - \omega^{n-a+1})\right\}, \quad a \le n \le d-1,$$

$$Q^-_{n,1} = \frac{1}{\lambda}\left\{Cr^n + \frac{B\phi\beta^{n+b-a+1}}{\mu - \mu\beta^b - \phi}\right\}, \quad n \ge 0.$$

2.3. Steady-state distribution at arbitrary epochs

The arbitrary epoch queue length distributions $P_{n,0}$ that an arrival sees n customers in the system and the server is on vacation, $Q_{n,0}$ that server is busy with an accessible batch and $Q_{n,1}$ that the server is busy with non-accessible batch are summarized in the following theorem.

Theorem 2.5. *The arbitrary epoch probabilities are given by*

$$P_{n,0} = \frac{B}{\lambda\phi(\mu - \mu\beta^b - \phi)}[\phi(\mu - \phi) - \mu\lambda\beta^{n+b-a}(1-\beta)]$$

$$+ \frac{C}{\lambda\mu(1 - r^b)^2}[\mu r^{a-1}(1 - r^b) - \lambda r^{n-1}(1-r)], \quad 1 \le n \le a-1,$$

$$P_{n,0} = \frac{B(1-\beta)\beta^{n-a}}{\phi}, \quad n \ge a,$$

$$Q_{n,0} = \frac{C(r^{n-b-1}(1-r) - r^{a-b-1}(1-r^b)(1-\omega)\omega^{n-a})}{\mu(1-r^b)}$$

$$+ \frac{B(\mu\beta^{n-a}(1-\beta) - \phi\omega^{n-a}(1-\omega))}{\mu(\mu - \mu\beta^b - \phi)}, \quad a \le n \le d-1,$$

$$Q_{0,1} = \frac{B(\mu\beta^{d-a}(1-\beta^{b-d+1}) - \phi\omega^{d-a})}{\mu(\mu - \mu\beta^{b} - \phi)} - \frac{C((1-r^{d-b-1}) + \omega^{d-a}r^{a-b-1}(1-r^{b}))}{\mu(1-r^{b})},$$

$$Q_{n,1} = \frac{Cr^{n-1}(1-r)}{\mu(1-r^{b})} + \frac{B(1-\beta)\beta^{n+b-a}}{\mu - \mu\beta^{b} - \phi}, \quad n \geq 1.$$

Finally, the only unknown quantity $P_{0,0}$ *is obtained by using the normalization condition*

$$P_{0,0} = 1 - \left(\sum_{n=1}^{\infty} P_{n,0} + \sum_{n=a}^{d-1} Q_{n,0} + \sum_{n=0}^{\infty} Q_{n,1}\right).$$

Proof. Differentiating (13) with respect to θ, and setting $\theta = 0$, we get the arbitrary epoch probabilities $\{P_{n,0}\}_{1}^{a-1}$ as

$$P_{n,0} = -\mu Q_{n,1}^{*(1)}(0) + P_{n-1,0}^{-}, \quad 1 \leq n \leq a-1,$$

where $Q_{n,1}^{*(1)}(0)$, $1 \leq n \leq a-1$, can be obtained by differentiating (26) with respect to θ and setting $\theta = 0$, we get

$$Q_{n,1}^{*(1)}(0) = \frac{Cr^{n-1}(\lambda(1-r) - \mu(1-r^{b}))}{\lambda\mu^{2}(1-r^{b})^{2}} - \frac{B\beta^{n+b-a}(\lambda(1-\beta) - \phi)}{\lambda\phi(\phi - \mu(1-\beta^{b}))}, \quad n \geq 1.$$

The other arbitrary epoch probabilities are obtained by setting $\theta = 0$ in (22), (26), (40) and (41).

3. Performance Measures

Performance measures are important features of queueing systems as they reflect the efficiency of the queueing system under consideration. Once the state probabilities at pre-arrival and arbitrary epochs are known, we can evaluate various performance measures. The average queue length when server is in dormancy (L_{q0}), the average queue length when server is on vacation (L_{q1}), the average queue length when the server is busy (L_{q2}) and the average number of customers in the queue at an arbitrary epoch (L_{q}) are given by

$$L_{q0} = \sum_{n=0}^{a-1} \delta n R_{n}, \quad L_{q1} = \sum_{n=0}^{\infty} nP_{n,0}, \quad L_{q2} = \sum_{n=0}^{\infty} nQ_{n,1},$$

$$L_{q} = \sum_{n=0}^{a-1} \delta n R_{n} + \sum_{n=0}^{\infty} n(P_{n,0} + Q_{n,1}).$$

The average waiting time in the queue (W_q) of a customer using Little's rule is given as $W_q = L_q / \lambda.$

4. Special Cases

In this section, some special cases which are available in the literature are deduced by taking specific values for the parameters a, d and b.

Case 1. $a = d = b = 1,$ that is, the batch size is one. In this case, the model reduces to $GI/M/1$ queue with single and multiple vacations. Using Theorem 2.4, the pre-arrival epoch probabilities in case of multiple vacation are given by

$$P^{-}_{n,0} = (1-r)\sigma\beta^{n}, \quad n \geq 0,$$

$$Q^{-}_{n,1} = (1-r)\sigma\gamma(\beta^{n+1} - r^{n+1}), \quad n \geq 0,$$

where $\sigma = \dfrac{\mu - \mu\beta - \phi}{\mu - \mu r - \phi}, \quad \gamma = \dfrac{\phi}{\mu - \mu\beta - \phi}$ and the above result matches with Chatterjee and Mukerjee [4]. Using Theorem 2.1, the pre-arrival epoch probabilities in case of single vacation are given by

$$P^{-}_{n,0} = B^{*}\mu(\beta - r)(\mu - \mu\beta - \phi)\beta^{n}, \quad n \geq 0,$$

$$Q^{-}_{n,1} = B^{*}[(\mu - \mu\beta - \phi + \mu\phi\psi)(\mu - \mu r - \phi)r^{n+1} - \mu(r - \beta)\phi\beta^{n+1}], \quad n \geq 0,$$

where

$$B^{*} = (1-r)/\{(\mu - \mu\beta - \phi + \mu\phi\psi)(\mu - \mu r - \phi) - \mu^{2}(1-r)(r-\beta)\}.$$

Case 2. $a = d = b = 1$ and taking vacation parameter ϕ sufficiently large, that is, non vacation queue with the batch size one. In this case, the model reduces to $GI/M/1$ queue without vacations and our results match with the results available in the literature. Using Theorem 2.1 and Theorem 2.4, we obtain pre-arrival epoch probabilities as

$$P^{-}_{0,0} = (1-r),$$

$$Q^{-}_{n,1} = (1-r)r^{n+1}, \quad n \geq 0,$$

in both the cases, results are same as it should be and in case of single vacation $R^{-}_{0} = P^{-}_{0,0}.$

Case 3. Taking vacation parameter ϕ sufficiently large, the model reduces to $GI/M^{(a,d,b)}/1$ queue without vacations.

Case 4. $a = d$, that is, the general batch service queue without accessibility and our model reduces to $GI/M^{(a,b)}/1$ queue with single and multiple vacation policies. Substituting $a = d$ in Theorem 2.1 and Theorem 2.4, we obtain the pre-arrival epoch probabilities in single and multiple vacation policies, respectively. In case of multiple vacation pre-arrival epoch probabilities match analytically with the one obtained by Choi and Han [7].

5. Numerical Results

To demonstrate the applicability of the results obtained in the previous sections, a variety of numerical results have been presented in tables and graphs by considering various inter-arrival time distributions such as exponential (M), deterministic (D), Eralng (E_k) and hyper-exponential $(HE_2$, with parameters $\sigma_1, \sigma_2, \lambda_1, \lambda_2)$.

In Tables 1 and 2, the pre-arrival and arbitrary epoch probabilities of multiple and single vacation queues have been presented along with some performance measures. As desired, the queue length distributions match exactly in case of exponential distribution. In Table 3, the comparison of multiple and single vacation models with regard to average queue lengths and waiting time has been made for $E_4/M^{(5,10,15)}/1/\infty$ model for different values of ρ. It is observed that L_{q1}, L_{q2}, L_q and W_q are less in case of single vacation as compared to that in multiple vacation as it is expected for all values of ρ.

In Figures 1 and 2, we have plotted average queue lengths (L_{q1}) and (L_{q2}) against service rate (μ) for HE_2 distribution with $\sigma_1 = \sigma_2 = 0.5$, $\lambda_1 = 4.0$, $\lambda_2 = 6.0$, $\lambda = 4.8$, $\phi = 0.7$, $a = 5$ and $b = 25$ for various values of accessible limit d.

Figure 3 shows the average waiting time in queue (W_q) versus traffic intensity (ρ) for deterministic distribution with $\lambda = 3.0$, $\phi = 0.3$, $d = 20$ and $b = 30$ for various services starting threshold value a.

The average waiting time in queue (W_q) versus traffic intensity (ρ) is presented in Figure 4 for E_3 distribution with $\lambda = 3.0$, $\phi = 1.5$, $a = 5$ and $d = 7$ for various values of maximum batch size b.

In Figure 5, we have plotted average waiting time in the queue W_q for single and multiple vacations against accessible limit (d) for M, E_4, D and HE_2 distributions with $\sigma_1 = 0.6$, $\sigma_2 = 0.4$, $\lambda_1 = 1.2$, $\lambda_2 = 3.2$, $\lambda = 1.6$, $\phi = 1.2$, $\rho = 0.4$, $a = 5$ and $b = 25$. The following observations can be made from these figures:

- As μ increases L_{q1} increases while L_{q2} decreases and asymptotically approaches its minimum value. Further, L_{q1} is low for small d and becomes high for large d but for L_{q2} this situation will be reversed.

Table 1. Queue length distributions at various epochs in case of single vacation

	$M/M^{(4,7,15)}/1/\infty$ $\lambda = 1.25,\ \phi = 0.7,$ $\rho = 0.5$		$E_5/M^{(4,7,15)}/1/\infty$ $\lambda = 1.25,\ \phi = 0.7,$ $\rho = 0.5$		$D/M^{(4,7,15)}/1/\infty$ $\lambda = 1.25,\ \phi = 0.7,$ $\rho = 0.5$		$HE_2/M^{(4,7,15)}/1/\infty$ $\lambda_1 = 2,\ \lambda_2 = 1,$ $\sigma_1 = 0.4,\ \sigma_2 = 0.6,$ $\rho = 0.5,\ \phi = 0.7.$	
	pre-arrival	arbitrary	pre-arrival	arbitrary	pre-arrival	arbitrary	pre-arrival	arbitrary
R_0	0.009070	0.009070	0.006504	0.003095	0.005700	0.001948	0.009483	0.010888
R_1	0.017731	0.017731	0.016488	0.012554	0.016107	0.010976	0.017907	0.018662
R_2	0.025868	0.025868	0.025702	0.022079	0.025647	0.020951	0.025843	0.026547
R_3	0.033428	0.033428	0.034131	0.030821	0.034327	0.030057	0.033236	0.033888
$P_{0,0}$	0.016196	0.016196	0.018604	0.011615	0.019347	0.010179	0.015757	0.016935
$P_{1,0}$	0.015468	0.015468	0.017291	0.017829	0.017814	0.018587	0.015121	0.015042
$P_{2,0}$	0.014529	0.014529	0.015890	0.016452	0.016257	0.017033	0.014261	0.014171
$P_{3,0}$	0.013500	0.013500	0.014499	0.015052	0.014755	0.015499	0.013296	0.013202
$P_{4,0}$	0.008654	0.008654	0.008527	0.010664	0.008428	0.011298	0.008632	0.008328
$P_{5,0}$	0.005547	0.005547	0.005015	0.006272	0.004814	0.006453	0.005604	0.005407
$P_{10,0}$	0.000600	0.000600	0.000353	0.000441	0.000293	0.000392	0.000646	0.000624
$P_{15,0}$	0.000065	0.000065	0.000025	0.000031	0.000018	0.000024	0.000075	0.000072
$P_{20,0}$	0.000007	0.000007	0.000001	0.000002	0.000001	0.000001	0.000008	0.000008
$P_{50,0}$	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
$Q_{4,0}$	0.038998	0.038998	0.041271	0.038791	0.041964	0.038581	0.038559	0.038854
$Q_{5,0}$	0.041893	0.041893	0.044690	0.043552	0.045516	0.043998	0.041360	0.041487
$Q_{6,0}$	0.043023	0.043023	0.045833	0.045516	0.046635	0.046230	0.042480	0.042494
$Q_{0,1}$	0.065575	0.065575	0.069195	0.060453	0.070162	0.059040	0.064837	0.066470
$Q_{1,1}$	0.059503	0.059503	0.062347	0.065030	0.063098	0.066567	0.058919	0.058404
$Q_{2,1}$	0.053987	0.053987	0.056174	0.058592	0.056742	0.059864	0.053534	0.053066
$Q_{3,1}$	0.048978	0.048978	0.050610	0.052789	0.051025	0.053833	0.048638	0.048212
$Q_{4,1}$	0.044432	0.044432	0.045596	0.047560	0.045884	0.048409	0.044186	0.043799
$Q_{5,1}$	0.040306	0.040306	0.041078	0.042847	0.041260	0.043531	0.040140	0.039789
$Q_{10,1}$	0.024752	0.024752	0.024377	0.025427	0.024257	0.025593	0.024826	0.024608
$Q_{20,1}$	0.009332	0.009332	0.008584	0.008954	0.008384	0.008845	0.009492	0.009409
$Q_{50,1}$	0.000500	0.000500	0.000375	0.000391	0.000346	0.000365	0.000531	0.000526
$Q_{100,1}$	0.000004	0.000004	0.000002	0.000002	0.000001	0.000001	0.000004	0.000004
Sum	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
L_q	7.287780		7.007780		6.936800		7.352420	
L_{q0}	0.169750		0.149170		0.143048		0.173420	
L_{q1}	0.224500		0.236425		0.239643		0.221906	
L_{q2}	6.893530		6.622170		6.554110		6.957100	
W_q	5.830220		5.606220		5.549440		5.881940	

Table 2. Queue length distributions at various epochs in case of multiple vacation

	$M/M^{(2,5,10)}/1/\infty$ $\lambda = 2, \phi = 1.5,$ $\rho = 0.4$		$D/M^{(4,6,10)}/1/\infty$ $\lambda = 1, \phi = 2.0,$ $\rho = 0.6$		$E_3/M^{(3,4,10)}/1/\infty$ $\lambda = 3, \phi = 1.0,$ $\rho = 0.8$		$HE_2/M^{(3,6,10)}/1/\infty$ $\lambda_1 = 2.0, \lambda_2 = 1.0,$ $\sigma_1 = 0.4,$ $\lambda = 1.25,$ $\rho = 0.6, \phi = 2.0.$	
(n, r)	pre-arrival	arbitrary	pre-arrival	arbitrary	pre-arrival	arbitrary	pre-arrival	arbitrary
$P_{0,0}$	0.067984	0.067984	0.026589	0.012303	0.004303	0.002521	0.032954	0.036412
$P_{1,0}$	0.087961	0.087961	0.038457	0.032634	0.009099	0.007513	0.042905	0.043768
$P_{2,0}$	0.050263	0.050263	0.049061	0.043858	0.013724	0.012195	0.051925	0.052707
$P_{3,0}$	0.028722	0.028722	0.058535	0.053887	0.010005	0.011157	0.020770	0.019472
$P_{4,0}$	0.016413	0.016413	0.007922	0.025307	0.007293	0.008134	0.008308	0.007789
$P_{5,0}$	0.009379	0.009379	0.001072	0.003425	0.005317	0.005929	0.003323	0.003115
$P_{10,0}$	0.000571	0.000571	0.000000	0.000000	0.001095	0.001221	0.000034	0.000032
$P_{20,0}$	0.000002	0.000002	0.000000	0.000000	0.000046	0.000052	0.000000	0.000000
$P_{50,0}$	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
$Q_{2,0}$	0.043347	0.043347						
$Q_{3,0}$	0.062788	0.062788			0.007522	0.005181	0.033526	0.034820
$Q_{4,0}$	0.069045	0.069045	0.053196	0.035292			0.045712	0.046102
$Q_{5,0}$			0.058131	0.056869			0.049206	0.049224
$Q_{0,1}$	0.096754	0.096754	0.075292	0.067373	0.039247	0.029246	0.066458	0.067580
$Q_{1,1}$	0.079910	0.079910	0.067273	0.071207	0.037916	0.038364	0.060246	0.059707
$Q_{2,1}$	0.065944	0.065944	0.060107	0.063623	0.036537	0.036999	0.054605	0.054116
$Q_{3,1}$	0.054387	0.054387	0.053705	0.056846	0.035143	0.035608	0.049489	0.049045
$Q_{4,1}$	0.044838	0.044838	0.047984	0.050791	0.033753	0.034215	0.044850	0.044448
$Q_{5,1}$	0.036955	0.036955	0.042873	0.045381	0.032384	0.032838	0.040645	0.040281
$Q_{6,1}$	0.030452	0.030452	0.038307	0.040547	0.031045	0.031489	0.036835	0.036504
$Q_{7,1}$	0.025090	0.025090	0.034226	0.036228	0.029743	0.030174	0.033381	0.033082
$Q_{8,1}$	0.020671	0.020671	0.030581	0.032370	0.028482	0.028899	0.030251	0.029980
$Q_{9,1}$	0.017028	0.017028	0.027323	0.028922	0.027265	0.027668	0.027415	0.027169
$Q_{10,1}$	0.014028	0.014027	0.024413	0.025841	0.026093	0.026480	0.024845	0.024622
$Q_{20,1}$	0.002017	0.002017	0.007916	0.008379	0.016714	0.016966	0.009283	0.009199
$Q_{50,1}$	0.000006	0.000006	0.000270	0.000286	0.004360	0.004425	0.000484	0.000480
$Q_{100,1}$	0.000000	0.000000	0.000000	0.000001	0.000464	0.000471	0.000004	0.000003
Sum	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1 .000000	1.000000
L_{q1}	0.478899		0.403661		0.266167		0.268175	
L_{q2}	2.583430		6.276250		20.688800		6.793510	
L_q	3.062330		6.679910		20.954900		7.061680	
W_q	1.531160		6.679910		6.984980		5.649350	

Table 3. Comparison of performance measures with single and multiple vacation policy for $E_4/M^{(5,10,15)}/1/\infty$ distribution with $\lambda = 1.5,\ \phi = 2.1$

	Single vacation policy				Multiple vacation policy			
ρ	L_{q1}	L_{q2}	L_q	W_q	L_{q1}	L_{q2}	L_q	W_q
0.1	0.083343	0.013312	1.639280	1.092850	2.037000	0.022457	2.059460	1.372970
0.3	0.089022	0.954965	1.966510	1.311000	1.298790	1.047000	2.345800	1.563860
0.5	0.066435	4.661810	5.222970	3.481980	0.731107	4.804690	5.535800	3.690530
0.7	0.036378	14.801500	15.059600	10.039700	0.340572	14.949100	15.289700	10.193100
0.9	0.010537	65.911400	65.978000	43.985300	0.088727	66.043200	66.131900	44.088000

• As μ or ρ increases the performance of the system increases for small values of *a* and large values of *d* and *b*.

• Among all arrival distributions under consideration the deterministic distribution gives better performance of the system.

• The single vacation policy outperforms the multiple vacation policy.

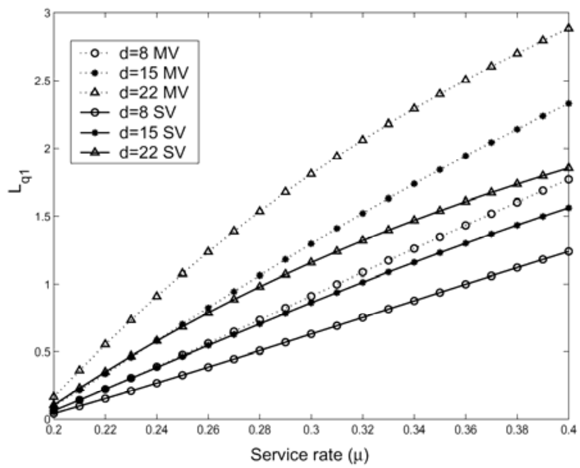

Figure 1. Effect of μ on L_{q1} for different values of *d*.

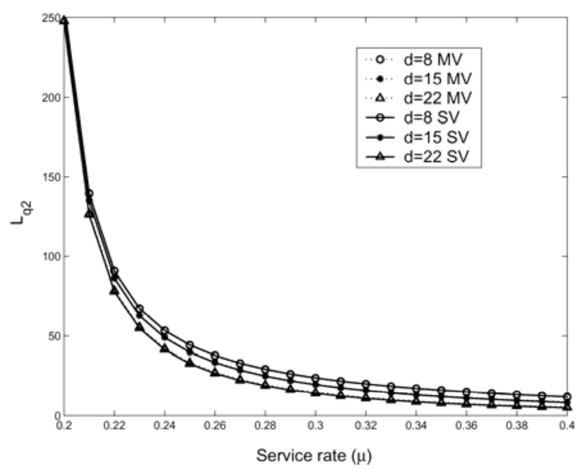

Figure 2. Effect of μ on L_{q2} for different values of *d*.

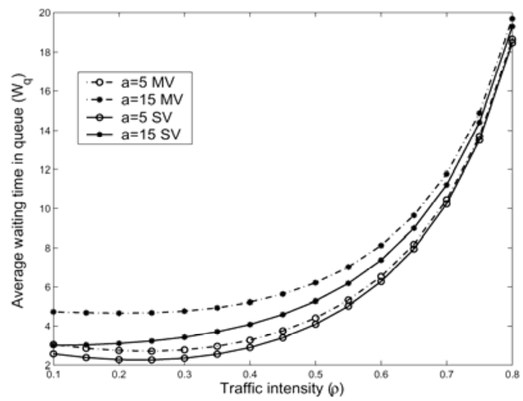

Figure 3. Effect of ρ on W_q for different values of *a*.

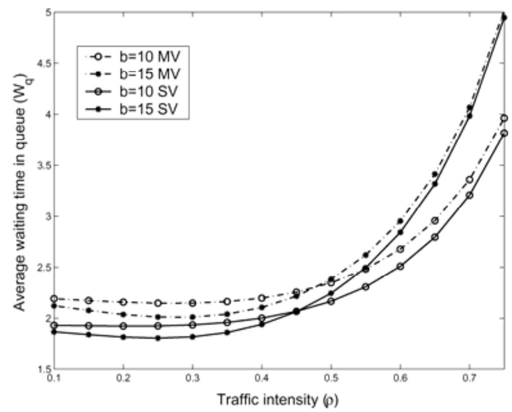

Figure 4. Effect of ρ on W_q for different values of *b*.

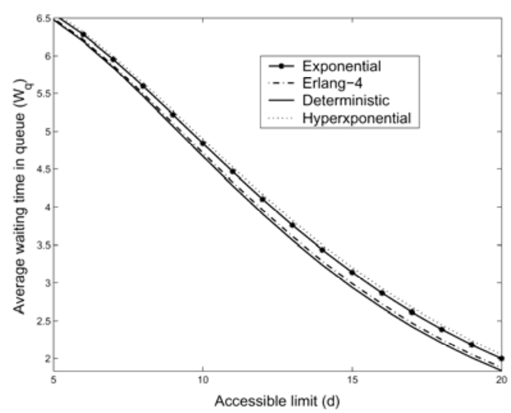

Figure 5. Effect of *d* on the average waiting time for SV policy.

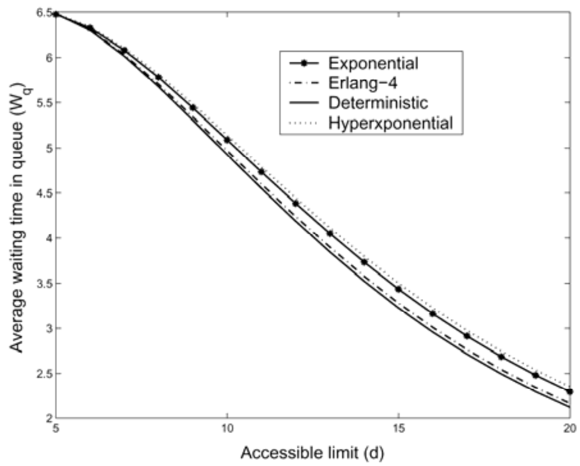

Figure 6. Effect of *d* on the average waiting time for MV policy.

Conclusion

In this paper, we have analyzed the general arrival infinite buffer queue with accessible and non-accessible batch services with both single and multiple vacation policies that have potential applications in modelling computer and telecommunication systems, computer networks, etc. We have developed the recursive method, using the supplementary variable technique and treating the remaining inter-arrival time as the supplementary variable, to find the steady-state queue/system length distributions at pre-arrival and arbitrary epochs. The recursive method is powerful and easy to implement. Various performance measures are obtained and it is noticed that the average queue length and average waiting time are less in the case of single vacation model as compared to that in multiple vacation case. The techniques used in this paper can be applied to analyze more complex models such as $GI^X/M^{(a,d,b)}/1$ and $GI/G^{(a,d,b)}/1$ queues which are left for future investigations.

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