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NEW EXACT SOLUTIONS OF THE $(1+1)$-DIMENSIONAL GROSS-PITAEVSKII EQUATION

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Abstract

An improved homogeneous balance principle and an F-expansion technique are used to construct exact periodic wave solutions to the $(1+1)$-dimensional Gross-Pitaevskii equation.

1. Introduction

Traditional studies of solitons are mainly focused on those nonlinear evolution equations where time variable does not appear explicitly. On the other hand, however, nonlinear systems with time-dependent nonlinearities and dispersions also attract lots of interests from physicists and mathematicians [2, 9], and are getting more and more absorbing. It has been reported that specific dependencies of the equation coefficients on time variables can enhance the stabilities of the solutions [7]. Moreover, time-modulated nonlinearities and/or dispersions can facilitate the manipulation of the soliton behaviors. These facts have greatly enlarged our knowledge on nonlinear excitations and gave an origin to the important concept, nonautonomous soliton, which was first proposed by Serkin et al. [5]. The BECs and

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nonlinear optics provide excellent proving grounds for exploring nonlinear systems with distributed coefficients. The well-known Feschbach resonance is used to control the nonlinearities of matter waves by manipulating the scattering length either in time or space, and have led to the proposal of many novel nonlinear phenomena [3]. Dispersion management (DM) for atomic matter waves is also proposed recently and has induced plenty of consequent studies [1]. Moreover, the evolution of matter waves in time-dependent traps has been addressed, and the modulation instability of a one-dimensional BEC system in a time-dependent harmonic potential has been investigated [8]. In nonlinear optics, nonlinear management (NM) and DM are also widely used for experiments and theories with temporal or spatial optical solitons, soliton lasers, ultrafast soliton switches [5]. Furthermore, the recent progresses on inhomogeneous nonlinear media have generated novel concepts such as the optical similariton [4].

In this paper, we present an analytical study on the dynamics of periodic waves of BEC with a time-varying atomic scattering length in a time-varying expulsive parabolic potential. New exact solutions of the 1D GP equation are obtained. The results show that the periodic waves can be compressed into a desired width and amplitude in a controllable manner by changing the scattering length and external potential.

2. New Exact Solutions of the 1D GP Equation

Now we discuss the controlled compression of a matter-wave bright soliton in an expulsive potential. The evolution of the weakly coupled BEC at zero temperature is governed by the 3D time-dependent GP equation. We assume that the trapping in the transverse directions is stronger and the longitudinal confinement frequency and the *s*-wave scattering length vary with time; then the BEC is cigar-shaped and the 3D GP equation can be reduced to the following dimensionless 1D GP equation [6]:

$$i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} - \frac{1}{2}k(t)\Omega^2 x^2\psi - g(t)|\psi|^2\psi, \tag{1}$$

where Ω is the normalized longitudinal confinement frequency at the initial stage. The wave function ψ, time t and variables x are, respectively, normalized to $(8\pi a_s\hbar/mw_\perp)^{1/2}$, the oscillation period $w_\perp^{-1}$, the harmonic oscillator length

$a_{\perp} = \sqrt{\hbar/mw_{\perp}}$, where a_s is the s-wave scattering length, $w_{\perp}$ is the harmonic frequency corresponding to the strong confinement cross-section and m is the mass of the atom.

Utilizing an F-expansion technique [13] and a procedure for balancing terms in the expansion [10], we present in this paper a method for finding analytical periodic traveling wave solutions to (1).

We define the complex periodic wave $\psi(x, t)$ of (1) in terms of its amplitude and phase [12]:

$$\psi(x, t) = u(x, t)\exp[i\phi(x, t)]. \tag{2}$$

Substituting $\psi(x, t)$ into (1), we find the following coupled equations:

$$u_t + 2u_x\phi_x + u\phi_{xx} = 0; \tag{3}$$

$$-u\phi_t + u_{xx} - u\phi_x^2 + \frac{1}{2}k\Omega^2x^2u + gu^3 = 0. \tag{4}$$

We seek traveling wave solutions to equations (3) and (4) and assume the functions to be of the form:

$$u = f_1(t)F(\theta) + f_2(t)F^{-1}(\theta); \tag{5}$$

$$\theta = p(t)x + l(t); \tag{6}$$

$$\phi = a(t)x^2 + b(t)x + e(t), \tag{7}$$

where f_1, f_2, p, l, a, b, and e are the parameter functions to be determined and F is a Jacobi elliptic function (JEF), which in general satisfy the following general first- and second-order nonlinear ordinary differential equations:

$$\left(\frac{dF}{d\theta}\right)^2 = c_0 + c_2F^2 + c_4F^4, \tag{8}$$

$$\frac{d^2F}{d\theta^2} = c_2F + 2c_4F^3, \tag{9}$$

where c_0, c_2 and c_4 are real constants related to the elliptic modulus of the JEFs (see Table 1).

Table 1. Jacobi elliptic functions

Solution	c_0	c_2	c_4	F	$M = 0$	$M = 1$
1	1	$-(1+M^2)$	M^2	*sn*	sin	tanh
2	$1-M^2$	$2M^2-1$	$-M^2$	*cn*	cos	sech
3	M^2-1	$2-M^2$	-1	*dn*	1	sech
4	M^2	$-(1+M^2)$	1	*ns*	cosec	coth
5	$-M^2$	$2M^2-1$	$1-M^2$	*nc*	sec	cosh
6	-1	$2-M^2$	M^2-1	*nd*	1	cosh
7	1	$2-M^2$	$1-M^2$	*sc*	tan	sinh
8	$1-M^2$	$2-M^2$	1	*cs*	cot	cosech
9	1	$-(1+M^2)$	M^2	*cd*	cos	1
10	M^2	$-1+M^2$	1	*dc*	sec	1

Substituting equations (5), (6), (7) into equations (3) and (4) and requiring that $x^q F^n (q = 0, 1, 2;\ n = 0, 1, 2, 3, 4, 5, 6)$ and $\sqrt{c_0 + c_2F^2 + c_4F^4}$ of each term be separately equal to zero, we obtain a system of algebraic or first order ordinary differential equations for f_1, f_2, p, l, a, b, and e:

$$\frac{df_j}{dt} + 2f_j a = 0, \tag{10}$$

$$f_j\left(\frac{dp}{dt} + 4pa\right) = 0, \tag{11}$$

$$f_j\left(\frac{dl}{dt} + 2lb\right) = 0, \tag{12}$$

$$f_j\left(2\frac{da}{dt} + 8a^2 - k\Omega^2\right) = 0, \tag{13}$$

$$f_j\left(\frac{da}{dt} + 4ab\right) = 0, \tag{14}$$

$$f_j\left(2\frac{de}{dt} + b^2 - c_2p^2 + 3gf_1f_2\right) = 0, \tag{15}$$

$$f_1(gf_1^2 + 2c_4 p^2) = 0, \tag{16}$$

$$f_2(gf_2^2 + 2c_0 p^2) = 0, \tag{17}$$

where $j = 1, 2,$ the constants c_0, c_2, and c_4 appearing in equations (15)-(17) are related to the square of the elliptic modulus M of JEFs (see Table 1). We consider the most generic case, in which f_1 and f_2 are assumed nonzero and a is arbitrary. The following set of exact solutions is found:

$$f_1 = f_{10}\alpha^{\frac{1}{2}}; \qquad f_2 = \varepsilon f_{10}\sqrt{\frac{c_0}{c_4}}, \tag{18}$$

$$p = p_0\alpha, \tag{19}$$

$$l = l_0 - 2p_0 b_0 \int_0^t \alpha^2 dt, \tag{20}$$

$$b = b_0\alpha, \tag{21}$$

$$e = e_0 + (c_2 p_0^2 - b_0^2 - 6\varepsilon p_0^2\sqrt{c_0 c_4})\int_0^t \alpha^2 dt, \tag{22}$$

$$k = \left(8a^2 + 2\frac{da}{dt}\right)\Big/\Omega^2, \tag{23}$$

$$g = -\frac{2c_4 p_0^2}{f_{10}^2}\alpha, \tag{24}$$

where $\alpha = \exp\left(-\int_0^t 4a\,dt\right)$. The subscript 0 denotes the value of the given function at $t = 0$. A parameter $\varepsilon = 0, 1, -1$ is introduced in equations (18) and (22).

Incorporating these solutions back into (2), we obtain the general periodic traveling wave solutions to (1):

$$\psi = f_{10}\alpha^{\frac{1}{2}}\left[F(\theta) + \varepsilon\sqrt{\frac{c_0}{c_4}}F(\theta)\right]\exp[i(ax^2 + bx + e)]. \tag{25}$$

3. Propagation Characteristics of Snaking Behavior

For simplicity in the following discussion, we merely analyze periodic traveling wave solutions ψ expressed by (25) and rewrite it in a simple form $(\varepsilon = 0,$ $c_0 = 1 - M^2,\ c_2 = 2M^2 - 1,\ c_4 = -M^2),$ namely

$$\psi = f_{10}\alpha^{\frac{1}{2}}F(\theta)\exp i[ax^2 + bx + e], \tag{26}$$

$$A \equiv |\psi|^2 = f_{10}^2\alpha cn(px + l,\ M)^2, \tag{27}$$

where M is modulus of Jacobi elliptic function.

It can be seen that in order to construct solutions ψ, nonlinearity g and potential k are not arbitrary, they are linked through the relations (23) and (24). Anyway, we still have a large freedom to choose those functions to obtain physical meaningful solutions.

For the simple example, we focus on the case

$$k(t) = \frac{k_0(k_0 + \sin t + k_0 \cos^2 t)}{2(1 + k_0 \sin t)^2\Omega^2}, \tag{28}$$

where k_0 is a constant. Actually, the case gives periodically time-modulated potential and diffraction. Substituting the expressions for $k(t)$ to (23) and solving the differential equation, we can obtain

$$a(t) = -\frac{k_0 \cos t}{4(1 + k_0 \sin t)}. \tag{29}$$

Consequently, the function α can be given as

$$\alpha = 1 + k_0 \sin t, \tag{30}$$

by integrating expression α. Substituting (30) and the expression for $g(t)$ into (24), we can obtain

$$g(t) = \frac{2M^2 p_0^2}{f_{10}}(1 + k_0 \sin t). \tag{31}$$

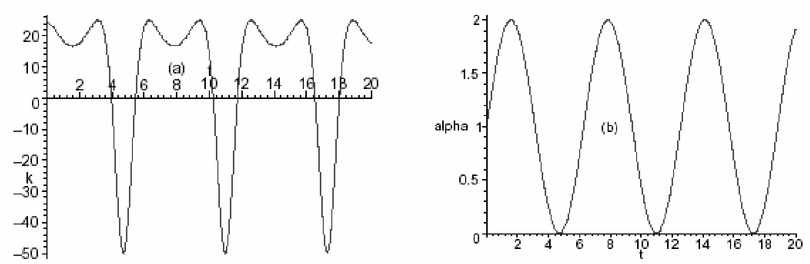

Figure 1. (a) and (b) given by equations (28) and (30), where $k_0 = 0.5$, $\Omega = 0.1$.

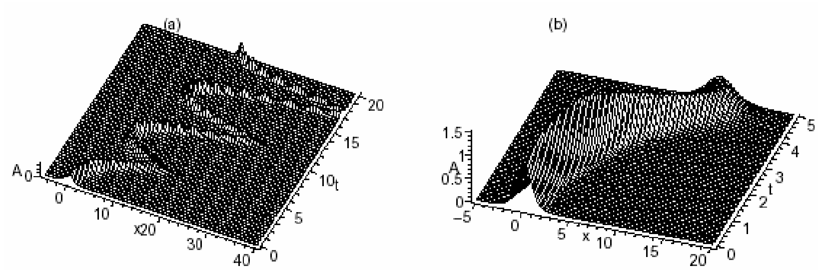

Figure 2. Density $A \equiv |\psi|^2$ as a function of propagation distance *x* and *t*, with $f_{10} = 0$, $p_0 = 1$, $l_0 = 1$, $k_0 = 0.5$, $M \to 1$, $b_0 = 1$.

In Figure 1, we show the picture of α and *k* as a function of *t*, with $k_0 = 0.5$, $\Omega = 0.1$. The function α has an important effect on modulating the amplitude of the solution *A*. It is seen in this situation that α is periodically oscillating along the *t*-axis. Consequently, the corresponding *A* will be explicit breathing solitary wave solution, and its density $|\psi|^2$ is depicted as a function of the propagation distance *x* and time *t*. The dynamics of the corresponding density $A \equiv |\psi|^2$ is shown in Figure 2(a). From the profile, we can see that the density of the solitary wave has strong variations on its propagation, and thus shows a snaking behavior. The variations are getting more and more significant as the wave is propagating, and the amplitude of the density also oscillates. In Figure 2(b), we plot the density $A \equiv |\psi|^2$ in a smaller time interval where *t* varies from 0 to 5, which clearly shows the snaking of the density along the *x*-axis and the oscillations of the amplitude. The solutions can help to understand matter wave dynamics in BECs with the interaction strength and the external potential changing with time.

4. Summary and Conclusion

In summary, an improved homogeneous balance principle and an *F*-expansion technique are applied to the generalized $(1+1)D$ Gross-Pitaevskii equation. Abundant exact analytical periodic wave solutions are obtained. Such exact solutions exist under certain conditions by imposing constraints on the functions describing nonlinearity, and potential function. The dynamics of the derived solutions can be manipulated by prescribing specific time-modulated nonlinearities and potentials. The results show that the spatiotemporal periodic waves and solitary waves are with breathing and snaking behaviors similar to the similaritons reported in other nonlinear systems. The present solution method provides a reliable technique that is more transparent and less tedious than the Jacobian elliptic function ansatz, or other expansion and variational methods. The technique is also applicable to other multidimensional nonlinear partial differential equation systems.

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