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ESSENTIAL NORM AND COMPACTNESS OF COMBINATIONS OF WEIGHTED COMPOSITION OPERATORS

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Abstract

In this paper, we determine the essential norm of a weighted composition operator on spaces of vector valued continuous functions defined on a compact Hausdorff space. We also provide necessary and sufficient conditions for a finite sum of compact weighted composition operators to be itself a compact operator.

1. Introduction

We consider the Banach space $\mathcal{C}(X, E)$ of all continuous functions defined on a compact Hausdorff topological space X and with values in a Banach space E. This space is equipped with the usual norm. The space $\mathcal{B}(E)$ denotes all the bounded operators on E and $\mathcal{K}(E)$ denotes the subspace of compact operators. We consider continuous functions $u : X \to \mathcal{B}(E)$ and $\varphi : X \to X$. We denote by (uC_φ) the weighted composition operator acting on $\mathcal{C}(X, E)$ and given by

$$(uC_\varphi)(f)(x) = u(x) f(\varphi(x)).$$

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Weighted composition operators on $\mathcal{C}(X)$ and $\mathcal{C}(X, E)$ are an important class of operators since they arise naturally in characterizations of surjective isometries (see [3]) and in the study of bi-separating and disjointness preserving operators, see [1] and [8]. Kamowitz [9] characterized compact weighted composition operators on $\mathcal{C}(X)$. Jamison and Rajagopalan [8] extended this characterization for operators on $\mathcal{C}(X, E)$, see also [11]. In this paper, we consider the essential norm of a weighted composition operator on $\mathcal{C}(X, E)$. We denote by $\mathcal{K}(X, E)$, the subspace of $\mathcal{B}(\mathcal{C}(X, E))$, of all compact operators. We recall that the essential norm of an operator $T \in \mathcal{B}(\mathcal{C}(X, E))$, is the distance from that operator to $\mathcal{K}(X, E)$, (cf. [15]), i.e.,

$$\| T \|_e = \inf\{\| T - S \| : S \in \mathcal{K}(X, E)\}.$$

In the first section of this paper, we obtain a formula for the essential norm of weighted composition operators on $\mathcal{C}(X, E)$

$$\|uC_\varphi\|_e = \inf\{r > 0 : \varphi(\{x \in X : \| u(x)\| \geq r\}) \text{ is finite}\},$$

under some constrains on the underlying spaces. This result generalizes Theorem 1, in [15] to spaces of vector valued functions.

In the second section, we use Kamowitz's characterization of compact weighted composition operators on $\mathcal{C}(X)$ to answer the question: Does compactness of the sum of two weighted composition operators imply the compactness of each operator?

2. Essential Norm of Composition Operators on $\mathcal{C}(X, E)$

We consider a weighted composition operator (uC_φ) on $\mathcal{C}(X, E)$. Our main theorem computes the essential norm of (uC_φ). We first prove some preliminary results to be employed in the proof of the main theorem. We apply a strategy introduced in [15], however, the operator valued multiplier of the composition operator requires different arguments from those followed in the scalar case. We set $\rho = \inf\{r > 0 : \varphi(\{x \in X : \| u(x)\| \geq r\}) \text{ is finite}\}$.

Proposition 2.1. *If X is a compact Hausdorff topological space and* (uC_φ) *is a weighted composition operator on* $\mathcal{C}(X, E)$, *then*

$$\| uC_\varphi \|_e \geq \rho.$$

Proof. The inequality is clear if $\rho = 0$, then we assume $\rho > 0$. For every positive number ε, we have that $\varphi\{x : \| u(x) \| > \rho - \varepsilon\}$ is infinite. We select a sequence $\{x_n\}$ so that $\varphi(x_n) \neq \varphi(x_m)$ (for $n \neq m$), $\| u(x_n) \| > \rho - \varepsilon$, and $\varphi(x_n) \in \mathcal{O}_n$, with $\{\mathcal{O}_n\}$ a family of pairwise disjoint open subsets of X. This is possible since a perfect subset of a compact Hausdorff space must be uncountable, see [12]. For every n, there exists $f_n \in \mathcal{C}(X, E)$ so that $\| f_n \|_\infty = 1$, f_n while restricted to $X \setminus \mathcal{O}_n$ is identically zero, and $f_n(\varphi(x_n)) = \nu_n \in E$, with ν_n chosen so that $\| \nu_n \|_E = 1$ and $| \| u(x_n)(\nu_n) \|_E - \| u(x_n) \| | < \varepsilon$. We observe that $\{f_n\}$ is pointwise convergent to zero. Proposition 1.7.1 (p. 36) in [4] (also cf. in [6]) implies that $\{f_n\}$ is weakly convergent to zero. It is well known (cf. [5]) that compact operators on a Banach space are completely continuous and hence, every compact operator S will transform the sequence $\{f_n\}$ into a sequence with a norm convergent subsequence. We then select S, a compact operator so that $\|uC_\varphi - S\| \leq \|uC_\varphi\|_e + \varepsilon$. Therefore, we have

$$\|uC_\varphi\|_e \geq \|uC_\varphi - S\| - \varepsilon \geq \|(uC_\varphi - S) f_n\|_\infty - \varepsilon \geq \|(uC_\varphi)(f_n)\|_\infty - \|Sf_n\|_\infty - \varepsilon.$$

Since $\{\|Sf_n\|_\infty\}_n$ converges to zero, we choose n so large that $\|Sf_n\|_\infty < \varepsilon$, hence,

$$\|(uC_\varphi)(f_n)\|_\infty \geq \|u(x_n) f_n(\varphi(x_n))\|_E \geq \| u(x_n) \| - \varepsilon$$

and

$$\|uC_\varphi\|_e \geq \| u(x_n) \| - 3\varepsilon > \rho - 4\varepsilon.$$

We then conclude that $\|uC_\varphi\|_e \geq \rho$.

Theorem 2.1. *Let E be a Banach space and X be a compact Hausdorff topological space. If* (uC_φ) *is a weighted composition operator on* $\mathcal{C}(X, E)$, *so that* $u(x) \in \mathcal{K}(E)$, *for all* $x \in X$, *and*

1. φ *is finite to one, or*

2. *X is a separable topological space, or*

3. *E is finite dimensional, then*

$$\|uC_{\varphi}\|_e = \rho.$$

We first prove a lemma addressing the finite dimensional case. This proof encompasses all of the essential constructions needed for the infinite dimensional spaces.

Lemma 2.1. *If* (uC_{φ}) *is a weighted composition operator on* $\mathcal{C}(X, E)$, *with E a finite dimensional Banach space, then*

$$\|uC_{\varphi}\|_e \leq \rho.$$

Proof. We suppose that $\|uC_{\varphi}\|_e > \rho,$ we choose $\varepsilon > 0$ so that $\|uC_{\varphi}\|_e > \rho + \varepsilon.$ As in [15], we define the following closed and disjoint subsets of *X*; $F = \{x \in X : \| u(x) \| \leq \rho + \varepsilon\}$ and $G = \{x \in X : \| u(x) \| \geq \rho + 2\varepsilon\}$. There exists a continuous function $g : X \to [0, 1]$ so that $g|_F = 0$ and $g|_G = 1$. We set $v = g \cdot u$, i.e., for $x \in X$, $v(x) = g(x)u(x) \in \mathcal{B}(E)$. We first show that v is uniformly continuous. Given a net $\{x_\alpha\}$ convergent to x_*, we have that

$$\begin{aligned} &\| g(x_\alpha)u(x_\alpha) - g(x_*)u(x_*) \| \\ &\leq | g(x_\alpha) | \| u(x_\alpha) - u(x_*) \| + | g(x_\alpha) - g(x_*) | \| u(x_*) \| \to 0. \end{aligned}$$

We now show that (vC_{φ}) is a compact operator. We consider a bounded sequence $\{g_n\}$ in $\mathcal{C}(X, E)$ and we prove that there exists a norm convergent subsequence of $\{(vC_{\varphi})(g_n)\}_n$. We notice that $X \setminus F \subseteq \left\{x : \| u(x) \| \geq \rho + \frac{\varepsilon}{2}\right\}$, therefore, $\varphi(X \setminus F)$ is finite. We set $\varphi(X \setminus F) = \{y_1, y_2, ..., y_k\}$ and $F_i = \{x \in X \setminus F : \varphi(x) = y_i\}$. Each set F_i is a closed and open subset of $X \setminus F$. Since $X \setminus F$ is open in *X*, each F_i is also open in *X*. We select a subsequence of $\{g_n\}$, $\{g_{n_k}\}$ so that, for every $i \in \{1, ..., k\}$, $\{g_{n_k}(y_i)\}$ converges to z_i in *E*. This is possible, since *E* is finite dimensional and $\varphi(X \setminus F)$ is finite. Therefore, we define h as follows: $h(x) = 0$, for $x \in F$ and $h(x) = v(x)(z_i)$, for $x \in F_i$. We start by showing that $h \in \mathcal{C}(X, E)$. Let $\{x_\alpha\}$ be a net in X, converging to x_*. If we assume that $x_* \notin F$, then x_* is in some F_i. Since F_i is open, there exists α_0 so that for every $\beta \geq \alpha_0$, $x_\beta \in F_i$. The continuity of v implies that $h(x_\alpha)$ converges to $h(x_*)$. On the other hand, if we assume that $x_* \in F$, and there exists a sub-net $\{x'_\alpha\}$ of $\{x_\alpha\}$ in $X \setminus F$, then without of generality $\{x'_\alpha\} \in F_i$, for some i. Thus, we have

$$\| h(x'_\alpha) \|_E = \| v(x'_\alpha)(z_i) \|_E = | g(x'_\alpha) | \| u(x'_\alpha)(z_i) \|_E$$

$$\to | g(x_*) | \| u(x_*)(z_i) \|_E = 0 = h(x_*).$$

We now show that $\{vC_\varphi(g_{n_k})\}$ converges to h. In fact,

$$\begin{aligned} &\|(vC_\varphi)(g_{n_k}) - h\|_\infty \\ &= \sup_x \|(vC_\varphi)(g_{n_k})(x) - h(x)\|_E \\ &= \sup_x \|v(x)(g_{n_k}(\varphi(x))) - h(x)\|_E. \end{aligned}$$

Therefore,

$$\|(vC_\varphi(g_{n_k}) - h\|_\infty = \sup_x \begin{cases} 0, & \text{if } x \in F, \\ \|v(x)(g_{n_k}(y_i)) - v(x)(z_i)\|_E, & \text{if } x \in F_i \end{cases}$$

and $\|v(x)(g_{n_k}(y_i)) - v(x)(z_i)\|_E \leq \| v(x) \| \| g_{n_k}(y_i) - z_i \|_E \leq \| v \| \| g_{n_k}(y_i) - z_i \|_E$ $\to 0.$ This establishes the compactness of vC_φ. Therefore,

$$\begin{aligned} \|uC_\varphi\|_e &\leq \|uC_\varphi - vC_\varphi\| \leq \sup_{\| f \|_\infty \leq 1} \|uC_\varphi f - vC_\varphi f\|_\infty \\ &= \sup_{\| f \|_\infty \leq 1} \sup_x \| u(x)(f(\varphi(x))) - v(x)(f(\varphi(x))) \|_E \\ &\leq \sup_x \| u(x) - v(x) \| = \sup_x | 1 - g(x) | \| u(x) \| \\ &\leq \sup_{x \in X \setminus G} \| u(x) \| \leq \rho + 2\varepsilon. \end{aligned}$$

This implies that $\|uC_\varphi\|_e \leq \rho$, contradicting our initial assumption.

Remark 2.1. In the previous proof the finite dimensionality of E was crucial for the construction of function h, and to establish the compactness of the weighted composition operator vC_{φ}. We now show that similar but more elaborate constructions also work in broader settings, as those listed in Theorem 2.1.

Proof of Theorem 2.1. If E is finite dimensional (item 3), then the statement in the theorem follows from Lemma 2.1 and Proposition 2.1. We now prove the theorem for the assumptions stated in items 1 and 2.

1. If φ is finite to one, then each F_i $(i = 1, \ldots, k)$ is finite, $F_i = \{x_{i,1}, \ldots, x_{i,k_i}\}$. We recall that compact operators map bounded sequences into sequences with a convergent subsequence. We consider a bounded sequence $\{g_n\}$ and select a subsequence of $\{g_{n_k}\}$, so that for every $x_{i,j}$, $\{u(x_{i,j})(g_{n_k}(\varphi(x_{i,j})))\}_k$, converges to $\omega_{i,j} \in E$. This is possible, since $X \setminus F$, is finite and for every x, $u(x)$ is a compact operator on E. We define h as follows: $h(x) = 0$, if $x \in F$, and $h(x) = \omega_{i,j}$, if $x = x_{i,j} \in F_i$. As in the proof of Lemma 2.1, we conclude that h is continuous. Furthermore, h is the uniform limit of $\{u(x_{i,j})(g_{n_k})\}_k$. This establishes the compactness of uC_{φ}. The remainder of the proof now follows as in Lemma 2.1.

2. We now assume that X is separable, i.e., there exists a countable and dense subset of $XA = \{a_n\}$. Each F_i is open in X, then $A_i = A \bigcap F_i$ $(i = 1, \ldots, k)$ is countable and dense in F_i. We set $A_i = \{a_{i,j}\}_{j=1,\ldots}$ and $\{g_n\}$ a sequence of functions in $\mathcal{C}(X, E)$ with norm less or equal to one. Since $\bigcup_{i=1}^{k} A_i$ is countable, we can select a subsequence (g_{n_k}) so that, for every $a_{i,j}$, $u(a_{i,j})(g_{n_k}(y_i))$ converges to $\omega_{i,j} \in E$. We define h on a dense set as follows: $h(x) = 0$ for $x \in F$ and $h(x) = \omega_{i,j}$ for $x = a_{i,j}$. We now show that h has a unique continuous extension to X. We set $D = F \bigcup A_1 \cdots \bigcup A_k$, and consider $x_* \notin D$. Hence, x_* is in F_{i_0} for some $i_0 \in \{1, 2, \ldots, k\}$. We select a net $\{x_\alpha\}$ in D converging to x_*. Since F_{i_0} is open there exists α_0 so that, for every $\alpha > \alpha_0$, $x_\alpha \in F_{i_0}$. Therefore, $x_\alpha = a_{i_0 j_\alpha} \in A_{i_0}$. We recall that u is continuous and hence, $u(x_\alpha)$ converges in

norm to $u(x_*)$, a compact operator on E. Therefore, there exists a subsequence $u(x_*)(g_{n_{\tau(k)}}(y_{i_0}))$ that converges in E. We claim that $u(x_*)(g_{n_k}(\varphi(x)))$ converges. We assume that there exist two subsequences of $\{g_{n_k}(y_{i_0})\}$, say $g_{n_{\tau_1(k)}}(y_{i_0})$ and $g_{n_{\tau_2(k)}}(y_{i_0})$ so that $u(x_*)(g_{n_{\tau_1(k)}}(y_{i_0}))$ and $u(x_*)(g_{n_{\tau_2(k)}}(y_{i_0}))$ converge to ω_1 and ω_2, respectively. If $\| \omega_1 - \omega_2 \| = r > 0$, then for a given positive number $\varepsilon < r$ we choose α so that $\| u(x_*) - u(x_\alpha) \| < \frac{\varepsilon}{5}$. The sequence $\{u(x_\alpha)(g_{n_k}(y_{i_0}))\}_k$ is a Cauchy sequence, therefore, there exists k_0 so that for every k_1 and k_2 greater than k_0 we have

$$\|u(x_\alpha)(g_{n_{k_1}}(y_{i_0}) - u(x_\alpha)(g_{n_{k_2}}(y_{i_0})\|_E < \frac{\varepsilon}{5}.$$

We also choose k so that $\tau_1(k)$ and $\tau_2(k)$ are greater than k_0, and

$$\|u(x_*)(g_{n_{\tau_1(k)}}(y_{i_0})) - \omega_1\|_E < \frac{\varepsilon}{5} \quad \text{and} \quad \|u(x_*)(g_{n_{\tau_2(k)}}(y_{i_0})) - \omega_2\|_E < \frac{\varepsilon}{5}.$$

Therefore, we have

$$\begin{aligned} \| \omega_1 - \omega_2 \|_E \le & \|\omega_1 - u(x_*)(g_{n_{\tau_1(k)}}(y_{i_0}))\|_E \\ & + \|u(x_*)(g_{n_{\tau_1(k)}}(y_{i_0})) - u(x_\alpha)(g_{n_{\tau_1(k)}}(y_{i_0})\|_E \\ & + \|u(x_\alpha)(g_{n_{\tau_1(k)}}(y_{i_0})) - u(x_\alpha)(g_{n_{\tau_2(k)}}(y_{i_0}))\|_E \\ & + \|u(x_\alpha)(g_{n_{\tau_2(k)}}(y_{i_0})) - u(x_*)(g_{n_{\tau_2(k)}}(y_{i_0}))\|_E \\ & + \|u(x_*)(g_{n_{\tau_2(k)}}(y_{i_0})) - \omega_2\|_E < \varepsilon. \end{aligned}$$

This contradiction implies that $\{u(x_*)(g_{n_k}(y_{i_0}))\}$ is convergent due to the compactness of $u(x_*)$. We set $h(x_*) = \lim_k u(x_*)(g_{n_k}(y_{i_0}))$. Thus, the uniform continuity of u implies that h is in $\mathcal{C}(X, E)$. The statement in the theorem now follows from similar considerations to those used in the proof of Lemma 2.1.

3. Compactness of the Sum of Two Compact Weighted Composition Operators

In this section, we investigate the problem of whether the sum of two compact weighted composition operators is compact. We follow some techniques employed for the characterization of compact weighted composition operators on $C(X)$ used by Kamowitz [8]. We consider two weighted composition operators T_1 and T_2, of the form $u_k C_{\varphi_k} (k = 1, 2)$ with $u_k \in C(X)$ and $\varphi_k \in C(X, X)$. We start with a definition and some preliminary results.

Definition 3.1. The maps φ_1 and φ_2 are said to be *locally distinct* provided that for every $x \in X$, and for every open neighborhood of *x*, $\mathcal{O}_x$, there exists $z \in \mathcal{O}_x$ so that $\varphi_1(z) \neq \varphi_2(z)$.

Lemma 3.1. *Let φ_1 and φ_2 be locally distinct, and $T_1 + T_2$ be a compact operator. For every $x \in X$ such that $|u_1(x)| + |u_2(x)| > 0$, there exists an open set $\mathcal{O}_x$, containing x, so that φ_1 or φ_2 restricted to $\mathcal{O}_x$ is constant.*

Proof. We suppose the claim is not true. There exists x_0, so that given an open set containing x_0, $\mathcal{O}_0$, φ_1 and φ_2, restricted to $\mathcal{O}_0$, are not constant. In addition, we first assume that $\varphi_1(x_0) \neq \varphi_2(x_0)$. We select two disjoint open sets W_1 and W_2, containing $\varphi_1(x_0)$ and $\varphi_2(x_0)$, respectively. The continuity of φ_1 and φ_2 implies that $\mathcal{O}_{x_0} = \varphi_1^{-1}(W_1) \cap \varphi_2^{-1}(W_2)$, is open. For every open set *W* containing x_0, we select a point $z_W \in \mathcal{O}_{x_0} \cap W$ so that $\varphi_1(z_W) \neq \varphi_1(x_0)$, since φ_1 is not locally constant. We have constructed a net $\{z_W\}$ converging to x_0. Clearly, we have $\varphi_1(z_W) \neq \varphi_2(z_W)$ and $\varphi_2(z_W) \neq \varphi_1(x_0) \neq \varphi_2(x_0)$. We now select a continuous function f_W, defined on *X* and with values on the interval $[0, 1]$, satisfying the following conditions: $f_W(\varphi_1(x_0)) = 1$, $f_W(\varphi_1(z_W)) = f_W(\varphi_2(z_W)) = f_W(\varphi_2(x_0)) = 0$. Similarly, we can select a net y_W, converging to x_0, so that $\varphi_2(y_W) \neq \varphi_2(x_0)$ and a net of functions g_W satisfying the conditions:

$$g_W(\varphi_2(x_0)) = 1, \ \ g_W(\varphi_1(y_W)) = g_W(\varphi_2(y_W)) = g_W(\varphi_1(x_0)) = 0.$$

Associated with $T_1 + T_2$ we define $\tau : X \to C(X)^*$, given by $\tau(x)(h) = u_1(x)h(\varphi_1(x)) + u_2(x)h(\varphi_2(x))$. It is shown in Dunford-Schwartz ([7] Theorem VI, 7.1) that the compactness of $T_1 + T_2$ is equivalent to the uniform continuity of τ. Therefore, we have

$$|\tau(z_W)(f_W) - \tau(x_0)(f_W)| = |u_1(x_0)|$$

and

$$|\tau(y_W)(g_W) - \tau(x_0)(g_W)| = |u_2(x_0)|.$$

This implies that $u_1(x_0) = u_2(x_0) = 0$, contradicting our initial assumption. Now we assume that $\varphi_1(x_0) = \varphi_2(x_0)$. Since φ_1 and φ_2 are locally distinct, we select a net $\{x_\alpha\}$ converging to x_0 so that $\varphi_1(x_\alpha) \neq \varphi_2(x_\alpha)$. Clearly, at least one $\varphi_1(x_\alpha)$ or $\varphi_2(x_\alpha)$ must be different from $\varphi_1(x_0)$. Without loss of generality, and by selecting a subnet, we may assume that $\varphi_1(x_\alpha) \neq \varphi_1(x_0) = \varphi_2(x_0)$, for all α. If $\varphi_2(x_\alpha) \neq \varphi_1(x_0)$, for a subnet of values α, then we choose two nets of functions, f_α and g_α, so that $f_\alpha(\varphi_1(x_\alpha)) = f_\alpha(\varphi_1(x_0)) = f_\alpha(\varphi_2(x_0)) = 0$, $f_\alpha(\varphi_2(x_\alpha)) = 1$, $g_\alpha(\varphi_2(x_\alpha)) = g_\alpha(\varphi_1(x_0)) = g_\alpha(\varphi_2(x_0)) = 0$, and $g_\alpha(\varphi_1(x_\alpha)) = 1$. The uniform continuity of τ implies that $u_1(x_0) = u_2(x_0) = 0$, contradicting our initial assumption. If for every α, we have $\varphi_2(x_\alpha) = \varphi_1(x_0) = \varphi_2(x_0) \neq \varphi_1(x_\alpha)$. We now select f_α so that $f_\alpha(\varphi_1(x_\alpha)) = 1$ and $f_\alpha(\varphi_2(x_\alpha)) = f_\alpha(\varphi_1(x_0)) = f_\alpha(\varphi_2(x_0)) = 0$. This implies that $u_1(x_0) = 0$.

Given a net $\{x_\beta\}$ converging to x_0 so that $\varphi_2(x_\beta) \neq \varphi_2(x_0)$ and $f \in \mathcal{C}(X)$ we have

$$\begin{aligned} \|\tau(x_\beta) - \tau(x_0)\| &\geq |\tau(x_\beta)(f) - \tau(u_0)(f)| \\ &= |u_1(x_\beta)f(\varphi_1(x_\beta)) + u_2(x_\beta)f(\varphi_2(x_\beta)) - u_2(x_0)f(\varphi_2(x_0))|. \end{aligned} \tag{1}$$

We choose a net $\{x_\beta\}$ converging to x_0 so that $\varphi_2(x_\beta) \neq \varphi_2(x_0)$. We have the following possibilities: **1.** There exists a subnet, also denoted by $\{x_\beta\}$, so that $\varphi_1(x_\beta) = \varphi_2(x_\beta)$. **2.** There exists a subnet, also denoted by $\{x_\beta\}$, so that $\varphi_1(x_\beta) = \varphi_2(x_0)$. **3.** For all β, $\varphi_2(x_0) \neq \varphi_1(x_\beta) \neq \varphi_2(x_\beta)$.

In case **1**, for every β we choose f_β so that $f_\beta(\varphi_1(x_\beta)) = f_\beta(\varphi_2(x_\beta)) = 0$, and $f_\beta(\varphi_2(x_0)) = 1$. Equation (1) implies that $u_2(x_0) = 0$, contradicting our assumption. Similarly, for case **2**, we select f_β so that $f_\beta(\varphi_1(x_\beta)) = f_\beta(\varphi_2(x_0)) = 0$, and $f_\beta(\varphi_2(x_\beta)) = 1$. For case **3**, we set $f_\beta(\varphi_1(x_\beta)) = f_\beta(\varphi_2(x_\beta)) = 0$, and $f_\beta(\varphi_2(x_0)) = 1$. These three cases also lead to $u_2(x_0) = 0$ and prove the claim.

Lemma 3.2. *If $T_1 + T_2$ is compact, φ_1 and φ_2 are locally distinct and x_0 is so that $u_1(x_0) \neq 0$, then there exists an open set $\mathcal{O}_{x_0}$, containing x_0, such that φ_1 restricted to $\mathcal{O}_{x_0}$ is constant.*

Proof. We assume the claim is not true. For every open set $\mathcal{W}_{x_0}$, φ_1, restricted to $\mathcal{W}_{x_0}$, is not constant. Lemma 3.1 implies that φ_2 must be constant on some neighborhood of x_0, say $\mathcal{O}_{x_0}$. We select a net $\{x_\alpha\}$ converging to x_0 so that $\varphi_1(x_\alpha) \neq \varphi_1(x_0)$ and $\varphi_2(x_\alpha) = \varphi_2(x_0)$. If $\varphi_1(x_0) \neq \varphi_2(x_0)$, then for each α we select f_α so that $f_\alpha(\varphi_1(x_0)) = 1$ and $f_\alpha(\varphi_1(x_\alpha)) = f_\alpha(\varphi_2(x_0)) = f_\alpha(\varphi_2(x_\alpha)) = 0$. The compactness of $T_1 + T_2$ and thus, the uniform continuity of τ implies that $u_1(x_0) = 0$. Therefore, $\varphi_1(x_0) = \varphi_2(x_0) = \varphi_2(x_\alpha)$. Then we set f_α so that $f_\alpha(\varphi_1(x_\alpha)) = 1$, and $f_\alpha(\varphi_1(x_0)) = f_\alpha(\varphi_2(x_0)) = f_\alpha(\varphi_2(x_\alpha)) = 0$, which also implies that $u_1(x_0) = 0$.

Remark 3.1. Under the same conditions of Lemma 3.2 we also have that $u_2(x_0) \neq 0$ implies the existence of an open set $\mathcal{O}_{x_0}$, containing x_0, such that φ_2 restricted to $\mathcal{O}_{x_0}$ is constant.

Lemma 3.3. *If $T_1 + T_2$ is compact, φ_1 and φ_2 are locally distinct and C is a connected component of $\{x : u_1(x) \neq 0\}$, then there exists an open set W, containing C, so that φ_1 restricted to W is constant.*

Proof. We first observe that φ_1 restricted to C is constant. Lemma 3.2 asserts that for each $x_0 \in C$ there exists $\mathcal{O}_{x_0}$, so that φ_1 restricted to $\mathcal{O}_{x_0}$ is constant. We let $W = \bigcup_{x \in C} \mathcal{O}_x$.

Theorem 3.1. *If X is a compact Hausdorff space, T_1 and T_2 are weighted composition operators on $\mathcal{C}(X)$ defined by $T_i = u_i C_{\varphi_i}$, and φ_1 and φ_2 are locally distinct, then $T_1 + T_2$ is compact if and only if T_1 and T_2 are compact.*

Proof. If $T_1 + T_2$ is compact, then Lemma 3.3 asserts that given a connected component of $\{x : u_i(x) \neq 0\}$, there exists an open set W_i, containing that component so that φ_i restricted to W_i is constant. The statement follows from Kamowitz's characterization of compact operators on $\mathcal{C}(X)$, see Theorem A in [9]. The reverse implication is clear.

Example 3.1. We observe that if φ_1 and φ_2 are not locally distinct we may have $T_1 + T_2$ compact but neither T_1 nor T_2 compact. As for example $T_i = u_i C\varphi_i$ in $\mathcal{C}([0, 1], R)$ with $u_1(x) = -u_2(x) = 2x$, for $x \in \left[0, \frac{1}{2}\right]$, $u_1(x) = 2 - 2x$ and $u_2(x) = -1$ for $x \in \left[\frac{1}{2}, 1\right]$, $\varphi_1 = \varphi_2 = 1$, over the interval $[1/4, 1]$, and equal to $4x$ over the interval $[0, 1/4]$.

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