Advances in Fuzzy Sets and Systems
Volume 5, Number 1, 2010, Pages 29-39
Published Online: February 15, 2010
This paper is available online at http://www.pphmj.com

S-PRODUCT OF *S*-ANTI-FUZZY RIGHT *R*-SUBGROUPS OF NEAR-RINGS

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Abstract

In this paper, we introduce the notion of S-anti-fuzzy right R-subgroups of near-rings and its basic properties are investigated. We also study the homomorphic image and preimage of S-anti-fuzzy right R-subgroups. Using S-norm, we introduce the notion on sensible anti-fuzzy right R-subgroups in near-rings and some related properties on a near-rings R are discussed.

1. Introduction

The concept of fuzzy subset was introduced by Zadeh [15]. Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. Schweizer and Sklar [13] introduced the notions of Triangular norm (t-norm) and Triangular co-norm (S-norm) are the most general families of binary operations that satisfy the requirement of the conjunction and disjunction operators, respectively. First, Abu Osman [2] introduced the notion of fuzzy subgroup with respect to t-norm. Abou-Zaid [1] introduced the concept of R-subgroups of a near-rings and Kim [8] introduced the concept of fuzzy

2010 Mathematics Subject Classification: 06F35, 03G25, 03E72.

Keywords and phrases: fuzzy right R-subgroups, near-rings, S-norm, sensible fuzzy right R-subgroups.

Received October 26, 2007

R-subgroups of a near-ring. Then Zhan [16] introduced the properties of fuzzy hyper ideals in hyper near-rings with t-norm. Recently, Cho et al. [5] introduced the notion of fuzzy subalgebras with respect to S-norm of BCK-algebras and Akram and Zhan [3] introduced the notion of sensible fuzzy ideal of [3] and [5]. In this paper, we will redefine anti-fuzzy right R-subgroups of a near-ring R with respect to an S-norm and investigate it is related properties. Also, we review several results described in [8] and [13] using S-norm.

2. Preliminaries

A *ring* S is a system consisting of a non-empty set S together with two binary operations on S called *addition* and *multiplication* such that

(i) S together with addition is a semi group.

(ii) S together with multiplication is a semi group.

(iii) $a(b+c) = ab + ac$ and $(a+b)c = ac + bc$ for all $a, b, c \in S$. A *semi ring* S is said to be additively commutative if $a + b = b + a$ for all $a, b \in S$. A zero element of a semi ring S is an element o such that $o \cdot x = x \cdot o = o$ and $o + x = x + o = x$ for all $x \in S$. By a near-ring, we mean a non-empty set R with two binary operations '+' and '$\cdot$' satisfies the following axioms:

(i) $(R, +)$ is a group.

(ii) $(R, \cdot)$ is a semi group.

(iii) $(b + c)a = ba + ca$ for all $a, b, c \in R$.

Precisely speaking, it is a right near-ring because it satisfies the right distribution law $x \cdot y$. Note that, $xo = o$ and $x(-y) = -(xy)$ but in general $ox \neq o$ for some $x \in R$. A two sided R-subgroups in a near-ring R is a subset N of R such that

(i) $(N, +)$ is a subgroup of $(R, +)$.

(ii) $RN \subset N$.

(iii) $NR \subset N$.

If N satisfies (i) and (ii), then it is called a *right R subgroup of R*. We now review some fuzzy logic concepts. A fuzzy set μ in a set R is a function

$$\mu : R \to [0, 1].$$

Let $\mathrm{Im}(\mu)$ denote the image set of μ. Let μ be a fuzzy set in R. For $t \in [0, 1]$, the set $L(\mu : \alpha) = \{x \in R / \mu(x) \leq \alpha\}$ is called a *lower level subset of* μ.

Let R be a near-ring and μ be a fuzzy set in R. We say that μ is a *fuzzy near-ring of R* if, for all $x, y \in R$,

(FS1) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$.

(FS2) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$. If a fuzzy set μ in a near-ring R satisfies the property (FS1), then $\mu(0) \geq \mu(x)$ for all $x \in R$.

Definition 2.1. By an s-norm S, we mean a function $S : [0, 1] \to [0, 1]$ satisfying the following conditions:

(S1) $S(x, 0) = x$.

(S2) $S(x, y) \leq S(x, z)$ if $y \leq z$.

(S3) $S(x, y) = S(y, x)$.

(S4) $S(x, S(y, z)) = S(S(x, y), z)$, for all $x, y, z \in [0, 1]$.

Replacing 0 by 1 in condition S1, we obtain the concept of t-norm T.

Proposition 2.2. *For an S-norm, then the following statement holds* $S(x, y) \geq \max\{x, y\}$, *for all* $x, y \in [0, 1]$.

Definition 2.3. Let S be an s-norm. A *fuzzy set* μ in R is said to be *sensible* with respect to S if $\mathrm{Im}(\mu) \subset \Delta s$, where $\Delta s = \{s(\alpha, \alpha) = \alpha / \alpha \in [0, 1]\}$.

Definition 2.4. Let $(R, +, \cdot)$ be a near-ring. A *fuzzy set* μ in R is called an *anti fuzzy right* (resp. *left*) *R-subgroup of R* if

(AF1) $\mu(x - y) \leq \max\{\mu(x), \mu(y)\}$, for all $x, y \in R$.

(AF2) $\mu(xr) \leq \mu(x)$, for all $r, x \in R$.

Definition 2.5. Let $(R, +, \cdot)$ be a near-ring. A *fuzzy set* μ in R is called a *fuzzy right* (resp. *left*) *R-subgroup of R* if

(FR1) μ is a fuzzy subgroup of $(R, +)$.

(FR2) $\mu(xr) \geq \mu(x)$ (resp. $\mu(rx) \geq \mu(x)$), for all $r, x \in R$.

Definition 2.6. Let S be an s-norm. A *function* $\mu : R \to [0, 1]$ is called a *fuzzy right* (resp. *left*) *R-subgroup of R* with respect to S if

(C1) $\mu(x - y) \leq S\{\mu(x), \mu(y)\}$.

(C2) $\mu(xr) \leq \mu(x)$ (resp. $\mu(rx) \leq \mu(x)$) for all $r, x \in R$. If a fuzzy R-subgroup μ of R with respect to S is sensible, we say that μ is a *sensible fuzzy R-subgroup of R* with respect to S.

Example 2.7. Let K be the set natural numbers including 0 and K be an R-subgroup with usual addition and multiplication.

Proposition 3.1. *Define a fuzzy subset* $\mu : R \to [0, 1]$ *by*

$$\mu(x) = 0 \ \text{ if } x \text{ is even } 0$$

$$= 1 \ \text{ otherwise.}$$

And let $Sm : [0,1] \to [0,1]$ *by a function defined by* $Sm(\alpha, \beta) = \min\{x + y, 1\}$ *for all* $x, y \in [0, 1]$. *Then Sm is a t-norm. By routine calculation, we know that* μ *is sensible R-fuzzy subgroup of R.*

3. Properties of Anti-fuzzy *R*-subgroups

Proposition 3.1. *Let S be an s-norm. Then every sensible S-anti-fuzzy right R-subgroups* μ *of R is an anti-fuzzy R-subgroups of R.*

Proof. Assume that μ is a sensible S-anti-fuzzy right R-subgroups of R. Then we have

(AF1) $\mu(x - y) \leq S(\mu(x), \mu(y))$ and (AF2) $\mu(xr) \leq \mu(x)$ for all $x, y \in S$.

Since μ is sensible, we have

$$\begin{aligned} \text{Max}\{\mu(x), \mu(y)\} &= S(\min\{\mu(x), \mu(y)\}, \min\{\mu(x), \mu(y)\}) \\ &\geq S\{\mu(x), \mu(y)\} \\ &\geq \max\{\mu(x), \mu(y)\} \end{aligned}$$

and so, $S\{\mu(x), \mu(y)\} = \max\{\mu(x), \mu(y)\}$. It follows that

$$\mu(x - y) \leq S\{\mu(x), \mu(y)\} = \max\{\mu(x), \mu(y)\} \quad \text{for all } x, y \text{ in } R.$$

Clearly $\mu(xr) \leq \mu(x)$ for all r, x in R. So μ is an anti-fuzzy R-subgroup of R.

Proposition 3.2. *If μ is an S-anti-fuzzy right R-subgroup of a near-ring R and θ is an endomorphism of R, then $\mu_{[\theta]}$ is an S-anti-fuzzy right R-subgroup of R.*

Proof. For any $x, y \in R$, we have

(i) $\mu_{[\theta]}(x - y) = \mu(\theta(x - y)) = \mu(\theta(x) - \theta(y)) \leq S(\mu_{[\theta]}(x), \mu_{[\theta]}(y))$.

(ii) $\mu_{[\theta]}(xr) = \mu(\theta(xr)) = \mu(\theta(x)r) \leq \mu(\theta(x)) \leq \mu_{[\theta]}(x)$, hence $\mu_{[\theta]}$ is an S-anti-fuzzy right R-subgroup of R.

Definition 3.3. Let f be a *mapping* defined on R. If ψ is a fuzzy subset in $f(R)$, then the fuzzy subset $\mu = \psi$ in R, i.e., $\mu(x) = \psi(f(x))$ for all x in R is called the *preimage of* ψ under f.

Proposition 3.4. *An onto homomorphic preimage of an S-anti-fuzzy right R-subgroup of a near-ring is S-anti-fuzzy right R-subgroups.*

Proof. Let $f : R \to R^1$ be an onto homomorphism of near-ring, ψ be an S-anti-fuzzy right R-subgroup of R and μ be the preimage of ψ under f. Then we have

(i) $\mu(x - y) = \psi(f(x - y)) = \psi(f(x) - f(y))$

$$\leq S(\psi(f(x)), \psi(f(y))) = S(\mu(x), \mu(y)).$$

(ii) $\mu(xr) = \psi(f(xr)) = \psi(f(x)r) \leq \psi(f(x)) = \mu(x)$.

Hence μ is an S-anti-fuzzy-right R-subgroup of R.

Proposition 3.5. *An onto homomorphic image of an anti-fuzzy right R-subgroup with the inf property is an anti-fuzzy right R-subgroup.*

Proof. *Let* $f : R \to R^1$ be an onto homomorphism of near-ring and μ be an S-anti-fuzzy right R-subgroup of R with inf property. Given

$x, y \in R,$ we let $x_0 \in f^{-1}(x^1),$ and $y_0 \in f^{-1}(y^1)$ be such that $\mu(x_0) = \inf_{h \in f^{-1}(x^1)} \mu(h)$ and $\mu(y_0) = \inf_{h \in f^{-1}(y^1)} \mu(h),$ respectively. Then we can deduce that

$$\mu f(x^1 - y^1) = \inf_{z \in f^{-1}(x^1 - y^1)} \mu(z) \leq \max\{\mu(x_0), \mu(y_0)\}$$

$$= \max\{\inf_{h \in f^{-1}(x^1)} \mu(h), \inf_{h \in f^{-1}(y^1)} \mu(h)\}$$

$$= \max\{\mu f(x^1), \mu f(y^1)\},$$

$$\mu f(xr) = \inf_{z \in f^{-1}(x^1 r^1)} \mu(z) \leq \mu(y_0) = \inf_{h \in f^{-1}(y^1)} \mu(h) = \mu f(y^1).$$

Hence, μ^f is anti-fuzzy right R-subgroups of R.

The above proposition can be further strengthened, we first give the following definitions:

Definition 3.6. An s-norm S on $[0, 1]$ is called a *continuous function* from $[0, 1] \times [0, 1] \to [0, 1]$ with respect to the usual topology. We observe that the function 'max' is always a continuous S-norm.

Proposition 3.7. *Let* $f : R \to R^1$ *be a homomorphism of near-rings. If* μ *is an S-anti-fuzzy right R-subgroup of* R^1, *then* μ^f *is S-anti-fuzzy right R-subgroup of R.*

Proof. Suppose μ is an S-anti-fuzzy right R-subgroup of R^1. Then

(i) for all $x, y \in R,$ we have

$$\mu^f(x - y) = \mu f(x - y) \leq S(\mu f(x), \mu f(y)) \leq S(\mu^f(x), \mu^f(y)).$$

(ii) for all $x, y \in R,$ we have

$$\mu^f(xr) = \mu f(xr) = \mu(f(x), r) \leq \mu(f(x)) \leq \mu^f(x).$$

Hence, μ^f is an S-anti-fuzzy right R-subgroup of R.

Proposition 3.8. *Let* $f : R \to R'$ *be a homomorphism of near-rings. If* μ^f *is an S-anti-fuzzy right R-subgroup of R, then* μ *is S-anti-fuzzy right R-subgroup* R' *right R-subgroup of* R^1, *then*:

Let x^1, y^1 in R^1, there exists $x, y \in R$, such that $f(x) = x^1$ and $f(y) = y^1$.

We have

$$\begin{aligned}\mu(x^1 - y^1) &= \mu(f(x) - f(y))\\ &= \mu(f(x - y))\\ &= \mu^f(x - y)\\ &\leq S(\mu f(x), \mu f(y))\\ &= S(\mu(f(x)), \mu f(y))\\ &= S(\mu(x^1), \mu(y^1)).\end{aligned}$$

(iii) Let $x^1, r^1 \in R$, there exists $x, r \in R$, such that $f(x) = x^1$, $f(y) = r^1$, we have

$$\mu(x^1 r^1) = \mu(f(x), f(y)) = \mu(f(xr)) \leq \mu^f(x) \leq \mu(f(x)) \leq \mu(x^1).$$

Proposition 3.9. *Let S be a continuous S-norm and f be a homomorphism on a near-ring R. If* μ *is an S-anti-fuzzy right R-subgroup of R, then* μ^f *is an S-anti-fuzzy right R-subgroup of* $f(R)$.

Proof. Let $A_1 = f^{-1}(y_1)$, $A_2 = f^{-1}(y_2)$ and $A_{12} = f^{-1}(y_1 - y_2)$, where $y_1 - y_2 \in f(R)$. Consider the set $A_1 - A_2 = \{x \in R / x = a_1 - a_2$ for some $a_1 \in A_1, a_2 \in A_2\}$. If $x \in A_1 - A_2$, then $x = x_1 - x_2$ for some $x_1 \in A_1$ and $x_2 \in A_2$. So that, we have $f(x) = f(x_1 - x_2) = f(x_1) - f(x_2) = y_1 - y_2$, i.e., $x \in f^{-1}(y_1 - y_2) = A_{12}$. We have $A_1 - A_2 \subset A_{12}$.

It follows that

$$\begin{aligned}\mu f(y_1 - y_2) &= \inf\{\mu(x)/x \in f^{-1}(x_1 - x_2)\}\\ &= \inf\{\mu(x)/x \in A_{12}\}\\ &\le \inf\{\mu(x)/x \in A_1 - A_2\}\\ &\le \inf\{\mu(x_1 - x_2)/x_1 \in A_1,\ x_2 \in A_2\}\\ &\le \inf\{S(\mu(x_1),\ \mu(x_2))/x_1 \in A_1,\ x_2 \in A_2\}.\end{aligned}$$

Since S is continuous for every $\varepsilon > 0$, we see that, if $\inf\{\mu(x_1)/x_1 \in A_1\} - x_1^* \le \delta$ and $\inf\{\mu(x_2)/x_2 \in A_2\} - x_2^* \le \delta$, then $S(\inf\{\mu(x_1)/x_1 \in A_1\}, \inf\{\mu(x_2)/x_2 \in A_2\}) - S(x_1^*, x_2^*) \le \varepsilon$. Choose $a_1 \in A_1$, and $a_2 \in A_2$, such that

$$\inf\{\mu(x_1)/x_1 \in A_1\} - \mu(a_1) \le \delta$$

and

$$\inf\{\mu(x_2)/x_2 \in A_2\} - \mu(a_2) \le \delta,$$

then

$$S(\inf\{\mu(x_1)/x_1 \in A_1\}, \inf\{\mu(x_2)/x_2 \in A_2\}) - S\{\mu(a_1), \mu(a_2)\} \le \varepsilon.$$

Thus, we have

$$\begin{aligned}\text{(i)}\ \mu^f(y_1 - y_2) &\le \inf\{S(\mu(x_1), \mu(x_2))/x_1 \in A_1,\ x_2 \in A_2\}\\ &= S(\inf\{\mu(x_1)/x_1 \in A_1\}, \inf\{\mu(x_2)/x_2 \in A_2\})\\ &= S\{\mu^f(y_1),\ \mu^f(y_2)\}.\end{aligned}$$

(ii) Similarly, we can prove that

$$\mu^f(xr) \le \mu^f(x).$$

Hence μ^f is an S-anti-fuzzy right R-subgroup of $f(R)$.

Lemma 3.10. *Let T be a t-norm. Then t-conorm S can be defined as*:

$$S(x,\ y) = 1 - T(1 - x, 1 - y).$$

Proof. Straightforward.

Proposition 3.11. *A fuzzy subset* μ *of* R *is a* T*-anti fuzzy right* R*-subgroup, if and only if* μ^c *is an* S*-anti-fuzzy right* R*-subgroup of* R.

Proof. Let μ be a T-anti-fuzzy right R-subgroup of R. Then for all $x, y \in R$, we have

(i) $\mu^c(x - y) = 1 - \mu(x - y) \leq 1 - T(\mu(x), \mu(y))$

$$= 1 - T(1 - \mu^c(x), 1 - \mu^c(y)) = S(\mu^c(x), \mu^c(y)).$$

(ii) $\mu^c(xr) = 1 - \mu(xr) \leq 1 - \mu(x) = \mu^c(x)$, μ^c is an anti fuzzy right R-subgroup of R.

4. S-product of S-anti-fuzzy Right R-subgroups

Definition 4.1. A fuzzy relation on any set X is a fuzzy set $\mu : X \times X \to [0, 1]$.

Definition 4.2. Let S be an s-norm. If μ is a fuzzy relation on a set R and χ is fuzzy set in R, then μ is an S-fuzzy relation on χ if $\mu_\chi(x, y) \geq S(\chi(x), \chi(y))$ for all x, y in R.

Definition 4.3. Let S be an s-norm, and μ and χ be fuzzy subsets of R. Then *direct* S*-product* of μ and χ is defined as

$$(\mu \times \chi)(x, y) = S(\mu(x), \chi(y)), \quad \text{for all } x, y \in R.$$

Lemma 4.4. *Let* S *be an* s*-norm, and* μ *and* χ *be a fuzzy set of* R*. Then*

(i) $\mu \times \chi$ *is an* S*-fuzzy relation on* S.

(ii) $L(\mu \times \chi; t) = L(\mu; t) \times L(\chi; t)$, *for all* $t \in [0, 1]$.

Proof. Obivious.

Definition 4.5. Let S be an s-norm and μ be a fuzzy subset of R. Then μ is called *strongest* S*-fuzzy relation on* R if

$$\mu_\chi(x, y) \geq S(\chi(x), \chi(y)), \quad \text{for all } x, y \text{ in } R.$$

Proposition 4.6. *Let* S *be an* s*-norm, and* μ *and* χ *be* S*-anti-fuzzy right* R*-subgroups of* R*. Then* $\mu \times \chi$ *is an anti fuzzy right* R*-subgroup of* R.

Proof.

(i)
$$\begin{aligned}(\mu \times \chi)(x - y) &= (\mu \times \chi)((x_1, x_2) - (y_1, y_2)) \\ &= (\mu \times \chi)((x_1 - y_1), (x_2 - y_2)) \\ &= S(\mu(x_1 - y_1), \chi(x_2 - y_2)) \\ &\leq S(S(\mu(x_1), \mu(y_1), S(\chi(x_2), \chi(y_2)))) \\ &= S(S(\mu(x_1), \chi(x_2)), S(\mu(y_1), \chi(y_2))) \\ &= S((\mu \times \chi)(x_1, x_2), (\mu \times \chi)(y_1, y_2)) \\ &= S((\mu \times \chi)(x), (\mu \times \chi)(y)).\end{aligned}$$

(ii)
$$\begin{aligned}(\mu \times \chi)(xr) &= (\mu \times \chi)((x_1, x_2)(r_1, r_2)) \\ &= (\mu \times \chi)(x_1 r_1, x_2 r_2) \\ &= S(\mu(x_1), \chi(x_2)) \\ &= (\mu \times \chi)(x_1, x_2) \\ &= (\mu \times \chi)(x).\end{aligned}$$

Proposition 4.7. *Let* μ *and* χ *be sensible* S*-anti-fuzzy right* R*-subgroups of a near-ring* R*. Then* $\mu \times \chi$ *is a sensible* S*-anti-fuzzy right* R*-subgroup of* $R \times R$.

Proof. By Proposition 4.6, we have $\mu \times \chi$ is S-anti-fuzzy right R-subgroup of $R \times R$. Let $x = (x_1, x_2)$ be any element of $S \times S$. Then

$$\begin{aligned}S((\mu \times \chi)(x), (\mu \times \chi)(x)) &= S((\mu \times \chi)(x_1, x_2), (\mu \times \chi)(x_1, x_2)) \\ &= S(S(\mu(x_1), \chi(x_2)), S(\mu(x_1), \chi(x_2))) \\ &= S(S(\mu(x_1), \mu(x_1)), S(\chi(x_2), \chi(x_2))) \\ &= S(\mu(x_1), \chi(x_2)) \\ &= (\mu \times \chi)(x_1, x_2) = (\mu \times \chi)(x).\end{aligned}$$

Remark 4.8. If $\mu \times \chi$ is a sensible S-anti-fuzzy right R-subgroup of $R \times R$, then $\mu \times \chi$ need not be sensible S-anti-fuzzy right R-subgroup of R.

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