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# ON THE COMPLETENESS AND SOME SPECTRAL PROPERTIES OF DEGENERATE NON-SELFADJOINT ELLIPTIC DIFFERENTIAL OPERATORS

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**Abstract**

Let $\Omega$ be a bounded domain in $R^n$ with smooth boundary $\partial\Omega$. In this paper, we investigate the spectral properties of a non-selfadjoint elliptic differential operator $(Au)(x) = -\sum_{i,\, j=1}^{n} (\rho^{2\alpha}(x) a_{ij}(x) q(x) u'_{x_i}(x))'_{x_j}$ acting on Hilbert space $H = L^2(\Omega)$ with Dirichlet-type boundary conditions. Here $\rho(x) = \operatorname{dist}\{x, \partial\Omega\}$, $0 \le \alpha < 1$, $q(x) \in C^2(\overline{\Omega})$, $a_{ij}(x) \in C^2(\overline{\Omega})$, $a_{ij}(x) = a_{ji}(x)$, and there exists $c > 0$, such that $c|s|^2 \le \sum_{i,\, j=1}^{n} a_{ij}(x) s_i \overline{s_j}$ $(s = (s_1, ..., s_n) \in \mathbf{C}^n,\ x \in \Omega)$. Assume that $\forall x \in \overline{\Omega}$, $q(x) \in \mathbf{C} \backslash \Phi$, where $\Phi = \{z \in \mathbf{C} : |\arg z| \le \varphi\}$, $\varphi \in (0, \pi)$.

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## 1. Introduction

Let $\Omega$ be a bounded domain in $R^n$ with smooth boundary $\partial\Omega$ (i.e., $\partial\Omega \in C^\infty$). We introduce the weighted Sobolev space $\mathcal{H} = W_{2,\alpha}^2(\Omega)$ as the space of complex value functions $u(x)$ defined on $\Omega$ with finite norm:

$$|u|_+ = \left(\sum_{i=1}^n \int_\Omega \rho^{2\alpha}(x)|u'_{x_i}(x)|^2 ds + \int_\Omega |u(x)|^2 dx\right)^{1/2},$$

where $0 \le \alpha < 1$, and $\rho(x) = \text{dist}\{x, \partial\Omega\}$. We denote by $\overset{\circ}{\mathcal{H}}$ the closure of $C_0^\infty(\Omega)$ in $\mathcal{H}$ with respect to the above norm, i.e., $\overset{\circ}{\mathcal{H}}$ is the closure of $C_0^\infty(\Omega)$ in $W_{2,\alpha}^2(\Omega)$. The notion $C_0^\infty(\Omega)$ stands for the space of infinitely differentiable functions with compact support in $\Omega$. In this paper, we investigate the spectral properties, in particular, we estimate the resolvent of a non-selfadjoint elliptic differential operator of type

$$(Au)(x) = -\sum_{i,j=1}^n (\rho^{2\alpha}(x)a_{ij}(x)q(x)u'_{x_i}(x))'_{x_j} \quad \text{defined on} \quad H = L_2(\Omega),$$

now, for the closed extension of the operator $A$ with respect to space $= W_{2,\alpha}^2(\Omega)$, we need to extend its domain to the closed domain

$$D(A) = \left\{y \in \overset{\circ}{\mathcal{H}} \cap W_{2,loc}^2(\Omega) : \sum_{i,j=1}^n (\rho^{2\alpha}a_{ij}qy'_{x_i})'_{x_j} \in H\right\},$$

(see [8]) where the local space $W_{2,loc}^2(\Omega)$ is the class of the functions $u(x)$ $(x \in \Omega)$ in this form $W_{2,loc}^2(\Omega) = \left\{u(x) : \sum_{i=0}^2 \int_J |u^{(i)}(x)|^2 dx < \infty, J \subset \Omega, \text{open}\right\}$. Here $\rho(x) = \text{dist}\{x, \partial\Omega\}$, $0 \le \alpha < 1$, $q(x) \in C^2(\overline{\Omega})$, $a_{ij}(x) \in C^2(\overline{\Omega})$, $a_{ij}(x) = a_{ji}(x)$, and the function $a_{ij}(x)$ satisfies the uniformly elliptic condition, i.e., there exists $c > 0$ such that $c|s|^2 \le \sum_{i,j=1}^n a_{ij}(x)s_i\overline{s_j}$, where $s = (s_1, ..., s_n) \in \mathbf{C}^n$, $x \in \Omega$. Assume that $q(x) \in \mathbf{C}\backslash\Phi$, $\forall x \in \overline{\Omega}$, where $\Phi = \{z \in \mathbf{C} : |\arg z| \le \varphi\}$, $\varphi \in (0, \pi)$ (i.e., the value of $q(x)$ lie on the complex plane and outside of the closed angle $\Phi$), and let $q(x) \in C^2(\overline{\Omega})$. Here, and in the sequel, the value of the function $\arg z \in (-\pi, \pi]$, and $\| T \|$ denotes the norm of the bounded operator $T : H \to H$.

To get a feeling for the history of the subject under study, refer to our earlier papers [9, 11]. Indeed this paper was written in continuing with our earlier papers, the paper is sufficiently more general than our earlier papers, which here, we obtain the estimate of the resolvent of the operator *A* that satisfies the special and general conditions, and so we estimate the resolvent of the operator *A* that satisfies system of roots of vector functions of the operator *A* in $\mathcal{H}$. Moreover, the Fourier series of any vector function $f \in \mathcal{H}$ by the system of the root vector functions of the operator *A* is summable to *f* by the Abelian method with brackets.

This paper has four sections: Section 1 is devoted to introduction. In Section 2, we have Theorem 2.1 on the resolvent estimate of the differential operator *A* acting on *H* in the certain case (i.e., in this case, we will study Theorem 2.1 under assumption (2.2)). In Section 3, we have Theorem 3.1 on the resolvent estimate of the differential operator *A* acting on *H* in the general case (i.e., in this case, we will study Theorem 3.1 in contrast with Theorem 2.1, in other words, Theorem 3.1 does not include assumption (2.2) of Theorem 2.1).

Note that it is necessary to note some remarks regarding Theorem 2.1 and Theorem 3.1: Theorem 3.1 follows from Theorem 2.1 by dropping assumption (2.2) from Theorem 2.1, and so another comment regarding the assertion of these two theorems: We will see that Theorem 2.1 under the assumption (2.2) leads to its assertion that includes two estimates (2.3) and (2.4). While Theorem 3.1 without including assumption (2.2) of Theorem 2.1, leads to its assertion that is similar to the assertion of Theorem 2.1, but asserts only statement (2.3) of Theorem 2.1, which becomes (3.2) (in other words, now here, it is an open question which arises for us, i.e., whether we can prove a theorem the same Theorem 2.1 for general case, i.e., without condition (2.2), which its assertion includes two estimates (2.3) and (2.4)?). Completeness of the system of root vector functions of the operator *P* by the Abelian method with brackets is studied in Section 4.

## 2. The Resolvent Estimate of Degenerate Elliptic Differential Operators on *H* in Some Special Case

**Theorem 2.1.** *Let A and $\Phi$ be defined as in Section* 1. *Choose a closed sector*

$S \subset \Phi$ *with its vertex at zero* (*for more explanation see* [8]), *such that* $S \cap R_+ = \varnothing$.

*Let the complex function* $q(x)$ *satisfies the following conditions*:

$$q(x) \in C^1(\overline{\Omega}), \qquad q(x) \in \mathbf{C} \backslash S, \qquad (\forall x \in \overline{\Omega}), \tag{2.1}$$

$$|\arg\{q(x_1)q^{-1}(x_2)\}| \leq \frac{\pi}{8}, \qquad (\forall x_1, x_2 \in \overline{\Omega}). \tag{2.2}$$

*Then*, *for sufficiently large in modulus* $\lambda \in S$, *the inverse operator* $(A - \lambda I)^{-1}$ *exists and is continuous in H*, *and the following estimates are valid*:

$$\|(A - \lambda I)^{-1}\| \leq M_S |\lambda|^{-1} (\lambda \in S, |\lambda| > C_S), \tag{2.3}$$

$$\left\| \rho^{\alpha} \frac{\partial}{\partial x_i} (A - \lambda I)^{-1} \right\| \leq M_S' |\lambda|^{-\frac{1}{2}} (\lambda \in S, |\lambda| > C_S),$$

$$\text{for } i = 1, \ldots, n, \tag{2.4}$$

*where* $M_S, C_S > 0$ *are sufficiently large numbers depending on S*. *The symbol* $\|\cdot\|$ *stands for the norm of a bounded arbitrary operator T in H*.

**Proof.** Here, to establish Theorem 2.1, we will first prove the assertion of Theorem 2.1 together with estimate (2.3). So, as in Section 1 for the closed extension the operator *A* (for more explanation see Chapter 6 of [8]), we need to extend its domain to the closed set

$$D(A) = \{v \in \overset{\circ}{\mathcal{H}} \cap W_{2, loc}^2(\Omega) : hu' \in H, (hqv')' \in H\}.$$

Let the operator *A*, now satisfy (2.1), (2.2). Then there exists a complex number $Z \in C$ (notice that we can take $Z = e^{i\gamma}$, for a fix real $\gamma \in (-\pi, \pi]$), such that we have $|Z = e^{i\gamma}| = 1$, and so

$$c' \leq Re\{Zq(x)\}, \qquad c'|\lambda| \leq -Re\{Z\lambda\}, \qquad c' > 0 \ (\forall x \in \overline{\Omega}, \lambda \in \Phi). \tag{2.5}$$

In view of the uniformly elliptic condition, we have

$$c|s|^2 = c\sum_{i=1}^{n} |s_i|^2 \leq \sum_{i, j=1}^{n} a_{ij}(x) s_i \overline{s_j}, \quad (c > 0, s = (s_1, \ldots, s_n) \in \mathbf{C}^n, x \in \Omega)$$

take $s_i = y'_{x_i}$ implies that $c\sum_{i=1}^{n}|y'_{x_i}(x)|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) y'_{x_i}(x)\overline{y'_{x_j}(x)}$. From this, and according to $c' \leq Re\{Zq(x)\}$ in (2.4), we then multiply these two positive relations with each other which implies that

$$c_1 \sum_{i=1}^{n} |y'_{x_i}(x)|^2 \leq ReZq(x) \sum_{i,j=1}^{n} a_{ij}(x) y'_{x_i}(x)\overline{y'_{x_j}(x)}, \quad \text{for} \quad y \in D(A).$$

Multiplying both sides of the latter relation by the positive term $\rho^{2\alpha}(x)$, and then integrating from both sides, we have

$$c_1 \sum_{i=1}^{n} \int_{\Omega} \rho^{2\alpha}(x)|y'_{x_i}(x)|^2 dx \leq ReZ \sum_{i,j=1}^{n} \int_{\Omega} \rho^{2\alpha}(x) a_{ij}(x) q(x) y'_{x_i}(x)\overline{y'_{x_j}(x)} dx.$$

Now by applying the integration by parts, and using Dirichlet-type condition, then the right side of the latter relation without multiple *ReZ* becomes:

$$\sum_{i,j=1}^{n} \int_{\Omega} \rho^{2\alpha}(x) a_{ij}(x) q(x) y'_{x_i}(x)\overline{y'_{x_j}(x)} dx$$

$$= -\sum_{i,j=1}^{n} \int_{\Omega} (\rho^{2\alpha}(x) a_{ij}(x) q(x) y'_{x_i}(x))'_{x_j} \bar{y}(x) dx$$

$$= \left( -\sum_{i,j=1}^{n} (\rho^{2\alpha}(x) a_{ij}(x) q(x) y'_{x_i}(x))'_{x_j}, \ y(x) \right) = (Ay, y). \tag{2.6}$$

Since $(Ay)(x) = -\sum_{i,j=1}^{n} (\rho^{2\alpha}(x) a_{ij}(x) q(x) u'_{x_i}(x))'_{x_j}$.

Here, the symbol (,) denotes the inner product in $H$.

Notice that the above equality in (2.6) obtained by the well-known theorem of the *m*-sectorial operators which are closed by extending its domain to the closed domain in $\mathcal{H}$. These operators are associated with the closed sectorial bilinear forms that are densely defined in $\mathcal{H}$ (for more explanation see the well-known Theorem 2.1, Chapter 6 of [8]). The reason we extend the domain of the operator $A$ to the closed domain in space $\mathcal{H}$, above is now specified.

Therefore

$$c_1 \sum_{i=1}^{n} \int_{\Omega} \rho^{2\alpha}(x) | y'_{x_i}(x) |^2 dx \leq ReZ(Ay,\ y),$$

from (2.4), we have: $c' | \lambda | \leq -Re\{Z\lambda\},\ c' > 0,\ \forall \lambda \in \Phi.$ Multiply this inequality by $\int_{\Omega} | y(x) |^2 dx = (y,\ y) = \| y \|^2 > 0.$ It follows that

$$c' | \lambda | \int_{\Omega} | y(x) |^2 dt \leq -Re\{Z\lambda\}(y,\ y).$$

From this and the above inequality, we have

$$\begin{aligned} & c_1 \sum_{i=1}^{n} \int_{\Omega} \rho^{2\alpha}(x) | y'_{x_i}(x) |^2 dx + c' | \lambda | \int_{\Omega} | y(x) |^2 dx \\ & \leq Re\{Z(Ay,\ y) - Z\lambda(y,\ y)\} \\ & = Re\{Z((A - \lambda I) y,\ y)\} \\ & \leq \| Z \| \| y \| \| (A - \lambda I) y \| \\ & = \| y \| \| (A - \lambda I) y \|, \end{aligned} \tag{2.7}$$

i.e.,

$$c_1 \sum_{i=1}^{n} \int_{\Omega} \rho^{2\alpha}(x) | y'_{x_i}(x) |^2 dx + c' | \lambda | \int_{\Omega} | y(x) |^2 dx \leq \| y \| \| (A - \lambda I) y \|.$$

Since $c_1 \sum_{i=1}^{n} \int_{\Omega} \rho^{2\alpha}(x) | y'_{x_i}(x) |^2 dx$ is positive, we have either $c' | \lambda | \| y(x) \|^2 = | \lambda | \int_{\Omega} | y(x) |^2 dx \leq \| y \| \| (A - \lambda I) y \|$ or

$$| \lambda | \| y(x) \| \leq M_S \| (A - \lambda I) y \|. \tag{2.8}$$

This inequality ensures that the operator $(A - \lambda I)$ is one to one, which implies that $ker(A - \lambda I) = 0$. Therefore, the inverse operator $(A - \lambda I)^{-1}$ exists, and its continuity follows from the proof of the estimate (2.3) of Theorem 2.1. To prove (2.3), we set $v = (A - \lambda I)^{-1} f,\ \ f \in H$ in (2.7) implies that

$$|\lambda|\int_{\Omega}|(A-\lambda I)^{-1}f|^2 dx \le M_S\|(A-\lambda I)^{-1}f\|\|(A-\lambda I)(A-\lambda I)^{-1}f\|.$$

Since $(A-\lambda I)(A-\lambda I)^{-1}f = I(f) = f$,

$$|\lambda|\int_{\Omega}|(A-\lambda I)^{-1}f|^2 dx \le M_S\|(A-\lambda I)^{-1}f\||f|.$$

So

$$|\lambda|\|(A-\lambda I)^{-1}(f)\|^2 \le M_S\|(A-\lambda I)^{-1}(f)\||f|,$$

which implies that $|\lambda|\|(A-\lambda I)^{-1}(f)\| \le M_S|f|$. Since $\lambda \ne 0$, $\|(A-\lambda I)^{-1}(f)\| \le M_S|\lambda|^{-1}|f|$, i.e., $\|(A-\lambda I)^{-1}\| \le M_S|\lambda|^{-1}$. This estimate completes the proof of the assertion of Theorem 2.1 together with the estimate (2.3). Now, we start to prove the estimate (2.4) of Theorem 2.1. As in the above argument, we drop the positive term $c'|\lambda|\int_{\Omega}|y(x)|^2 dx$ from

$$c_1\sum_{i=1}^{n}\int_{\Omega}\rho^{2\alpha}(x)|y'_{x_i}(x)|^2 dx + c'|\lambda|\int_{\Omega}|y(x)|^2 dx \le \|y\|\|(A-\lambda I)y\|.$$

It follows that

$$c_1\sum_{i=1}^{n}\int_{\Omega}\rho^{2\alpha}(x)|y'_{x_i}(x)|^2 dx \le \|y\|\|(A-\lambda I)y\|.$$

Eminently,

$$c_1\left\|\rho^{\alpha}\frac{\partial}{\partial x_i}(A-\lambda I)^{-1}f\right\|^2 \le \|y\|\|(A-\lambda I)y\|.$$

Set $y = (A-\lambda I)^{-1}f$, $f \in H$ in the latter relation, and proceeding by similar calculation as in the proof (2.3) we then obtain:

$$c_1\left\|\rho^{\alpha}\frac{\partial}{\partial x_i}(A-\lambda I)^{-1}f\right\|^2 \le \|(A-\lambda I)^{-1}f\|\|(A-\lambda I)(A-\lambda I)^{-1}f\|.$$

Since $(A - \lambda I)(A - \lambda I)^{-1} f = I(f) = f,$

$$c_1 \left\| \rho^{\alpha} \frac{\partial}{\partial x_i} (A - \lambda I)^{-1} f \right\|^2 \leq \| (A - \lambda I)^{-1} \| \| f \|^2,$$

consequently, by (2.3) this implies that

$$c_1 \left\| \rho^{\alpha} \frac{\partial}{\partial x_i} (A - \lambda I)^{-1} f \right\|^2 \leq M_S | \lambda |^{-1} \| f \|^2,$$

to this end, we have

$$\left\| \rho^{\alpha} \frac{\partial}{\partial x_i} (A - \lambda I)^{-1} \right\| \leq M'_S | \lambda |^{-\frac{1}{2}}. \tag{2.7}$$

Thus, here the proof of the estimate (2.4) is finished, i.e., this completes the proof of Theorem 2.1.

Now let the condition (2.2) does not hold. Then we have the following statement.

## 3. The Resolvent Estimate of Some Classes of Degenerate Elliptic Differential Operators on *H*

In this section by dropping the assumption (2.2) from Theorem 2.1 in Section 2, we will derive the following new general theorem:

**Theorem 3.1.** *Let A and* $\Phi$ *be defined as in Section* 1, *and let* $S \subset \Phi \backslash R_+$ *be some closed sector with vertex at* 0. *Assume that the complex function* $q(x)$ *satisfies*

$$q(x) \in C^1(\overline{\Omega}), \quad q(x) \in \mathbf{C} \backslash S, \quad (\forall x \in \overline{\Omega}). \tag{3.1}$$

*Then*, *for sufficiently large in modulus* $\lambda \in S$, *the inverse operator* $(A - \lambda I)^{-1}$ *exists and is continuous in H*, *and the following estimates hold*:

$$\| (A - \lambda I)^{-1} \| \leq M_S | \lambda |^{-1}, \quad (\lambda \in S, | \lambda | > C_S), \tag{3.2}$$

*where* $M_S, C_S > 0$ *are sufficiently large numbers depending on S.*

**Proof.** Let us assume that (2.3) does not satisfy. To prove the assertion of Theorem 3.1 together with (3.2), we construct the functions $\varphi_1(x), \ldots, \varphi_m(x),$

$q_1(x), \ldots, q_m(x)$ so that each one of the functions $q_1(x), \ldots, q_m(x)$ $(x \in \overline{\Omega})$, as the function $q(x)$ in Theorem 2.1 satisfies (2.2). Therefore, let

$$\varphi_1(x), \ldots, \varphi_m(x), \qquad q_1(x), \ldots, q_m(x) \in C_0^\infty(\Omega),$$

satisfy

$$0 \le \varphi_r(x), \qquad r = 1, \ldots, m, \qquad \varphi_1^2(x) + \cdots + \varphi_m^2(x) \equiv 1 \qquad (x \in \overline{\Omega}),$$

$$\frac{d}{dt}\varphi_r(x) \in C_0^\infty(\Omega), \qquad q_r(x) = q(x), \qquad \forall x \in \operatorname{supp}\varphi_r,$$

$$q_r(x) \in \mathbf{C}\backslash\Phi, \qquad (\forall x \in \overline{\Omega}), \qquad r = 1, \ldots, m,$$

$$|\arg\{q_r(x_1)q_r^{-1}(x_2)\}| \le \frac{\pi}{8}, \qquad (\forall x_1, x_2 \in \operatorname{supp}\varphi_r), \qquad r = 1, \ldots, m.$$

In view of Theorem 2.1, and by (2.3) and (2.4), set $A_r = A$ in the definition of the differential operator, implies that

$$A_r u(x) = -\sum_{i,\, j=1}^{n} (\rho^{2\alpha}(x) a_{ij}(x) q_r(x) u'_{x_i}(x))'_{x_j},$$

acting on $H$, where

$$D(A_r) = \left\{ u \in \overset{\circ}{\mathcal{H}} \cap W_{2,\, loc}^2(\Omega) : \sum_{i,\, j=1}^{n} (\rho^{2\alpha} a_{ij} q_r u'_{x_i})'_{x_j} \in H \right\}.$$

Due to the assertion of Theorem 2.1, for $0 \ne \lambda \in S$, the inverse operator $(A - \lambda I)^{-1}$ exists and is continuous in space $H = L^2(0, 1)$, and satisfies

$$\|(A_r - \lambda I)^{-1}\| \le M_S |\lambda|^{-1},$$

$$\left\| \rho^\alpha \frac{\partial}{\partial x_i}(A_r - \lambda I)^{-1} \right\| \le M'_S |\lambda|^{-\frac{1}{2}} \qquad (\lambda \in S, |\lambda| > C_S),$$

$$(0 \ne \lambda \in S). \tag{3.3}$$

Let us introduce

$$G(\lambda) = \sum_{r=1}^{m} \varphi_r (A_r - \lambda I)^{-1} \varphi_r. \tag{3.4}$$

Here $\varphi_r$ is the multiplication operator in $H$ by the function $\varphi_r(x)$. Consequently, easily we have

$$(A - \lambda I)G(\lambda) = I + \rho^{2\alpha-1}(x)\sum_{r=1}^{m}\beta_r(x)(A_r - \lambda I)^{-1}\varphi_r$$

$$+ \rho^{2\alpha}(x)\sum_{i=1}^{n}\sum_{r=1}^{m}\gamma_{i_r}(x)\frac{\partial}{\partial x_i}(A_r - \lambda I)^{-1}\varphi_r, \tag{3.5}$$

where $\beta_r, \gamma_{i_r} \in L_\infty(\Omega)$, supp $\beta_r$ and supp $\gamma_{i_r}$ are contained in supp $\varphi_r$.

Let us take the right side of (3.5) equals to $I + T(\lambda)$. Thus, we have

$$(A - \lambda I)G(\lambda) = I + T(\lambda). \tag{3.6}$$

Now according to Section 2 if we put $A = A_r$, $r = 1, \ldots, m$ in (2.2), we have

$$\| (A_r - \lambda I)^{-1} \| \leq M1_S |\lambda|^{-1}, \qquad \left\| \rho^\alpha \frac{\partial}{\partial x_i}(A_r - \lambda I)^{-1} \right\| \leq M'_S \| \lambda \|^{-\frac{1}{2}}.$$

Owing to the definition of $T(\lambda)$ in the (3.5), it easily follows that

$$\| T(\lambda) \| \leq M_S |\lambda|^{-\frac{1}{2}} \qquad (\lambda \in \Phi, |\lambda| > 1). \tag{3.7}$$

Since $|\lambda|$ is sufficiently large number which easily implies that $\| T(\lambda) \| < \frac{1}{2} < 1$, from this and using the well-known theorem in the operator theory, we conclude that $I + T(\lambda)$ and so $(A - \lambda I)G(\lambda)$ are invertible. Hence, $((A - \lambda I)G(\lambda))^{-1}$ exists and equals to

$$(G(\lambda))^{-1}(A - \lambda I)^{-1} = (I + T(\lambda))^{-1}. \tag{3.8}$$

By adding $+I$ and $-I$ to the right side of the (3.8), it follows that

$$(G(\lambda))^{-1}(A - \lambda I)^{-1} = (I + T(\lambda))^{-1} - I + I.$$

Set

$$F(\lambda) = (I + T(\lambda))^{-1} - I.$$

Then

$$(G(\lambda))^{-1}(A - \lambda I)^{-1} = I + F(\lambda).$$

In view of (3.7) and $\| T(\lambda) \| < 1,$ we estimate the following geometric series:

$$\| F(\lambda) \| \leq \sum_{i=2}^{+\infty} \| T^k(\lambda) \| \leq \| T(\lambda) \|^2 (1 + \| T(\lambda) \| + \| T(\lambda) \|^2 + \cdots)$$

$$\leq \| T(\lambda) \|^2 M_S (1 + 1/2 + \cdots) \leq 2M_S (M'_S | \lambda |^{-1/2})^2.$$

Consequently, $\| F(\lambda) \| \leq 2M1_S | \lambda |^{-1}.$ By (3.3) and $\| (A_r - \lambda I)^{-1} \| \leq M1_S | \lambda |^{-1},$ we have

$$\| G(\lambda) \| = \left\| \sum_{r=1}^{m} \varphi_r (A_r - \lambda I)^{-1} \varphi_r \right\| \leq M''_S \| (A_r - \lambda I)^{-1} \| \leq M''_S M1_S | \lambda |^{-1},$$

i.e., $\| G(\lambda) \| \leq M2_S | \lambda |^{-1}.$ Now from (3.8), we have

$$(A - \lambda I)^{-1} = G(\lambda)(I + T(\lambda))^{-1} = G(\lambda)(I + F(\lambda)).$$

Therefore

$$\| (A - \lambda I)^{-1} \| = \| G(\lambda) \| \| (I + F(\lambda)) \| \leq M2_S | \lambda |^{-1} \| (1 + 2M1_S | \lambda |^{-1}) \|.$$

To complete the proof of Theorem 3.1, we must prove the estimate in (3.2), to the end we have according to latter inequality

$$\| (A - \lambda I)^{-1} \| \leq M2_S | \lambda |^{-1} + 2M2_S M1_S | \lambda |^{-1} | \lambda |^{-1},$$

and since $| \lambda |^{-1} | \lambda |^{-1} = | \lambda |^{-2} \leq | \lambda |^{-1},$ it follows that

$$\| (A - \lambda I)^{-1} \| \leq M_S | \lambda |^{-1}, \qquad (| \lambda | \geq C, \lambda \in \Phi),$$

i.e., here the estimate in (3.2) of Theorem 3.1 is proved and the above arguments complete the proof of the assertion of Theorem 3.1 together with (3.2). Therefore, now the proof of Theorem 3.1 is finished.

## 4. Completeness of the System of Root Vector Functions of the Operator $A$ in $\mathcal{H}$; and Summability of the Fourier Series of Elements $f \in \mathcal{H}$

**Theorem 4.1.** *Let* $\rho^{2\alpha}(x) \geq ct^{\gamma}(1 - t)^{\gamma}, (\gamma < 2).$ *Also let* $S \subset \overline{\Phi} \backslash R_+$ *be some* *closed sector with origin at zero. Then for sufficiently large in modulus* $\lambda \in S$, *the operator* $(A - \lambda I)^{-1}$ *is a compact operator of order not larger than* $1/2$. *The spectrum of the operator A consists of discrete countable eigenvalues counted according to multiplicity with perhaps an isolated point at infinity. The Fourier series of an arbitrary vector function* $f \in \mathcal{H}$ *of the system of the root of vector functions of the operator A converges to f by the Abelian method with brackets of order* $\gamma' = \frac{1}{2} + \varepsilon$ (*where* $\varepsilon > 0$ *is a sufficiently small number*). *For the definition of summability by Abelian method* (*introduced by Lidskij see* [1]). *The symbol* $N_0(x)$ *denotes the number of eigenvalues of the operator A does not exceed t* (*counted according to multiplicity*), *and satisfies the following inequality*:

$$N_0(x) \leq M(1 + t)^{1/2}.$$

**Proof.** Let $Z, \lambda \in \mathbf{C}$ be the same numbers as in Section 2, and let the operator $A$ have the same conditions as in (2.2) and (2.3), such that $\lambda \geq 0$. Since, as in Theorem 2.1, we showed that the operator $A - \lambda I$ has a continuous inverse and by (2.3), it follows that the operator $L(\lambda) = Z(A - \lambda I)$ is an *m*-sectorial operator in space $\mathcal{H}$ (see [8]), and so has the bilinear form

$$L_\lambda[v_1, v_2] = Z\int_0^1 \rho^{2\alpha}(x) q(x) v_1'(x)\overline{v_2'(x)}\,dt - \lambda Z\int_0^1 v_1(x)\overline{v_2(x)}\,dt,$$

where its domain is $D[L_\lambda] = \overset{\circ}{\mathcal{H}}$. Recall that as in Section 2, by closed extension its domain implies that this bilinear form is a closed and sectorial form which is densely defined in $\mathcal{H}$. Now by Theorem 3.2 ([8, Volume VI, 8.3.1]), the following relation is true:

$$L(\lambda) = L_0^{\frac{1}{2}}(\lambda) K(\lambda) L_0^{\frac{1}{2}}(\lambda), \tag{4.1}$$

in which $K(\lambda)$ is a bounded operator having continuous inverse, such that

$$\| K(\lambda) \| \leq M, \quad \| K^{-1}(\lambda) \| \leq M, \quad (0 \neq \lambda \in \Phi) \tag{4.2}$$

and $L_0(\lambda)$ is a self-adjoint operator in $\mathcal{H}$ associated with the bilinear form

$$\mathcal{L}_{0,\lambda}[v_1, v_2] = \int_0^1 Re[Z\rho^{2\alpha}(x)q(x)]v_1'(x)\overline{v_2'(x)}\,dt - Re(Z\lambda)\int_0^1 v_1(x)\overline{v_2(x)}\,dt,$$

$$D[\mathcal{L}_0, \lambda] = \overset{\circ}{\mathcal{H}}.$$

By (2.2), we have

$$\left\| (L' + |\lambda| I)^{\frac{1}{2}} L_0^{-\frac{1}{2}}(\lambda) \right\| \le M \qquad (|\lambda| > 1,\ \lambda \in S), \tag{4.3}$$

in which $L'$ is a positive self-adjoint operator in $\mathcal{H}$ associated with the bilinear form

$$\mathcal{L}'[v_1, v_2] = \int_0^1 \rho^{2\alpha}(x)q(x)v_1'(x)\overline{v_2'(x)}\,dt, \qquad D[\mathcal{L}'] = \overset{\circ}{\mathcal{H}}.$$

Now suppose that $\rho^{2\alpha}(x) \ge ct^{\gamma}(1-t)^{\gamma}$, $(\gamma < 2)$. Then

$$\left\| (L'' + |\lambda| I)^{\frac{1}{2}} (L' + |\lambda| I)^{-\frac{1}{2}} \right\| \le M \qquad (|\lambda| > 1,\ \lambda \in S), \tag{4.4}$$

in which $L''$ is a positive self-adjoint operator in $\mathcal{H}$ associated with the bilinear form

$$\mathcal{L}''[v_1, v_2] = \int_0^1 t^{\gamma}(1-t)^{\gamma} q(x)v_1'(x)\overline{v_2'(x)}\,dt, \qquad D[\mathcal{L}''] = \overset{\circ}{\mathcal{H}}.$$

Since $\gamma < 2$, $(L'' + T)^{-1} \in \sigma_{\infty}(H)$. Here $\sigma_{\infty}(H)$ denotes the space of compact linear operators on $H$, and for $N(\tau, L'')$ the corresponding eigenvalues of the operator $L''$ do not exceed $\tau$, we have (see [8, Chapter 7])

$$N(\tau, L'') \sim const(\tau)^{\frac{1}{2}} \qquad (\tau \to +\infty) \tag{4.5}$$

since $L''$ is a positive self-adjoint operator, we have

$$|(L'' + |\lambda| I)_2^{-\frac{1}{2}} \le M|\lambda|^{-\frac{1}{4}} \qquad (0 \ne \lambda \in S) \tag{4.6}$$

and

$$(L'' + |\lambda| I)^{-\frac{1}{2}} \in \sigma_p(H), \qquad \forall p > 1. \tag{4.7}$$

For the definition of $\sigma_p(H)$ and the norm $|\ |_p$ refer to [8]. According to (4.1)-(4.4), we have

$$L^{-1}(\lambda) = Z^{-1}(A - \lambda I)^{-1} = (L'' + |\lambda| I)^{\frac{1}{2}} K_1(\lambda)(L'' + |\lambda| I)^{-\frac{1}{2}} K_2(\lambda),$$

in which the functions $K_1(\lambda)$ and $K_2(\lambda)$ satisfy the following inequalities:

$$\| K_1(\lambda) \| \leq M, \quad \| K_2(\lambda) \| \leq M, \quad (|\lambda| \geq 1, \lambda \in S).$$

From these inequalities and from (4.5)-(4.7), we have $(A - \lambda I)^{-1} \in \sigma_q(H)$, $\forall q > \frac{1}{2}$ and

$$|(A - \lambda I)^{-1}|_1 \leq M_1 \left| (L'' + |\lambda| I)^{-\frac{1}{2}} \right|_2^2 \leq M_2 |\lambda|^{-\frac{1}{2}} \quad (|\lambda| \geq 1, \lambda \in S).$$

Consequently,

$$|(A - \lambda I)^{-1}|_1 \leq M_2 |\lambda|^{-\frac{1}{2}} \quad (|\lambda| \geq 1, \lambda \in S). \tag{4.8}$$

Notice that the above estimates are fulfilled when the operator *A* satisfies the condition (2.3) in Theorem 2.1 (as we saw in the proof of the first step of Theorem 2.1). Analogously, when the operator *A* does not satisfy (2.3), these estimates are again satisfied for the operator *A* (this was shown in the proof of Theorem 3.1).

Therefore, the order of the resolvent of the operator *A* does not exceed $\frac{1}{2}$.

To show discrete countable spectrum of the operator *A*, we know that this is proved in the [9, 11], here we recall that from the closed extension of the domain of the operator *A* to the closed set, we note that $D(A)$ is a closed set in the compact space $\mathcal{H}$, which this implies that the imbedding $\overset{\circ}{\mathcal{H}} \subset \mathcal{H}$ is compact (for the definition see [1]). Consequently, the operator *A* has a discrete countable spectrum (because, compact operators have a discrete countable spectrum).

Now, according to Theorems 6.4.1 and 6.4.2 of [1], we obtain the proof of Theorem 5.1, this is done by helping of the system of root vector functions of the operator *A* in $\mathcal{H}$, and for the summability of the Fourier series of elements $f \in \mathcal{H}$, by applying the Abelian methods with brackets.

Indeed, from (4.8) and the above observations, by applying Theorem 6.4.1 of [1], we can establish that the system of root vector functions of the operator $A$ is completed in $\mathcal{H}$. Analogously, by applying Theorem 6.4.2 of [1].

Analogously, by applying Theorem 6.4.2 of [1], we can prove that the Fourier series of any vector function $f \in \mathcal{H}$, by the system of root vector functions of the operator $A$, is summable to $f$ by the Abelian methods with brackets of order $\gamma' = \frac{1}{2} + \varepsilon$ (for sufficiently small $\varepsilon > 0$). Moreover, we now get the estimate $N_0(t) \leq M(1+t)^{\frac{1}{2}}$, $t > 0$, in which $N_0(t) = card\{j : |\lambda_j| \leq t\}$, and $\lambda_1, \lambda_2, \ldots,$ is the sequence of eigenvalues of the operator $A$. Since for sufficiently large absolute values of $\lambda \in S$, the operator $A - \lambda I$ has continuous inverse, for sufficiently large $j \geq j_0$, we have $\lambda_j \notin S$. Without loss of generality, we can suppose that

$$\{z \in \mathbf{C} : \arg z \leq \theta\} \subset \{0\} \cup \mathrm{Int}(S), \qquad 0 < \theta < \frac{\pi}{8},$$

where

$$|\lambda_j + t| \geq C_\theta(t + |\lambda_j|), \qquad \forall t > 1, \quad j \geq j_0(*).$$

Since the sum of the eigenvalues of the kernel operator does not exceed its kernel norm (see [8]), therefore (see (3.7)), now, by $(*)$ and $\frac{2t}{\tau + t} \leq 1$, we have

$$N_0(t) \leq \int_0^t dN_0(\tau) \leq 2t\int_0^t \frac{dN_0(\tau)}{\tau + t} \leq 2t\int_0^{+\infty} \frac{dN_0(\tau)}{\tau + t}$$

$$= 2t\sum_{j=1}^{+\infty} \frac{1}{|\lambda_j| + t} \leq 2tC_\theta^{-1}\sum_{j=t}^{+\infty}\left|\frac{1}{\lambda_j + t}\right|$$

$$\leq 2tC_\theta^{-1}|(A + tI)^{-1}|_1$$

$$\leq 2tC_\theta^{-1}(1+t)^{-\frac{1}{2}} \leq M(1+t)^{\frac{1}{2}},$$

this completes the proof of Theorem 4.1.

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