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A NUMERICAL APPROACH TO 3-SAT

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Abstract

We give a necessary and sufficient condition that a Boolean formula in 3-CNF is satisfiable in the form of a multiple integral and apply the Monte Carlo method to its evaluation.

1. Introduction

3-SAT is the problem deciding whether a given Boolean formula in 3-CNF is satisfiable, which was shown to be NP-complete by Cook [1]. 3-SAT can also be stated as follows.

Let $\varphi(x_1, \ldots, x_n)$ be a Boolean formula in 3-CNF

$$(y_{11} \vee y_{12} \vee y_{13}) \wedge \cdots \wedge (y_{m1} \vee y_{m2} \vee y_{m3}),$$

where $y_{11}, \ldots, y_{m3} \in \{x_1, \neg x_1, \ldots, x_n, \neg x_n\}$. By the equivalence relations

$$x \wedge y \Leftrightarrow xy, \quad x \vee y \Leftrightarrow 1 - (1 - x)(1 - y), \quad \neg x \Leftrightarrow 1 - x,$$

$\varphi(x_1, \ldots, x_n)$ can be transformed into

$$\prod_{i=1}^{m} (1 - (1 - y_{i1})(1 - y_{i2})(1 - y_{i3})),$$

where $y_{11}, \ldots, y_{m3} \in \{x_1, 1 - x_1, \ldots, x_n, 1 - x_n\}$ [3]. Then 3-SAT is equivalent to the

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problem deciding whether

$$T\varphi(x_1, \ldots, x_n) = \sum_{i=1}^{m} ((1 - y_{i1})(1 - y_{i2})(1 - y_{i3}))^{2n+1} = 0 \tag{1}$$

has a solution for $x_1, \ldots, x_n \in \{0, 1\}$.

In this paper, we give a necessary and sufficient condition that (1) has a solution for $x_1, \ldots, x_n \in \{0, 1\}$ in the form of a multiple integral and apply the Monte Carlo method to its evaluation.

2. The Criterion

We write, for $\varepsilon > 0$,

$$D_\varepsilon = \{(x_1, \ldots, x_n) : \varepsilon \le x_i \le 1 - \varepsilon,\ i = 1, \ldots, n\},$$

$$D_\varepsilon(a_1, \ldots, a_n) = \{(x_1, \ldots, x_n) \in [0, 1]^n : \varepsilon \le |x_i - a_i| \le 1/2,\ i = 1, \ldots, n\}.$$

Theorem 1. *Suppose that* $\varepsilon < (3n+1)^n/(2^{6n^2+2n} mn)$. *Then* $T\varphi(x_1, \ldots, x_n) = 0$ *has a solution* $x_1, \ldots, x_n \in \{0, 1\}$ *if and only if*

$$\int \cdots \int_{D_{\varepsilon - \varepsilon^2}} \frac{dx_1 \cdots dx_n}{T\varphi(x_1, \ldots, x_n)} > \frac{1}{mn\varepsilon}.$$

Proof. If $T\varphi(a_1, \ldots, a_n) = 0$ for $a_1, \ldots, a_n \in \{0, 1\}$, then

$$\begin{aligned} I(a_1, \ldots, a_n) &= \int \cdots \int_{D_{\varepsilon - \varepsilon^2}(a_1, \ldots, a_n)} \frac{dx_1 \cdots dx_n}{T\varphi(x_1, \ldots, x_n)} \\ &> \int \cdots \int_{D_{\varepsilon - \varepsilon^2}(0, \ldots, 0)} \frac{dx_1 \cdots dx_n}{m(x_1^{2n+1} + \cdots + x_n^{2n+1})} \\ &> \int_{\varepsilon - \varepsilon^2}^{\varepsilon} \cdots \int_{\varepsilon - \varepsilon^2}^{\varepsilon} \frac{dx_1 \cdots dx_n}{mn\varepsilon^{2n+1}} \\ &= \frac{1}{mn\varepsilon}. \end{aligned}$$

Hence, if (1) has at least one solution, then

$$\int \cdots \int_{D_{\varepsilon - \varepsilon^2}} \frac{dx_1 \cdots dx_n}{T\varphi(x_1, \ldots, x_n)} > \frac{1}{mn\varepsilon}. \tag{2}$$

If $T\varphi(a_1, ..., a_n) \neq 0$ for $a_1, ..., a_n \in \{0, 1\}$, then

$$I(a_1, ..., a_n) < \int \cdots \int_{D_{\varepsilon - \varepsilon^2}(0, ..., 0)} \frac{dx_1 \cdots dx_n}{((1 - x_1) \cdots (1 - x_n))^{6n+3}}$$

$$= \frac{1}{(6n + 2)^n} \left(\frac{1}{(1 - 1/2)^{6n+2}} - \frac{1}{(1 - \varepsilon + \varepsilon^2)^{6n+2}} \right)^n$$

$$< \frac{2^{6n^2 + 2n}}{(6n + 2)^n}.$$

Hence, if (1) has no solution, then

$$\int \cdots \int_{D_{\varepsilon - \varepsilon^2}} \frac{dx_1 \cdots dx_n}{T\varphi(x_1, ..., x_n)} < \frac{2^{6n^2 + 2n}}{(3n + 1)^n}. \tag{3}$$

By (2), (3) and $\varepsilon < (3n + 1)^n / (2^{6n^2 + 2n} mn)$, the theorem follows. □

3. Numerical Integration

From Theorem 1, to check satisfiability for $\varphi(x_1, ..., x_n)$ it suffices to evaluate

$$I = \int \cdots \int_{D_{\varepsilon - \varepsilon^2}} \frac{dx_1 \cdots dx_n}{T\varphi(x_1, ..., x_n)}$$

with the error

$$\frac{1}{2} \left(\frac{1}{mn\varepsilon} - \frac{2^{6n^2 + 2n}}{(3n + 1)^n} \right).$$

Using the Monte Carlo method [2], we can estimate

$$I \approx \frac{(1 - 2\varepsilon + 2\varepsilon^2)^n}{N} \sum_{i=1}^{N} \frac{1}{T\varphi(x_1(i), ..., x_n(i))},$$

where $(x_1(1), ..., x_n(1)), ..., (x_1(N), ..., x_n(N))$ are points selected at random in $D_{\varepsilon - \varepsilon^2}$. The error is $O(N^{-1/2})$ independent of the dimension n.

References

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