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AN IMPROVED APPROXIMATE ANALYTIC SOLUTION FOR BOUNDARY LAYER BETWEEN PARALLEL STREAMS

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Abstract

In the present paper, the mixing boundary layer equations between parallel streams are investigated for laminar flow. The method consists essentially in solving boundary layer equations using a hybrid analytic scheme, Meksyn approximation and Dirichlet series method are used. The accuracy of the analytics results has been tested by the comparison with numerical methods.

Introduction

The behavior of a two-dimensional mixing boundary layer between parallel streams is of basic fluid dynamics interest with many applications in engineering. Several authors have studied the boundary layer equations to mixing between parallel streams [1-5]. Lock [5] studied numerical solutions of the laminar boundary layer equations for the steady flow of a stream of viscous incompressible fluid over a parallel stream of different density and viscosity. Lessen [4] has obtained the velocity distribution of steady motion in the free laminar boundary layer separating the two streams, using a method equivalent to that of Blasius for the boundary layer on a flat plate. Chow [3] demonstrated that the technique suggested by Meksyn [6]

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may be employed to solve the jet mixing problem. Chen [2], based on the linearized theory of Blasius equation, analyzed the problem of two parallel streams of different properties.

Analysis

It is well known that the flow in a mixing layer is described by the boundary layer equation for the stream function with a zero pressure gradient [8]. In this case, the boundary layer equation may be stated as

$$f''' + \frac{1}{2} ff'' = 0 \tag{1}$$

with the boundary conditions

$$f(\eta = 0) = 0, \quad f'(\eta \to \infty) = 1, \quad f'(\eta \to -\infty) = 0. \tag{2}$$

The flow field may be interpreted as two separate flow regions, namely, the upper and lower half-planes. The two parts of the flow field will be joined together along the axis, so, the problem is transformed in:

Lower half-plane, $\eta \leq 0$

$$f_1''' + \frac{1}{2} f_1 f_1'' = 0 \tag{3}$$

with the boundary conditions:

$$f_1'(\eta \to -\infty) = 0, \quad f_1(\eta = 0) = 0. \tag{4}$$

Upper half-plane, $\eta \geq 0$

$$f_2''' + \frac{1}{2} f_2 f_2'' = 0 \tag{5}$$

with the boundary conditions:

$$f_2'(\eta \to \infty) = 1, \quad f_2(\eta = 0) = 0. \tag{6}$$

Furthermore, in $\eta = 0$ must be satisfied

$$f_1'(\eta = 0) = f_2'(\eta = 0), \tag{7}$$

$$f_1''(\eta = 0) = f_2''(\eta = 0). \tag{8}$$

Solution in lower half-plane, $\eta \le 0$

In this region, we will seek a solution by using Dirichlet series [4, 7],

$$f_1(\eta) = A_0 + A_1 e^{\frac{1}{2}a\eta} + A_2 e^{a\eta} + A_3 e^{\frac{3}{2}a\eta} + \cdots . \tag{9}$$

Substituting (9) in (3) and equating to zero the coefficients of successive powers of $e^{\frac{1}{2}a\eta}$, the following recurrence relations are obtained

$$A_0 = -a, \quad A_n = -\frac{1}{n^2}\frac{1}{(n-1)}\sum_{k=1}^{n-1} A_k A_{n-k}(n-k)^2, \quad n \ge 2, \tag{10}$$

where the coefficients a and A_1 are free. Introducing (10) in (9) gives the following expression for the current function

$$f_1(\eta) = -a + A_1\left(e^{\frac{1}{2}a\eta} - \frac{1}{4}\frac{A_1}{a}e^{a\eta} + \frac{5}{72}\left(\frac{A_1}{a}\right)^2 e^{\frac{3}{2}a\eta}\right.$$

$$\left. - \frac{17}{864}\left(\frac{A_1}{a}\right)^3 e^{2a\eta} + \frac{121}{21600}\left(\frac{A_1}{a}\right)^4 e^{\frac{5}{2}a\eta} + \cdots\right). \tag{11}$$

Defining $\beta = \frac{A_1}{a}$, $f_1(\eta)$ and its derivative can be written as

$$f_1(\eta) = -a + a\beta\left(e^{\frac{1}{2}a\eta} - \frac{1}{4}\beta e^{a\eta} + \frac{5}{72}\beta^2 e^{\frac{3}{2}a\eta}\right.$$

$$\left. - \frac{17}{864}\beta^3 e^{2a\eta} + \frac{121}{21600}\beta^4 e^{\frac{5}{2}a\eta} + \cdots\right), \tag{12}$$

$$f_1'(\eta) = \frac{a^2}{2}\beta\left(e^{\frac{1}{2}a\eta} - \frac{1}{4}2\beta e^{a\eta} + \frac{5}{72}3\beta^2 e^{\frac{3}{2}a\eta}\right.$$

$$\left. - \frac{17}{864}4\beta^3 e^{2a\eta} + \frac{121}{21600}5\beta^4 e^{\frac{5}{2}a\eta} + \cdots\right), \tag{13}$$

$$f_1''(\eta) = \frac{a^3}{2\cdot 2}\beta\Bigg(e^{\frac{1}{2}a\eta} - \frac{1}{4}2^2\beta e^{a\eta} + \frac{5}{72}3^2\beta^2 e^{\frac{3}{2}a\eta}$$

$$- \frac{17}{864}4^2\beta^3 e^{2a\eta} + \frac{121}{21600}5^2\beta^4 e^{\frac{5}{2}a\eta} + \cdots\Bigg). \tag{14}$$

The boundary condition $f_1(\eta = 0) = 0$ yields

$$-1 + \beta\left(1 - \frac{1}{4}\beta + \frac{5}{72}\beta^2 - \frac{17}{864}\beta^3 + \frac{121}{21600}\beta^4 + \cdots\right) = 0. \tag{15}$$

The solution of this equation delivers the following value $\beta_0 = 1.3188.$

Solution in the upper half-plane, $\eta \geq 0$

In this region, we can apply the Meksyn method [6], we may assume that

$$f_2(\eta) = a_1\eta + a_2\eta^2 + a_3\eta^3 + a_4\eta^4 + \cdots. \tag{16}$$

Substituting (16) and its derivatives into equation (5), we obtain

$$a_3 = 0, \quad a_4 = -\frac{1}{4\cdot 3\cdot 2}a_1 a_2, \quad a_5 = -\frac{1}{5\cdot 4\cdot 3}a_2^2, \ldots, \tag{17}$$

where the coefficients a_1 and a_2 are free. Introducing equation (16) in (5) and integrating, we obtain

$$f_2''(\eta) = 2a_2 e^{-F(\eta)}, \tag{18}$$

where

$$F(\eta) = \frac{1}{2}\int_0^\eta f_2(\eta)d\eta = \frac{1}{2}\left(\frac{a_1}{2}\eta^2 + \frac{a_2}{3}\eta^3 + \frac{a_4}{5}\eta^5 + \cdots\right), \tag{19}$$

i.e.,

$$f_2''(\eta) = 2a_2 e^{-\frac{1}{2}\left(\frac{a_1}{2}\eta^2 + \frac{a_2}{3}\eta^3 + \cdots\right)}. \tag{20}$$

Integrating equation (20), the following expression to $f_2'(\eta)$ is obtained

$$f_2'(\eta) = 2a_2\int_0^\eta e^{-\frac{1}{2}\left(\frac{a_1}{2}\eta^2 + \frac{a_2}{3}\eta^3 + \cdots\right)}d\eta + a_1. \tag{21}$$

The integral in equation (21) is evaluated by first transforming the variable through the relationship

$$\tau = \frac{1}{2}\left(\frac{a_1}{2}\eta^2 + \frac{a_2}{3}\eta^3 + \frac{a_4}{5}\eta^5 + \cdots\right), \tag{22}$$

where $\eta = b_1\tau^{\frac{1}{2}} + b_2\tau^1 + b_3\tau^{\frac{3}{2}} + b_4\tau^2 + \cdots$.

Inverting the series, the coefficients b_i are found to be given by

$$b_1 = \frac{2}{\sqrt{a_1}}, \quad b_2 = -\frac{4}{3}\frac{a_2}{a_1^2}, \quad b_3 = \frac{20\sqrt{a_1}}{9}\frac{a_2^2}{a_1^4}, \quad b_4 = -\frac{128}{27}\frac{a_2^3}{a_1^5} - \frac{16}{5}\frac{a_4}{a_1^3}, \dots . \tag{23}$$

Equation (21) now becomes

$$f_2'(\eta) = 2a_2\left(\frac{1}{2}b_1\gamma\left(\frac{1}{2}, \tau\right) + b_2(1 - e^{-\tau}) + \frac{3}{2}b_3\gamma\left(\frac{3}{2}, \tau\right) + 2b_4\gamma(2, \tau) + \cdots\right) + a_1, \tag{24}$$

where $\gamma(\alpha, \tau) = \Gamma(\alpha) - \Gamma(\alpha, \tau)$, and $\Gamma(\alpha, \tau) = \int_\tau^\infty e^{-\tau}\tau^{\alpha-1}d\tau$ is the incomplete Gamma function.

Patching in $\eta = 0$

The boundary conditions (7) and (8) yield

$$\frac{a^2}{2}\beta_0\left(1 - \frac{1}{4}2\beta_0 + \frac{5}{72}3\beta_0^2 - \frac{17}{864}4\beta_0^3 + \frac{121}{21600}5\beta_0^4 + \cdots\right) = a_1, \tag{25}$$

$$\frac{a^3}{2\cdot 2}\beta_0\left(1 - \frac{1}{4}2\cdot 2\beta_0 + \frac{5}{72}3\cdot 3\beta_0^2 - \frac{17}{864}4\cdot 4\beta_0^3 + \frac{121}{21600}5\cdot 5\beta_0^4 + \cdots\right) = 2a_2. \tag{26}$$

The coefficients a_1 and a_2 are found to be given by

$$a_1 = \lambda_1\frac{a^2}{2}, \tag{27}$$

where $\lambda_1 = \beta_0\left(1 - \frac{1}{4}2\beta_0 + \frac{5}{72}3\beta_0^2 - \frac{17}{864}4\beta_0^3 + \frac{121}{21600}5\beta_0^4 + \cdots\right) = 0.765737,$

$$a_2 = \lambda_2\frac{a^3}{8}, \tag{28}$$

where

$$\lambda_2 = \beta_0\left(1 - \frac{1}{4}\,2\cdot 2\beta_0 + \frac{5}{72}\,3\cdot 3\beta_0^2 - \frac{17}{864}\,4\cdot 4\beta_0^3 + \frac{121}{21600}\,5\cdot 5\beta_0^4 + \cdots\right) = 0.420505.$$

By imposing the boundary conditions (7) and (8), the whole problem is reduced to find the coefficient *a*.

The boundary condition $f_2'(\eta \to \infty) = 1$ will be assessed through patching in $\eta = \eta^* \gg 1$ between the solution (24) and an asymptotic expansion of the equation (5), there is a common region of validity of the solution (24) and the asymptotic expansion. The corresponding asymptotic expansions for $f_2(\eta)$ when η is large and positive, and when $\lim_{\eta\to\infty} f_2'(\eta) = 1,\ \lim_{\eta\to\infty}(f_2(\eta) - \eta) = B,$ where B is a constant, is given approximately by $f_2''' + \frac{1}{2}(\eta + B)\,f_2'' = 0,$ the solution of which may be written as

$$f_2(\eta) \approx \eta + B + C\,\frac{e^{-\omega^2}}{\omega^2}\left(1 - \frac{3}{2\omega^2} + \frac{15}{4\omega^4} - \cdots\right) + C^2\,\frac{e^{-2\omega^2}}{8\omega^5} + \cdots, \tag{29}$$

$$f_2'(\eta) \approx 1 + \sqrt{\pi}C(erf\omega - 1) - C^2\,\frac{e^{-2\omega^2}}{4\omega^4} + \cdots, \tag{30}$$

$$f_2''(\eta) \approx Ce^{-\omega^2} + C^2\,\frac{e^{-2\omega^2}}{2\omega^3} + \cdots, \tag{31}$$

where $\omega = \frac{1}{2}(\eta + B)$ and C is a constant.

The matching between the equations (20), (24) and equations (29-31) in $\eta = \eta^*,$ yield the following algebraic equations for the coefficients *a*, *B* and *C*,

$$2a_2\left(\frac{1}{2}\,b_1\gamma\left(\frac{1}{2},\,\tau^*\right) + b_2(1 - e^{-\tau^*}) + \frac{3}{2}\,b_3\gamma\left(\frac{3}{2},\,\tau^*\right) + 2b_4\gamma(2,\,\tau^*) + \cdots\right) + a_1$$

$$= 1 + \sqrt{\pi}C(erf\omega^* - 1) - C^2\,\frac{e^{-2\omega^{*2}}}{4\omega^{*4}}, \tag{32}$$

$$2a_2 e^{-\tau^*} = Ce^{-\omega^{*2}} + C^2 \frac{e^{-2\omega^{*2}}}{2\omega^{*3}}, \tag{33}$$

$$\int_0^{\eta^*} f_2'(\eta)d\eta = \eta^* + B + C\frac{e^{-\omega^{*2}}}{\omega^{*2}}\left(1 - \frac{3}{2\omega^{*2}} + \frac{15}{4\omega^{*4}} - \cdots\right) + C^2 \frac{e^{-2\omega^{*2}}}{8\omega^{*5}}. \tag{34}$$

Results

Solving the system (32-34) with $\eta^* = 3$, it has been found that $a = 1.2388$, $B = -0.5267$ and $C = 1.1648$. The valor obtained for the velocity in the axis is $a_1 = 0.5876$ that is very close to 0.5873 obtained by Lock [5] using numerical integration of equation (1).

Table 1 shows the results obtained to the velocity in the axis $f'(0)$, the slop in the axis $f''(0)$, and compared with other results found in the literature. The results show that the hybrid scheme proposed in this paper provides results with high accuracy.

Table 1. Results obtained to the velocity in the axis $f'(0)$ and the slop $f''(0)$

	$-a$	$f'(0)$	$f''(0)$
Numerical Solution [5]	1.2386	0.5873	0.1996
Approximate Solution [5]	-	0.5871	0.1901
Chow [3]	-	0.5885	0.2829
Chapman [1]	1.2330	0.5870	-
Present Work	1.2388	0.5876	0.1998

Conclusions

This paper shows an improved approximate analytic solution for boundary layer between parallel streams, demonstrating that the present method delivers results with a better accuracy than the standard analytic methods. The present method can be easily extended to more complex flows, for example, jets with heat and mass transfer and combustion problems.

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