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ON SOME FULL EXTENSION OF HARDY-HILBERT'S INTEGRAL INEQUALITY

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Abstract

Some full extension of Hardy-Hilbert's integral inequality is obtained.

1. Introduction

Suppose that f and g are real functions such that $0 < \int_0^\infty f^2(t)dt < \infty$ and $0 < \int_0^\infty g^2(t)dt < \infty$. Then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} < \pi\left(\int_0^\infty f^2(t)dt \int_0^\infty g^2(t)dt\right)^{\frac{1}{2}}, \tag{1.1}$$

where π is the best possible. If (a_n) and (b_n) are sequences of real numbers such that $0 < \sum_{n=1}^\infty a_n^2 < \infty$ and $0 < \sum_{n=1}^\infty b_n^2 < \infty$, then

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} < \pi\left(\sum_{n=1}^\infty a_n^2 \sum_{n=1}^\infty b_n^2\right)^{\frac{1}{2}}. \tag{1.2}$$

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The inequalities (1.1) and (1.2) are called *Hilbert's inequalities*. These inequalities play an important role in analysis (cf. [3, Chap. 9]). In their recent papers Hu [4] and Gao et al. [2] gave two distinct improvements of (1.1) and Gao [1] gave (1.2) a strenthened version.

The following definitions are given:

$$\varphi_\lambda(r) = \frac{r + \lambda - 2}{r}\, (r = p,\, q), \quad k_\lambda(p) = B(\varphi_\lambda(p),\, \varphi_\lambda(q)),$$

and B is the beta function.

Very recently, by introducing some parameters, Yang and Debnath [6] gave the following extensions:

Theorem A. *If* $f,\, g \geq 0,\;\; p > 1,\;\; 1/p + 1/q = 1,\;\; \lambda > 2 - \min\{p,\, q\},$ *such that* $0 < \int_0^\infty t^{1-\lambda} f^p(t)\,dt < \infty$ *and* $0 < \int_0^\infty t^{1-\lambda} g^q(t)\,dt < \infty,$ *then*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(Ax + By)^\lambda}\,dxdy$$

$$< \frac{k_\lambda(p)}{A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}} \left(\int_0^\infty x^{1-\lambda} f^p(x)\,dx\right)^{1/p} \left(\int_0^\infty y^{1-\lambda} g^q(y)\,dy\right)^{1/q}, \tag{1.3}$$

where the constant factor $[k_\lambda(p)/A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}]$ *is the best possible.*

Theorem B. *If* $f \geq 0,\;\; p > 1,\;\; 1/p + 1/q = 1,\;\; \lambda > 2 - \min\{p,\, q\},\;\; A,\, B > 0$ *such that* $0 < \int_0^\infty t^{1-\lambda} f^p(t)\,dt < \infty,$ *then*

$$\int_0^\infty y^{(\lambda-1)(p-1)} \left(\int_0^\infty \frac{f(x)}{(Ax + By)^\lambda}\,dx\right)^p dy$$

$$< \left(\frac{k_\lambda(p)}{A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}}\right)^p \int_0^\infty x^{1-\lambda} f^p(x)\,dx, \tag{1.4}$$

where the constant factor $\lfloor k_\lambda(p)/A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)} \rfloor$ *is the best possible. The inequalities* (1.3) *and* (1.4) *are equivalent.*

Theorem C. *If* $a_n,\ b_n > 0\ (n \in N),\ \ p > 1,\ \ 1/p + 1/q = 1,\ \ 2 - \min\{p, q\} < \lambda < 2,$ $A,\ B > 0,$ *such that* $0 < \sum_{n=1}^{\infty} n^{1-\lambda} a_n^p < \infty$ *and* $0 < \sum_{n=1}^{\infty} n^{1-\lambda} b_n^q < \infty,$ *then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(Am + Bn)^{\lambda}}$$

$$< \frac{k_{\lambda}(p)}{A^{\varphi_{\lambda}(p)} B^{\varphi_{\lambda}(q)}} \left(\sum_{n=1}^{\infty} n^{1-\lambda} a_n^p\right)^{1/p} \left(\sum_{n=1}^{\infty} n^{1-\lambda} b_n^q\right)^{1/q}, \tag{1.5}$$

where the constant factor $\lfloor k_{\lambda}(p)/A^{\varphi_{\lambda}(p)} B^{\varphi_{\lambda}(q)} \rfloor$ *is the best possible.*

Theorem D. *If* $a_n \geq 0\ (n \in N),\ \ p > 1,\ \ 1/p + 1/q = 1,\ \ 2 - \min\{p,\ q\} < \lambda \leq 2,$ $A,\ B > 0$ *such that* $0 < \sum_{n=1}^{\infty} n^{1-\lambda} a_n^p < \infty,$ *then*

$$\sum_{n=1}^{\infty} n^{(\lambda-1)(p-1)} \left(\sum_{m=1}^{\infty} \frac{a_m}{(Am + Bn)^{\lambda}}\right)^p$$

$$< \left(\frac{k_{\lambda}(p)}{A^{\varphi_{\lambda}(p)} B^{\varphi_{\lambda}(q)}}\right)^p \sum_{n=1}^{\infty} n^{1-\lambda} a_n^p, \tag{1.6}$$

where the constant factor $\lfloor k_{\lambda}(p)/A^{\varphi_{\lambda}(p)} B^{\varphi_{\lambda}(q)} \rfloor$ *is the best possible. The inequalities* (1.5) *and* (1.6) *are equivalent.*

2. Lemma

The following lemma is required for our aim:

Lemma. *Let* $T \geq 1,\ 0 < \lambda/2 \leq \alpha.$ *Then*

$$\int_0^T \frac{u^{\frac{\lambda}{2}-1}}{(1+u)^{\lambda}}\, du \leq B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)\left(1 - \frac{1}{2} T^{-\alpha}\right).$$

Proof. Define for $x \leq 1,$

$$f(x) = x^{-\alpha} \int_0^x \frac{u^{\frac{\lambda}{2}-1}}{(1+u)^{\lambda}}\, du,$$

then we have

$$f'(x) = x^{-\alpha} \frac{x^{\frac{\lambda}{2}-1}}{(1+x)^{\lambda}} + \int_0^x \frac{u^{\frac{\lambda}{2}-1}}{(1+u)^{\lambda}}\, du(-\alpha)\, x^{-\alpha-1}$$

$$\leq \frac{x^{\frac{\lambda}{2}-\alpha-1}}{(1+x)^{\lambda}} - \frac{\alpha x^{-\alpha-1}}{(1+x)^{\lambda}} \int_0^x u^{\frac{\lambda}{2}-1} du$$

$$= \frac{x^{\frac{\lambda}{2}-\alpha-1}}{(1+x)^{\lambda}} \left(1 - \frac{\alpha}{\lambda/2}\right) \leq 0.$$

This shows that f is nonincreasing which implies

$$f(x) \geq f(1) = \int_0^1 \frac{u^{\frac{\lambda}{2}-1}}{(1+u)^{\lambda}}\, du = \frac{1}{2} B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right).$$

Therefore

$$\int_0^T \frac{u^{\frac{\lambda}{2}-1}}{(1+u)^{\lambda}}\, du = \left(\int_0^{\infty} - \int_T^{\infty}\right) \frac{u^{\frac{\lambda}{2}-1}}{(1+u)^{\lambda}}\, du$$

$$= B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) - \int_0^{1/T} \frac{u^{\frac{\lambda}{2}-1}}{(1+u)^{\lambda}}\, du$$

$$\leq B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)\left(1 - \frac{1}{2} T^{-\alpha}\right).$$

3. Main Result

We prove the following result:

Theorem. *Let* $f_i \geq 0, \quad i = 1, \ldots, n, \quad n \geq 2, \quad 0 < \lambda/2 \leq \alpha, \quad T \geq 1, \quad p_i > 1,$

$$\sum_{i=1}^{n} 1/p_i = 1, \quad 0 < \int_0^T x_i^{-\frac{\lambda}{2^n} - p_i\left(\sum_{j=1}^{n} \frac{\lambda}{2^{n+j-i} p_j} - 1\right) - 1} \left(1 - \frac{1}{2}\left(\frac{x_i}{T}\right)^{\alpha}\right)^{n-1} f_i^{p_i}(x_i)\, dx_i$$

$< \infty$. *Then*

$$\int_0^T \cdots \int_0^T \frac{f_1(x_1)\ \dots\ f_n(x_n)}{(x_1 + \dots + x_n)^{\lambda}}\, dx_1\ \dots\ dx_n$$

$$\leq \prod_{j=1}^{n-1} B\left(\frac{\lambda}{2^j}, \frac{\lambda}{2^j}\right) \prod_{i=1}^{n} \left(\int_0^T x_i^{-\frac{\lambda}{2^n} - p_i\left(\sum_{j=1}^{n} \frac{\lambda}{2^{n+j-i} p_j} - 1\right) - 1} \right.$$

$$\left. \times \left(1 - \frac{1}{2}\left(\frac{x_i}{T}\right)^{\alpha}\right)^{n-1} f_i^{p_i}(x_i)\, dx_i \right)^{1/p_i} \tag{3.1}$$

and

$$\int_0^T \frac{x_n^{\frac{1}{p_n - 1}\left(\frac{\lambda}{2^n} + p_n\left(\sum_{j=1}^{n} \frac{\lambda}{2^j p_j} - 1\right) + 1\right)}}{\left(1 - \frac{1}{2}\left(\frac{x_n}{T}\right)^{\alpha}\right)^{\frac{1-n}{1-p_n}}}$$

$$\times \left(\int_0^T \cdots \int_0^T \frac{f_1(x_1)\ \dots\ f_{n-1}(x_{n-1})}{(x_1 + \dots + x_n)^{\lambda}}\, dx_1\ \dots\ dx_n \right)^{\frac{p_n}{p_n - 1}} dx_n$$

$$\leq \prod_{j=1}^{n-1} B^{\frac{p_n}{1-p_n}}\left(\frac{\lambda}{2^j}, \frac{\lambda}{2^j}\right) \prod_{i=1}^{n-1} \left(\int_0^T x_1^{-\frac{\lambda}{2^n} - p_i\left(\sum_{j=1}^{n} \frac{\lambda}{2^{n+j-i} p_j} - 1\right) - 1} \right.$$

$$\left. \times \left(1 - \frac{1}{2}\left(\frac{x_i}{T}\right)^{\alpha}\right)^{n-1} f_i^{p_i}(x_i)\, dx_i \right)^{\frac{p_n}{(p_n - 1) p_i}}. \tag{3.2}$$

The inequalities (3.1) *and* (3.2) *are equivalent.*

Proof. We have

$$\int_0^T \cdots \int_0^T \frac{f_1(x_1) \ldots f_n(x_n)}{(x_1 + \cdots + x_n)^\lambda} dx_1 \ldots dx_n$$

$$= \int_0^T \cdots \int_0^T \prod_{i=1}^n \frac{\prod_{j=1}^n x_{j+i-1}^{\left(\frac{\lambda}{2^{n+1-j}}-1\right)\frac{1}{p_i}} f_i(x_i)}{x_i^{\left(\sum_{j=1}^n \frac{\lambda}{2^{n+j-i}p_j}-1\right)} \left(\sum_{j=1}^n x_{1+j-i}\right)^{\lambda/p_i}} dx_1 \ldots dx_n$$

$$\leq \prod_{i=1}^n \left(\int_0^T \cdots \int_0^T \frac{\prod_{j=1}^n x_{j+i-1}^{\frac{\lambda}{2^{n+1-j}}-1} f_i^{p_i}(x_i)}{x_i^{p_i\left(\sum_{j=1}^n \frac{\lambda}{2^{n+j-i}p_j}-1\right)} \left(\sum_{j=1}^n x_{1+j-i}\right)^{\lambda}} dx_1 \ldots dx_n \right)^{1/p_i},$$

where $x_{i\pm n}^{\frac{\lambda}{2^{n+i}}} = x_i^{\frac{\lambda}{2^j}}$, $i, j = 1, 2, \ldots, n.$

$$= \prod_{i=1}^n P_i^{1/p_i}, \text{ say.}$$

Let us first consider

$$P_1 = \int_0^T \cdots \int_0^T \frac{\prod_{j=1}^n x_j^{\left(\frac{\lambda}{2^{n+1-j}}-1\right)} f_1^{p_1}(x_1)}{x_1^{p_1\left(\sum_{j=1}^n \frac{\lambda}{2^{n+j-1}p_1}-1\right)} \left(\sum_{j=1}^n x_j\right)^{\lambda}} dx_1 \ldots dx_n$$

$$= \int_0^T \frac{x_1^{\frac{\lambda}{2^n}-1} f_1^{p_1}(x_1)}{x_1^{p_1\left(\sum_{j=1}^n \frac{\lambda}{2^{n+j-1}p_j}-1\right)}} dx_1 \int_0^T x_2^{\frac{\lambda}{2^{n-1}}-1} dx_2$$

$$\cdots \int_0^T x_{n-1}^{\frac{\lambda}{2^2}-1} dx_{n-1} \int_0^T \frac{x_n^{\frac{\lambda}{2}-1}}{(x_1 + \cdots + x_n)^\lambda} dx_n$$

$$= \int_0^T x_1^{\frac{\lambda}{2^n} - p_1\left(\sum_{j=1}^n \frac{\lambda}{2^{n+j-1} p_j} - 1\right) - 1} f_1^{p_1}(x_1) dx_1$$

$$\times \int_0^T x_2^{\frac{\lambda}{2^{n-1}}-1} dx_2 \cdots \int_0^T \frac{x_{n-1}^{\frac{\lambda}{2^2}-1}}{(x_1 + \cdots + x_{n-1})^{\lambda/2}} dx_{n-1}$$

$$\times \int_0^T \frac{\left(\frac{x_n}{x_1 + \cdots + x_{n-1}}\right)^{\frac{\lambda}{2}-1} d\left(\frac{x_n}{x_1 + \cdots + x_{n-1}}\right)}{\left(1 + \frac{x_n}{x_1 + \cdots + x_{n-1}}\right)^\lambda}.$$

As

$$\int_0^T \frac{\left(\frac{x_n}{x_1 + \cdots + x_{n-1}}\right)^{\frac{\lambda}{2}-1} d\left(\frac{x_n}{x_1 + \cdots + x_{n-1}}\right)}{\left(1 + \frac{x_n}{x_1 + \cdots + x_{n-1}}\right)^\lambda}$$

$$= \int_0^{\frac{T}{x_1 + \cdots + x_{n-1}}} \frac{u^{\frac{\lambda}{2}-1}}{(1+u)^\lambda} du$$

$$\le \int_0^{T/x_1} \frac{u^{\frac{\lambda}{2}-1}}{(1+u)^\lambda} du$$

$$\le B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)\left(1 - \frac{1}{2}\left(\frac{x_1}{T}\right)^\alpha\right),$$

by virtue of Lemma, then

$$\int_0^T \cdots \int_0^T P_1^n$$

$$\le B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)\int_0^T x_1^{\frac{\lambda}{2} - p_1\left(\sum_{j=1}^n \frac{\lambda}{2^{n+j-1} p_j} - 1\right) - 1}$$

$$\times \left(1 - \frac{1}{2}\left(\frac{x_1}{T}\right)^{\alpha}\right) f_1^{p_1}(x_1)\, dx_1 \int_0^T x_2^{\frac{\lambda}{2^{n-1}}} dx_2 \times \cdots \times \int_0^T \frac{x_{n-1}^{\frac{\lambda}{2^2} - 1}}{(x_1 + \cdots + x_{n-1})^{\lambda/2}}\, dx_{n-1}$$

$\cdots$

$$\le B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \ldots B\left(\frac{\lambda}{2^{n-2}}, \frac{\lambda}{2^{n-2}}\right)\int_0^T x_1^{\frac{\lambda}{2^n} - p_1\left(\sum_{j=1}^n \frac{\lambda}{2^{n+j-1} p_j} - 1\right) - 1}$$

$$\times \left(1 - \frac{1}{2}\left(\frac{x_1}{T}\right)^{\alpha}\right)^{n-2} f_1^n(x_1)\, dx_1 \int_0^T \frac{x_2^{\frac{\lambda}{2^{n-1}} - 1}}{(x_1 + x_2)^{\frac{\lambda}{2^{n-2}}}}\, dx_2$$

$$= B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \ldots B\left(\frac{\lambda}{2^{n-1}}, \frac{\lambda}{2^{n-1}}\right)\int_0^T x_1^{-\frac{\lambda}{2^n} - p_1\left(\sum_{j=1}^n \frac{\lambda}{2^{n+j-1} p_j} - 1\right) - 1}$$

$$\times \left(1 - \frac{1}{2}\left(\frac{x_1}{T}\right)^{\alpha}\right)^{n-1} f_1^{p_1}(x_1)\, dx_1.$$

Therefore, we can proceed in the same manner to have

$$\int_0^T \cdots \int_0^T \frac{f_1(x_1) \ldots f_n(x_n)}{(x_1 + \cdots + x_n)^{\lambda}}\, dx_1 \ldots dx_n$$

$$\le \prod_{j=1}^{n-1} B\left(\frac{\lambda}{2^j}, \frac{\lambda}{2^j}\right) \prod_{i=1}^{n} \left(\int_0^T x_1^{-\frac{\lambda}{2^n} - p_i\left(\sum_{j=1}^n \frac{\lambda}{2^{n+j-i} p_j} - 1\right) - 1} \right.$$

$$\left. \times \left(1 - \frac{1}{2}\left(\frac{x_i}{T}\right)^{\alpha}\right)^{n-1} f_i^{p_1}(x_i)\, dx_i \right)^{1/p_i}.$$

In order to prove the equivalence of (3.1) and (3.2), suppose that (3.1) is satisfied, then write $s_i = -\frac{\lambda}{2^n} - p_i\left(\sum_{j=1}^{n} \frac{\lambda}{2^{n+j-i} p_j} - 1\right) - 1,$ we have

$$\int_0^T \frac{x_n^{\frac{s_n}{1-p_n}}}{\left(1-\frac{1}{2}\left(\frac{x_n}{T}\right)^{\alpha}\right)^{\frac{1-n}{1-p_n}}} \left(\int_0^T \cdots \int_0^T \frac{f_1(x_1) \ldots f_{n-1}(x_{n-1})}{(x_1 + \cdots + x_n)^{\lambda}} dx_1 \ldots dx_{n-1}\right)^{\frac{p_n}{p_n-1}} dx_n$$

$$= \int_0^T \left(\int_0^T \cdots \int_0^T \frac{f_1(x_1) \ldots f_{n-1}(x_{n-1})}{(x_1 + \cdots + x_{n-1})^{\lambda}}\right) \frac{x_n^{\frac{p_n}{1-p_n}}}{\left(1-\frac{1}{2}\left(\frac{x_n}{T}\right)^{\alpha}\right)^{\frac{1-n}{1-p_n}}}$$

$$\times \left(\int_0^T \cdots \int_0^T \frac{f_1(x_1) \ldots f_{n-1}(x_{n-1})}{(x_1 + \cdots + x_n)^{\lambda}} dx_1 \ldots dx_{n-1}\right)^{\frac{1}{p_n-1}} dx_1 \ldots dx_n$$

$$\leq \prod_{j=1}^{n-1} B\left(\frac{\lambda}{2^j}, \frac{\lambda}{2^j}\right) \prod_{i=1}^{n-1} \left(\int_0^T x_i^{s_i} \left(1-\frac{1}{2}\left(\frac{x_i}{T}\right)^{\alpha}\right)^{n-1} f_i^{p_i}(x_i)\, dx_i\right)^{1/p_i}$$

$$\times \left(\int_0^T x_x^{s_n} x_n^{\frac{s_n}{1-p_n} p_n} \left(1-\frac{1}{2}\left(\frac{x_n}{T}\right)^{\alpha}\right)^{\frac{n-1}{1-p_n} p_n}\right.$$

$$\left. \times \left(1-\frac{1}{2}\left(\frac{x_n}{T}\right)^{\alpha}\right)^{n-1} \left(\int_0^T \cdots \int_0^T \frac{f_1(x_1) \ldots f_{n-1}(x_{n-1})}{(x_1 + \cdots + x_n)^{\lambda}} dx_1 \ldots dx_{n-1}\right)^{\frac{p_n}{1-p_n}} dx_n\right)^{1/p_n}$$

$$= \prod_{j=1}^{n-1} B\left(\frac{\lambda}{2^j}, \frac{\lambda}{2^j}\right) \prod_{i=1}^{n-1} \left(\int_0^T x_i^{s_i} \left(1-\frac{1}{2}\left(\frac{x_i}{T}\right)^{\alpha}\right)^{n-1} f_i^{p_i}(x_i)\, dx_i\right)^{1/p_i}$$

$$\times \left(\int_0^T \frac{x_n^{\frac{s_n}{1-p_n}}}{\left(1-\frac{1}{2}\left(\frac{x_n}{T}\right)^{\alpha}\right)^{\frac{1-n}{1-p_n}}} \left(\int_0^T \cdots \int_0^T \frac{f_1(x_1) \ldots f_{n-1}(x_{n-1})}{(x_1 + \cdots + x_n)^{\lambda}} dx_1 \ldots dx_{n-1}\right)^{\frac{p_n}{1-p_n}} dx_n\right)^{1/p_n}$$

which implies

$$\int_0^T \frac{x_n^{\frac{s_n}{1-p_n}}}{\left(1-\frac{1}{2}\left(\frac{x_n}{T}\right)^{\alpha}\right)^{\frac{1-n}{1-p_n}}}\left(\int_0^T \cdots \int_0^T \frac{f_1(x_1)\ldots f_{n-1}(x_{n-1})}{(x_1+\cdots+x_n)^{\lambda}}\,dx_1 \ldots dx_{n-1}\right)^{\frac{p_n}{1-p_n}} dx_n$$

$$\leq \prod_{j=1}^{n-1} B^{\frac{p_n}{p_n-1}}\left(\frac{\lambda}{2^j}, \frac{\lambda}{2^j}\right)\prod_{i=1}^{n-1}\left(\int_0^T x_i^{s_i}\left(1-\frac{1}{2}\left(\frac{x_i}{T}\right)^{\alpha}\right)^{n-1} f_i^{p_i}(x_i)\,dx_i\right)^{\frac{p_n}{(p_n-1)p_i}}.$$

Now suppose that (3.2) is satisfied, then

$$\int_0^T \cdots \int_0^T \frac{f_1(x_1)\ldots f_n(x_n)}{(x_1+\cdots+x_n)^{\lambda}}\,dx_1 \ldots dx_n$$

$$= \int_0^T x_n^{-\frac{s_n}{p_n}}\left(1-\frac{1}{2}\left(\frac{x_n}{T}\right)^{\alpha}\right)^{-\frac{n-1}{p_n}}\left(\int_0^T \cdots \int_0^T \frac{f_1(x_1)\ldots f_{n-1}(x_{n-1})}{(x_1+\cdots+x_n)^{\lambda}}\,dx_1 \ldots dx_{n-1}\right)$$

$$\times\, x_n^{\frac{s_n}{p_n}}\left(1-\frac{1}{2}\left(\frac{x_n}{T}\right)^{\alpha}\right)^{\frac{n-1}{p_n}} f_n(x_n)\,dx_n$$

$$\leq \left(\int_0^T x_n^{\frac{s_n}{1-p_n}}\left(1-\frac{1}{2}\left(\frac{x_n}{T}\right)^{\alpha}\right)^{\frac{n-1}{1-p_n}}\right.$$

$$\left.\times\left(\int_0^T \cdots \int_0^T \frac{f_1(x_1)\ldots f_{n-1}(x_{n-1})}{(x_1+\cdots+x_n)^{\lambda}}\,dx_1 \ldots dx_{n-1}\right)^{\frac{p_n}{p_n-1}} dx_n\right)^{\frac{p_n-1}{p_n}}$$

$$\times\left(\int_0^T x_n^{s_n}\left(1-\frac{1}{2}\left(\frac{x_n}{T}\right)^{\alpha}\right)^{n-1} f_n^{n}(x_n)\,dx_n\right)^{1/p_n}$$

$$\leq \prod_{j=1}^{n-1} B\left(\frac{\lambda}{2^j}, \frac{\lambda}{2^j}\right)\prod_{i=1}^{n}\left(\int_0^T x_i^{s_i}\left(1-\frac{1}{2}\left(\frac{x_i}{T}\right)^{\alpha}\right)^{n-1} f_i^{p_i}(x_i)\,dx_i\right)^{1/p_i}.$$

This completes the proof of Theorem.

4. Applications

Corollary 1. *On putting* $n = 2, T = \infty,$ *in Theorem, we obtain the following inequalities*

$$\int_0^\infty \int_0^\infty \frac{f_1(x_1) f_2(x_2)}{(x_1 + x_2)^\lambda} dx_1 dx_2$$

$$\leq B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)\left(\int_0^\infty t^{p_1 - \frac{\lambda}{2}\left(1 + \frac{p_1}{p_2}\right) - 1} f_1^{p_1}(t)\, dt\right)^{1/p_1}\left(\int_0^\infty t^{p_2 - \frac{\lambda}{2}\left(1 + \frac{p_2}{p_1}\right) - 1} f_2^{p_2}(t)\, dt\right)^{1/p_2}$$

$$(4.1)$$

and

$$\int_0^\infty x_2^{\frac{1}{p_2 - 1}\left(\frac{\lambda}{2}\left(1 + \frac{p_2}{p_1}\right) - p_2 + 1\right)}\left(\int_0^\infty \frac{f_1(x_1)}{(x_1 + x_2)^\lambda} dx_1\right)^{\frac{p_2}{p_2 - 1}} dx_2$$

$$\leq B^{\frac{p_2}{1 - p_2}}\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)\int_0^\infty t^{p_1 - \frac{\lambda}{2}\left(1 + \frac{p_2}{p_1}\right) - 1} f_1^{p_1}(t)\, dt. \tag{4.2}$$

The inequalities (4.1) *and* (4.2) *are equivalent.*

Corollary 2. *On putting* $n = 3,\ T = \infty,\ 1/p_1 + 1/p_2 + 1/p_3 = 1,\ p_i > 1,$ *in Theorem, we get the following*:

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{f_1(x_1) f_2(x_2) f_3(x_3)}{(x_1 + x_2 + x_3)^\lambda} dx_1 dx_2 dx_3$$

$$\leq B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) B\left(\frac{\lambda}{4}, \frac{\lambda}{4}\right) B\left(\frac{\lambda}{8}, \frac{\lambda}{8}\right)$$

$$\times \prod_{i=1}^{3}\left(\int_0^T x_i^{-\frac{\lambda}{8} - p_i\left(\sum_{j=1}^{3} \frac{\lambda}{2^{2+j}} - 1\right) - 1}\left(1 - \frac{1}{2}\left(\frac{x_i}{T}\right)^\alpha\right)^4 f_i^{p_i}(x)\, dx_i\right)^{1/p_i} \tag{4.3}$$

and

$$\int_0^\infty x_3^{\frac{1}{p_3-1}\left(\frac{\lambda}{8}+p_3\left(\sum_{j=1}^{3}\frac{\lambda}{2^j p_j}-1\right)+1\right)}\left(1-\frac{1}{2}\left(\frac{x_n}{T}\right)^\alpha\right)^{\frac{n-1}{1-p_n}}$$

$$\times\left(\int_0^\infty\int_0^\infty \frac{f_1(x_1)\,f_2(x_2)}{(x_1+x_2+x_3)^3}\,dx_1dx_2\right)^{\frac{p_3}{p_3-1}}dx_3$$

$$\leq \prod_{j=1}^{2} B^{\frac{p_3}{1-p_3}}\prod_{i=1}^{2}\left(\int_0 x_i^{-\frac{\lambda}{8}-p_i\left(\sum_{j=1}^{2}\frac{\lambda}{2^{3+j-i}p_j}-1\right)-1}\right.$$

$$\left.\times\left(1-\frac{1}{2}\left(\frac{x_i}{T}\right)^\alpha\right)^2 f_i^{\,p_i}(x_i)\,dx_i\right)^{\frac{p_3}{(p_3-1)\,p_i}}. \tag{4.4}$$

The inequalities (4.3) *and* (4.4) *are equivalent*.

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