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INFINITESIMAL HOLOMORPHICALLY PROJECTIVE TRANSFORMATIONS ON TANGENT BUNDLES WITH RESPECT TO THE SYNECTIC METRIC TENSOR

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Abstract

In this paper, we find solutions of a system of partial differential equations which characterize infinitesimal holomorphically projective transformation on *TM* with Levi-Civita connection of the synectic metric and an adapted almost complex structure. Further, we investigate necessary conditions in order that *TM* admits a non-affine infinitesimal holomorphically projective transformation.

1. Introduction

Let *M* be an *n*-dimensional connected manifold and *TM* its tangent bundle. In the present paper, everything will be discussed in the C^{∞}-category. We denote by $\Im_s^r(M)$ the set of all tensor fields of type (r, s) on *M*, and by $\Im_s^r(TM)$ the corresponding set on *TM*.

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Let ∇ be an affine connection on M. Then a vector field V on M is called an *infinitesimal projective transformation* if there exists a 1-form Ω on M such that

$$(L_V \nabla)(X, Y) = \Omega(X)Y + \Omega(Y)X,$$

for any $X, Y \in \Im_0^1(M)$, where L_V is the Lie derivation with respect to V. In this case, Ω is called the *associated* 1-*form* of V. Especially, if $\Omega = 0$, then the vector field V is called an *infinitesimal affine transformation*.

Next, let (M, J) be an almost complex manifold with an affine connection ∇. Then a vector field V on M is called an *infinitesimal holomorphically projective transformation* if there exists a 1-form Ω on M such that

$$(L_V \nabla)(X, Y) = \Omega(X)Y + \Omega(Y)X - \Omega(JX)JY - \Omega(JY)JX,$$

for any $X, Y \in \Im_0^1(M)$. In this case, Ω is also called the *associated* 1-*form* of V, and if $\Omega = 0$, then V is an infinitesimal affine transformation, too.

The problems of determining infinitesimal holomorphically projective transformation on M and on TM have been considered by several authors. In 1957, Ishihara [5] has introduced the notion of infinitesimal holomorphically projective transformation, and Tachibana and Ishihara [7] investigated infinitesimal holomorphically transformation on Kählerian manifolds. In [1], Aminova and Kalinin studied the Lie algebras of infinitesimal H-projective (holomorphically-projective) transformation of $2n$-dimensional Kähler manifolds with constant holomorphic sectional curvature. In [2, 4], Hasegawa and Yamauchi investigated infinitesimal holomorphically projective transformation on TM with respect to the horizontal and complete lift connections. Recently, Tarakci et al. [9] have studied a similar problem on TM with respect to the metric $II + III$. Therefore, in this paper, we use the method of adapted frames to investigate the case of the Levi-Civita connection of the synectic metric on TM, introduced by Talantova and Shirokov [8], and prove the following two theorems:

Theorem 1. *Let (M, g) be a Riemannian manifold and TM its tangent bundle with the Levi-Civita connection of the synectic metric and an adapted almost complex structure. Then A vector field $\widetilde{V}$ is an infinitesimal holomorphically projective transformation with associated* 1-*form $\widetilde{\Omega}$ on TM if and only if there* *exist* $\varphi, \psi \in \Im_0^0(M)$, $B = (B^h)$, $D = (D^h) \in \Im_0^1(M)$, $A = (A_i^h)$, $C = (C_i^h) \in \Im_1^1(M)$ *satisfying*

1. $(\widetilde{V}^h, \widetilde{V}^{\bar{h}}) = (B^h + y^a A_a^h + 2\varphi y^h - y^a \Psi_a y^h \quad D^h + y^a C_a^h + 2\psi y^h + y^a \Phi_a y^h)$,

2. $(\widetilde{\Omega}_i, \widetilde{\Omega}_{\bar{i}}) = (\partial_i \psi, \partial_i \varphi) = (\Psi_i, \Phi_i)$,

3. $\nabla_i \Phi_j = 0, \quad \nabla_i \Psi_j = 0$,

4. $\nabla_i A_j^a = \Phi_j \delta_i^a - \Phi_i \delta_j^a$,

5. $\nabla_i C_j^a = \Psi_j \delta_i^a - \Psi_i \delta_j^a - B^h R_{hij}^a - H_{hi}^a A_j^h - 2\varphi H_{ji}^a$,

6. $L_B \Gamma_{ji}^a = \nabla_j \nabla_i B^a + B^h R_{hji}^a = \Psi_j \delta_i^a + \Psi_i \delta_j^a + H_{ji}^h A_h^a + 2\varphi H_{ji}^a$,

7. $L_D \nabla = \nabla_j \nabla_i D^a + D^h R_{hji}^a = -\Phi_j \delta_i^a - \Phi_i \delta_j^a - B^h \nabla_h H_{ji}^a + H_{ji}^h C_h^a$

$$+ 2\psi H_{ji}^a - H_{jh}^a \nabla_i B^h - H_{hi}^a \nabla_j B^h,$$

8. $B^h \nabla_h R_{bji}^a = R_{bji}^h C_h^a - C_b^h R_{hji}^a - R_{bjh}^a \nabla_i B^h - R_{bhi}^a \nabla_j B^h$

$$- A_b^h (\nabla_h H_{ji}^a - \nabla_j H_{hi}^a) - 2\varphi \nabla_b H_{ji}^a,$$

9. $A_b^h R_{hij}^a + 2\varphi R_{bij}^a = 0$,

10. $\Psi_l H_{ji}^a = 0, \ \Phi_l H_{ji}^a = 0$,

11. $\Phi_l R_{kji}^a = 0, \ \Psi_l R_{kji}^a = 0$,

where $\widetilde{V} = (\widetilde{V}^h \quad \widetilde{V}^{\bar{h}}) = \widetilde{V}^a E_a + \widetilde{V}^{\bar{a}} E_{\bar{a}}$ *and* $\widetilde{\Omega} = (\widetilde{\Omega}_i, \widetilde{\Omega}_{\bar{i}}) = \widetilde{\Omega}_a dx^a + \widetilde{\Omega}_{\bar{a}} \delta y^a$.

Theorem 2. *Let* (M, g) *be a Riemannian manifold and TM its tangent bundle with Levi-Civita connection of the synectic metric and an adapted almost complex structure. If TM admits non-affine infinitesimal holomorphically projective transformation, then the covariant derivative of symmetric tensor field* (a_{ji}) *of type* $(0, 2)$ *is zero and M is locally flat.*

2. Preliminaries

In this section, we shall summarize all the basic definitions and results on *TM* that are needed later. Most of them are well-known and details can be found in [11, 12]. Indices $a, b, c, i, j, h, \ldots$ have range in $\{1, \ldots, n\}$ while indices $\alpha, \beta, \lambda, \mu, \ldots$ have range in $\{1, \ldots, n; n+1, \ldots, 2n\}$. We put $\bar{i} = n + i$. Summation over repeated indices is always implied.

Coordinate systems on M are denoted by (U, x^h), where U is the coordinate neighborhood and x^h are the coordinate functions. Components in (U, x^h) of geometric objects on M will be referred to simply as components. We denote partial differentiation $\frac{\partial}{\partial x^h}$ by ∂_h.

Let (M, g) be a Riemannian manifold, ∇ be the Riemannian connection of g and Γ_{ji}^a be the coefficients of ∇, i.e., $\nabla_{\partial_j}\partial_i = \Gamma_{ji}^a \partial_a$ with respect to natural frame $\{\partial_h\}$. Then the curvature tensor R of ∇ has components R_{kji}^h. With the Riemannian connection ∇ given on M, we can introduce on each induced coordinate neighborhood $\pi^{-1}(U)$ of *TM* a frame field which is very useful in our computation. In each local chart $U(x^h)$ of M, we put

$$X_{(j)} = \frac{\partial}{\partial x^j} = \delta_j^h \frac{\partial}{\partial x^h} \in \Im_0^1(M).$$

Then $2n$ local vector fields ${}^H X_{(j)}$ and ${}^V X_{(j)}$ form a basis of the tangent space $T_P(TM)$ at each point $\widetilde{P} = \pi^{-1}(P)$ and their components are given respectively by

$${}^H X_{(j)} = \delta_j^h \partial_h - y^s \Gamma_{sj}^h \partial_{\bar{h}}, \tag{2.1}$$

$${}^V X_{(\bar{j})} = \delta_j^h \partial_{\bar{h}} \tag{2.2}$$

with respect to the natural frame $\left\{\frac{\partial}{\partial x^H}\right\} = \left\{\frac{\partial}{\partial x^h}, \frac{\partial}{\partial x^{\bar{h}}}\right\}$ on *TM*, where δ_i^j-Kronecker delta. These $2n$ vector fields are linear independent and generate, respectively, the horizontal distribution of ∇ and the vertical distribution of *TM*. We have called the set $\{{}^{H}X_{(j)}, {}^{V}X_{(\bar{j})}\}$ the frame adapted to the affine connection ∇ in $\pi^{-1}(U) \subset TM$. On putting $E_j = {}^{H}X_{(j)}$, $E_{\bar{j}} = {}^{V}X_{(\bar{j})}$, we write the adapted frame as $\{E_\beta\} = \{E_j, E_{\bar{j}}\}$.

By the straightforward calculation, we have the following:

Lemma 1 [11, p. 159]**.** *The Lie brackets of the adapted frame of TM satisfy the following identities*:

$$[E_j, E_i] = y^b R_{ijb}^a E_{\bar{a}}, \quad [E_j, E_{\bar{i}}] = \Gamma_{ji}^a E_{\bar{a}}, \quad [E_{\bar{j}}, E_{\bar{i}}] = 0. \tag{2.3}$$

Lemma 2 [3]**.** *Let* $\widetilde{V}$ *be a vector field on TM. Then*

$$\begin{cases} [\widetilde{V}, E_i] = -(E_i \widetilde{V}^a) E_a + (\widetilde{V}^c y^b R_{icb}^a - \widetilde{V}^{\bar{b}} \Gamma_{bi}^a - E_i \widetilde{V}^{\bar{a}}) E_{\bar{a}}, \\ [\widetilde{V}, E_{\bar{i}}] = -(E_{\bar{i}} \widetilde{V}^a) E_a + (\widetilde{V}^b \Gamma_{bi}^a - E_{\bar{i}} \widetilde{V}^{\bar{a}}) E_{\bar{a}}, \end{cases} \tag{2.4}$$

where $(\widetilde{V}^h \quad \widetilde{V}^{\bar{h}}) = \widetilde{V}^a E_a + \widetilde{V}^{\bar{a}} E_{\bar{a}}$.

Let g be a Riemannian metric with components g_{ji}. Then we see that

$$\widetilde{g} = a_{ji} dx^j dx^i + 2 g_{ji} dx^j \delta y^i \tag{2.5}$$

is non-singular and can be regarded as pseudo-Riemannian metric on *TM*, where $a = (a_{ji})$ is a symmetric tensor field of the type $(0, 2)$ on M and $\delta y^i = dy^i + \Gamma_{lk}^i dx^l y^k$, Γ_{lk}^i being Christoffel symbols formed with g. The metric (2.5) has components

$$\widetilde{g} = (\widetilde{g}_{\beta\gamma}) = \begin{pmatrix} a_{ji} & g_{ji} \\ g_{ji} & 0 \end{pmatrix}$$

with respect to the adapted frame on *TM*, that is, it coincides with $\widetilde{g} = {}^{C}g + {}^{V}a$, where ${}^{C}g$ and ${}^{V}a$ denote the complete and vertical lifts of g and a to *TM*, respectively. The synectic metric $\widetilde{g}$ was determined by Talantova and Shirokov [8] to study the differential geometry of tangent bundles of Riemannian manifolds. Their paper is related to the geometry of the space of n dual variables. The concept of a dual number is the analogue of a complex number: $x + jy$ with $j^2 = 0$. Since the set of dual numbers is represented geometrically by R^2, the set of n dual variables is represented by $R^{2n} = R^n \times R^n = TR^n$. They showed that the space TR^n with a certain metric represents a space of n dual variables with purely dual constant curvature. This special metric on TR^n is related projectively to the complete lift of the standard metric on R^n. Afterwards, Pavlov [6] studied the tangent bundles with a metric $\lambda^C g + {}^V a$ and also proved that the substitution of the metric ${}^C g \to \lambda^C g + {}^V a$ is a necessary and sufficient condition on preserving the "angles" between holomorphic planes.

Remark. In the case of $a = g$, the synectic metric $\widetilde{g}$ on *TM* coincides with the lift metric $I + II$ on *TM*, where $a = (a_{ji})$ is a symmetric tensor field of the type $(0, 2)$ on M and $g = (g_{ij})$ is a Riemannian metric on M. The metric $I + II$ is introduced by Yano and Ishihara [11, p. 147-155]. Also, they proved that the tangent bundle *TM* with the metric $I + II$ has vanishing scalar curvature.

We now consider local 1-forms ω^α defined by $\omega^\alpha = \widetilde{\mathcal{A}}^\alpha_B dx^B$ in $\pi^{-1}(U)$, where

$$\widetilde{\mathcal{A}}^\alpha_B = \begin{pmatrix} \widetilde{\mathcal{A}}^h_j & \widetilde{\mathcal{A}}^h_{\bar{j}} \\ \widetilde{\mathcal{A}}^{\bar{h}}_j & \widetilde{\mathcal{A}}^{\bar{h}}_{\bar{j}} \end{pmatrix} = \begin{pmatrix} \delta^h_j & 0 \\ y^s \Gamma^h_{sj} & \delta^h_j \end{pmatrix} \tag{2.6}$$

is the inverse matrix of the matrix

$$\mathcal{A}^A_\beta = \begin{pmatrix} \mathcal{A}^h_j & \mathcal{A}^h_{\bar{j}} \\ \mathcal{A}^{\bar{h}}_j & \mathcal{A}^{\bar{h}}_{\bar{j}} \end{pmatrix} = \begin{pmatrix} \delta^h_j & 0 \\ -y^s \Gamma^h_{sj} & \delta^h_j \end{pmatrix} \tag{2.7}$$

of frames changes $E_\beta = \mathcal{A}^A_\beta \partial_A$. These $2n$ 1-forms ω^α are linearly independent on *TM*. We call the set $\{\omega^\alpha\}$ the *dual adapted coframe*.

For various types of indices, we have

$$\begin{cases} E_j = \mathcal{A}^A_j \partial_A = \partial_j - y^s \Gamma^h_{sj} \partial_{\bar{h}}, \\ E_{\bar{j}} = \mathcal{A}^A_{\bar{j}} \partial_A = \partial_{\bar{j}}, \end{cases} \tag{2.8}$$

and

$$\begin{cases} \omega^j = \tilde{\mathcal{A}}^j_B dx^B = dx^j, \\ \omega^{\bar{j}} = \tilde{\mathcal{A}}^{\bar{j}}_B dx^B = \delta y^h, \end{cases} \tag{2.9}$$

where $\delta y^h = dy^h + y^b \Gamma^h_{ba} dx^a$.

Since the adapted frame field $\{E_\beta\}$ is non-holonomic, we put

$$[E_\alpha, E_\beta] = \Omega^\gamma_{\alpha\beta} E_\gamma$$

from which we have

$$\Omega^\alpha_{\gamma\beta} = (E_\gamma \mathcal{A}^A_\beta - E_\beta \mathcal{A}^A_\gamma) \tilde{\mathcal{A}}^\alpha_A.$$

Thus, according to equations (2.6), (2.7) and (2.8), the components of non-holonomic object $\Omega^\alpha_{\gamma\beta}$ are given by

$$\begin{cases} \Omega^{\bar{r}}_{l\bar{j}} = -\Omega^{\bar{r}}_{\bar{j}l} = \Gamma^r_{jl}, \\ \Omega^{\bar{r}}_{lj} = -\Omega^{\bar{r}}_{jl} = -R^r_{ljk}, \end{cases} \tag{2.10}$$

all the others being zero, with respect to the adapted frame.

If $\tilde{\nabla}$ denote the Levi-Civita connection of $\tilde{g}$ from $\tilde{T}(\tilde{X}, \tilde{Y}) = \tilde{\nabla}_{\tilde{X}} \tilde{Y} - \tilde{\nabla}_{\tilde{Y}} \tilde{X} - [\tilde{X}, \tilde{Y}] = 0$, $\forall \tilde{X}, \tilde{Y} \in \Im^1_0(TM)$, then we have

$$\tilde{\Gamma}^\alpha_{\gamma\beta} - \tilde{\Gamma}^\alpha_{\beta\gamma} = \Omega^\alpha_{\gamma\beta} \tag{2.11}$$

with respect to the adapted frame, where $\tilde{\Gamma}^\alpha_{\gamma\beta}$ are components of the Levi-Civita connection $\tilde{\nabla}$.

The equation $(\tilde{\nabla}_{\tilde{X}} \tilde{g})(\tilde{Y}, \tilde{Z}) = 0$, $\forall \tilde{X}, \tilde{Y}, \tilde{Z} \in \Im^l_0(TM)$ has form

$$E_\delta \tilde{g}_{\gamma\beta} - \tilde{\Gamma}^\varepsilon_{\delta\gamma} \tilde{g}_{\varepsilon\beta} - \tilde{\Gamma}^\varepsilon_{\delta\beta} \tilde{g}_{\gamma\varepsilon} = 0 \tag{2.12}$$

with respect to the adapted frame. Thus, we have from equations (2.11) and (2.12)

$$\tilde{\Gamma}^\alpha_{\beta\gamma} = \frac{1}{2} \tilde{g}^{\alpha\varepsilon} (E_\beta \tilde{g}_{\varepsilon\gamma} + E_\gamma \tilde{g}_{\beta\varepsilon} - E_\varepsilon \tilde{g}_{\beta\gamma}) + \frac{1}{2} (\Omega^\alpha_{\beta\gamma} + \Omega^\alpha_{\beta\gamma} + \Omega^\alpha_{\gamma\beta}), \tag{2.13}$$

where $\Omega^{\alpha}_{\gamma\beta} = \widetilde{g}^{\alpha\varepsilon}\widetilde{g}_{\delta\beta}\Omega^{\delta}_{\varepsilon\gamma}$, $\widetilde{g}^{\alpha\varepsilon}$ are the contravariant components of the metric $\widetilde{g}$ with respect to the adapted frame:

$$(\widetilde{g}^{\alpha\varepsilon}) = \begin{pmatrix} 0 & g^{hr} \\ g^{hr} & -a^{hr} \end{pmatrix}. \tag{2.14}$$

Taking account of equations (2.10), (2.13) and (2.14), for various types of indices, we find

$$\widetilde{\Gamma}^{h}_{ji} = \Gamma^{h}_{ji}, \quad \widetilde{\Gamma}^{\overline{h}}_{ji} = y^b R^{h}_{bji} + H^{h}_{ji}, \quad \widetilde{\Gamma}^{\overline{h}}_{\overline{ji}} = 0,$$

$$\widetilde{\Gamma}^{h}_{j\overline{i}} = 0, \quad \widetilde{\Gamma}^{\overline{h}}_{j\overline{i}} = \Gamma^{h}_{ji}, \quad \widetilde{\Gamma}^{h}_{j\overline{i}} = 0, \quad \widetilde{\Gamma}^{\overline{h}}_{\overline{j}i} = 0, \quad \widetilde{\Gamma}^{h}_{\overline{j}i} = 0 \tag{2.15}$$

with respect to the adapted frame, where Γ^{h}_{ji} denote the Levi-Civita connection components constructed with g on M with respect to the natural frame $\{\partial_i\}$ and H^{h}_{ji} is a tensor field of type $(1, 2)$ defined by $H^{h}_{ji} = \frac{1}{2} g^{hr}(\nabla_j a_{ri} + \nabla_i a_{jr} - \nabla_r a_{ji})$ [8] (see [10, p. 166]).

If $\widetilde{X}$ is a vector field on *TM* with frame components $\widetilde{X}^{\alpha}$, then it can be written that the frame components

$$\widetilde{\nabla}_{\lambda}\widetilde{X}_{\alpha} = E_{\lambda}(\widetilde{X}_{\alpha}) - \widetilde{\Gamma}^{\mu}_{\lambda\alpha}\widetilde{X}_{\mu}, \tag{2.16}$$

where $\widetilde{\Gamma}^{\alpha}_{\lambda\mu}$ being given by equation (2.15).

From equations (2.15) and (2.16), we have

Lemma 3. *Let* $\widetilde{\nabla}$ *be a Levi-Civita connection of the synectic metric on TM defined as follows*:

$$\begin{cases} \widetilde{\nabla}_{E_j} E_i = \Gamma^{a}_{ji} E_a + (y^b R^{a}_{bji} + H^{a}_{ji}) E_{\overline{a}}, \\ \widetilde{\nabla}_{E_j} E_{\overline{i}} = \Gamma^{a}_{ji} E_{\overline{a}}, \\ \widetilde{\nabla}_{E_{\overline{j}}} E_i = 0, \quad \widetilde{\nabla}_{E_{\overline{j}}} E_{\overline{i}} = 0. \end{cases} \tag{2.17}$$

Let us consider a tensor field $\widetilde{J}$ of type (1, 1) on *TM* by

$$\widetilde{J}\,{}^H X = {}^V X, \quad \widetilde{J}\,{}^V X = -{}^H X,$$

for any $X \in \Im_0^1(M)$, i.e., $\widetilde{J}E_i = E_{\bar{i}}$, $\widetilde{J}E_{\bar{i}} = -E_i$. Then we obtain $\widetilde{J}^2 = -I$. Therefore, $\widetilde{J}$ is an almost complex structure on *TM*. This almost complex structure is called the *adapted almost complex structure*. It is known that $\widetilde{J}$ is integrable if and only if *M* is locally flat [11, p. 118].

3. Proofs of the Theorems

Proof of Theorem 1. Here we prove only the necessary condition because it is easy to prove the sufficient condition. Let $\widetilde{V}$ be an infinitesimal holomorphically projective transformation with the associated 1-form $\widetilde{\Omega}$ on *TM*

$$(L_{\widetilde{V}}\widetilde{\nabla})(\widetilde{X}, \widetilde{Y}) = \widetilde{\Omega}(\widetilde{X})\widetilde{Y} + \widetilde{\Omega}(\widetilde{Y})\widetilde{X} - \widetilde{\Omega}(\widetilde{J}\widetilde{X})\widetilde{J}\widetilde{Y} - \widetilde{\Omega}(\widetilde{J}\widetilde{Y})\widetilde{J}\widetilde{X}, \tag{3.1}$$

for any $\widetilde{X}, \widetilde{Y} \in \Im_0^1(M)$.

From $(L_{\widetilde{V}}\widetilde{\nabla})(E_{\bar{j}}, E_{\bar{i}}) = \widetilde{\Omega}_{\bar{j}}E_{\bar{i}} + \widetilde{\Omega}_{\bar{i}}E_{\bar{j}} - \widetilde{\Omega}_jE_i - \widetilde{\Omega}_iE_j$, we obtain

$$\partial_{\bar{j}}\partial_{\bar{i}}\widetilde{V}^h = -\widetilde{\Omega}_j\delta_i^h - \widetilde{\Omega}_i\delta_j^h \tag{3.2}$$

and

$$\partial_{\bar{j}}\partial_{\bar{i}}\widetilde{V}^{\bar{h}} = \widetilde{\Omega}_{\bar{j}}\delta_i^h + \widetilde{\Omega}_{\bar{i}}\delta_j^h. \tag{3.3}$$

Contracting *i* and *h* in equation (3.2), we have

$$\widetilde{\Omega}_j = \partial_{\bar{j}}\widetilde{\psi}, \tag{3.4}$$

where $\widetilde{\psi} = -\frac{1}{n+1}\partial_{\bar{a}}\widetilde{V}^a$. Hence equation (3.2) is rewritten as follows:

$$\partial_{\bar{j}}\partial_{\bar{i}}\widetilde{V}^h = -(\partial_{\bar{j}}\widetilde{\psi})\delta_i^h - (\partial_{\bar{i}}\widetilde{\psi})\delta_j^h. \tag{3.5}$$

Differentiating equation (3.5) partially, we have

$$\begin{aligned}\partial_{\bar{k}}\partial_{\bar{j}}\partial_{\bar{i}}\widetilde{V}^h &= -(\partial_{\bar{k}}\partial_{\bar{j}}\widetilde{\psi})\delta_i^h - (\partial_{\bar{k}}\partial_{\bar{i}}\widetilde{\psi})\delta_j^h \\ &= -(\partial_{\bar{j}}\partial_{\bar{i}}\widetilde{\psi})\delta_k^h - (\partial_{\bar{i}}\partial_{\bar{k}}\widetilde{\psi})\delta_i^h \\ &= -(\partial_{\bar{i}}\partial_{\bar{k}}\widetilde{\psi})\delta_j^h - (\partial_{\bar{i}}\partial_{\bar{j}}\widetilde{\psi})\delta_k^h,\end{aligned} \tag{3.6}$$

from which we obtain

$$\partial_{\bar{k}}\partial_{\bar{j}}(\partial_{\bar{i}}\widetilde{V}^h + 2\widetilde{\psi}\delta_i^h) = 0. \tag{3.7}$$

Therefore, we can put

$$P_{ji}^h = \partial_{\bar{j}}(\partial_{\bar{i}}\widetilde{V}^h + 2\widetilde{\psi}\delta_i^h) \tag{3.8}$$

and

$$A_i^h + y^a P_{ai}^h = \partial_{\bar{i}}\widetilde{V}^h + 2\widetilde{\psi}\delta_i^h, \tag{3.9}$$

where A_i^h and P_{ji}^h are certain functions which depend only on the variables (x^h). The coordinate transformation rule implies that $A = (A_i^h) \in \Im_1^1(M)$ and $P = (P_{ji}^h) \in \Im_2^1(M)$.

Using equation (3.5), we have

$$P_{ji}^h + P_{ij}^h = 2\{\partial_{\bar{j}}\partial_{\bar{i}}\widetilde{V}^h + (\partial_{\bar{j}}\widetilde{\psi})\delta_i^h + (\partial_{\bar{i}}\widetilde{\psi})\delta_j^h\} = 0 \tag{3.10}$$

from which, using equation (3.8), we obtain

$$P_{ji}^h = \frac{1}{2}(P_{ji}^h - P_{ij}^h) = (\partial_{\bar{j}}\widetilde{\psi})\delta_i^h - (\partial_{\bar{i}}\widetilde{\psi})\delta_j^h. \tag{3.11}$$

On the other hand, using equation (3.9), we have

$$\widetilde{\psi} = -\varphi + y^a\Psi_a, \tag{3.12}$$

where $\varphi = -\frac{1}{n-1}A_a^a$ and $\Psi_i = \frac{1}{n-1}P_{ia}^a$, from which

$$\widetilde{\Omega}_i = \partial_{\bar{i}}\widetilde{\psi} = \Psi_i. \tag{3.13}$$

Using equations (3.9), (3.11) and (3.13), we obtain

$$\partial_{\bar{i}}\widetilde{V}^h = A_i^h + 2\varphi\delta_i^h - y^h\Psi_i - y^a\Psi_a\delta_i^h,$$

from which

$$\widetilde{V}^h = B^h + y^a A_a^h + 2\varphi y^h - y^a\Psi_a y^h, \tag{3.14}$$

where B^h are certain functions which depend only on (x^h). The coordinate transformation rule implies that $B = (B^h) \in \Im_0^1(M)$.

Similarly, from equation (3.3), there exist $\psi \in \Im_0^0(M)$, $\Phi = (\Phi_i) \in \Im_1^0(M_n)$, $D = (D^h) \in \Im_0^1(M)$ and $C = (C_i^h) \in \Im_1^1(M)$ satisfying

$$\widetilde{\varphi} = \psi + y^a \Phi_a, \tag{3.15}$$

$$\widetilde{\Omega}_{\bar{i}} = \partial_{\bar{i}} \widetilde{\varphi} = \Phi_i \tag{3.16}$$

and

$$\widetilde{V}^{\bar{h}} = D^h + y^a C_a^h + 2\psi y^h + y^a \Phi_a y^h, \tag{3.17}$$

where $\widetilde{\varphi} = \frac{1}{n+1} \partial_{\bar{a}} \widetilde{V}^{\bar{a}}$ and $\psi = -\frac{1}{n-1} C_a^a$.

Next, from equation (3.1), we have

$$(L_{\widetilde{V}} \widetilde{\nabla})(E_{\bar{j}}, E_i) = \Phi_j E_i + \Phi_i E_j + \Psi_j E_{\bar{i}} + \Psi_i E_{\bar{j}} \tag{3.18}$$

or

$$(L_{\widetilde{V}} \widetilde{\nabla})(E_j, E_{\bar{i}}) = \Phi_j E_i + \Phi_i E_j + \Psi_j E_{\bar{i}} + \Psi_i E_j.$$

From equation (3.18), we get

$$\begin{aligned}
&(\Phi_j \delta_i^a + \Phi_i \delta_j^a) E_a + (\Psi_j \delta_i^a + \Psi_i \delta_j^a) E_{\bar{a}} \\
&= \{(\nabla_i A_j^a + 2\delta_j^a \partial_i \varphi) - y^b (\delta_b^a \nabla_i \Psi_j + \delta_j^a \nabla_i \Psi_b)\} E_a \\
&\quad + \{(\nabla_i C_j^a + 2\delta_j^a \partial_i \psi + B^h R_{hij}^a + H_{hi}^a A_j^h + 2\varphi H_{ji}^a) + y^b (A_b^h R_{hij}^a + A_j^h R_{bih}^a + 4\varphi R_{bij}^a \\
&\quad + \delta_a^b \nabla_i \Phi_j + \delta_j^a \nabla_i \Phi_b - H_{bi}^a \Psi_j - H_{ji}^a \Psi_b) + y^b y^c (\Psi_j R_{icb}^a - 2\Psi_c R_{bij}^a)\} E_{\bar{a}}.
\end{aligned} \tag{3.19}$$

Comparing both hands of the above equation, we obtain

$$\begin{aligned}
&\Phi_j = \partial_j \varphi, \quad \nabla_i \Phi_j = 0, \\
&\Psi_j = \partial_j \psi, \quad \nabla_i \Psi_j = 0, \\
&\nabla_i A_j^a = \Phi_j \delta_i^a - \Phi_i \delta_j^a, \\
&\nabla_i C_j^a = \Psi_j \delta_i^a - \Psi_i \delta_j^a - B^h R_{hij}^a - H_{hi}^a A_j^h - 2\varphi H_{ji}^a, \\
&A_b^h R_{hij}^a = -2\varphi R_{bij}^a, \quad \Psi_l R_{kji}^a = 0, \quad \Psi_l H_{ji}^a = 0.
\end{aligned} \tag{3.20}$$

Lastly, from $(L_{\widetilde{V}}\widetilde{\nabla})(E_j, E_i) = \Psi_j E_i + \Psi_i E_j - \Phi_j E_{\bar{i}} - \Phi_i E_{\bar{j}},$ we obtain

$$(\Psi_j\delta_i^a + \Psi_i\delta_j^a)E_a - (\Phi_j\delta_i^a + \Phi_i\delta_j^a)E_{\bar{a}}$$

$$= \{L_B\Gamma_{ji}^a - H_{ji}^h A_h^a - 2\varphi H_{ji}^a\}E_a$$

$$+ \{(L_D\Gamma_{ji}^a + B^h\nabla_h H_{ji}^a - H_{ji}^h C_h^a - 2\psi H_{ji}^a + H_{jh}^a\nabla_i B^h + H_{hi}^a\nabla_j B^h)$$

$$+ y^b(C_b^h R_{hji}^a + B^h\nabla_h R_{bji}^a - R_{bji}^h C_h^a + R_{bjh}^a\nabla_i B^h + R_{bhi}^a\nabla_j B^h + A_b^h(\nabla_h H_{ji}^a$$

$$- \nabla_j H_{hi}^a) + 2\varphi\nabla_b H_{ji}^a - \Phi_i H_{jb}^a - 3\Phi_j H_{bi}^a) + y^b y^c(\Phi_c R_{bji}^a + \Phi_c R_{bij}^a - \Phi_j R_{cib}^a$$

$$- \Phi_i R_{cjb}^a + 2\varphi\nabla_c R_{bji}^a + 2\varphi\nabla_j R_{cib}^a + A_c^h\nabla_h R_{bji}^a + A_c^h\nabla_j R_{hib}^a)\}E_{\bar{a}}, \tag{3.21}$$

from which, we get the following important information:

$$L_B\Gamma_{ji}^a = \nabla_j\nabla_i B^a + R_{hji}^a B^h = \Psi_j\delta_i^a + \Psi_i\delta_j^a + H_{ji}^h A_h^a + 2\varphi H_{ji}^a, \tag{3.22}$$

$$L_D\nabla = \nabla_j\nabla_i D^a + D^h R_{hji}^a$$

$$= -\Phi_j\delta_i^a - \Phi_i\delta_j^a - B^h\nabla_h H_{ji}^a + H_{ji}^h C_h^a$$

$$+ 2\psi H_{ji}^a - H_{jh}^a\nabla_i B^h - H_{hi}^a\nabla_j B^h, \tag{3.23}$$

$$B^h\nabla_h R_{bji}^a = R_{bji}^h C_h^a - C_b^h R_{hji}^a - R_{bjh}^a\nabla_i B^h - R_{bhi}^a\nabla_j B^h$$

$$- A_b^h(\nabla_h H_{ji}^a - \nabla_j H_{hi}^a) - 2\varphi\nabla_b H_{ji}^a, \tag{3.24}$$

$$\Phi_l R_{bji}^a = 0, \quad \Phi_l H_{ji}^a = 0. \tag{3.25}$$

This completes the proof.

Using this Theorem 1, we at last come to the following:

Proof of Theorem 2. Let $\widetilde{V}$ be a non-affine infinitesimal holomorphically projective transformation on M. Using equation (3) in Theorem 1, we have $\nabla_i\|\Phi\|^2 = \nabla_i\|\Psi\|^2 = 0.$ Hence, $\|\Phi\|$ and $\|\Psi\|$ are constants on M. Suppose that M is not locally flat and the covariant derivative of symmetric tensor field (a_{ji}) of type $(0, 2)$ is non-zero, then $\Phi = \Psi = 0$ by virtue of equations (10) and (11) in Theorem 1, that is, $\widetilde{V}$ is an infinitesimal affine transformation. This is a contradiction. Therefore, M is locally flat and the covariant derivative of symmetric tensor field (a_{ji}) of type $(0, 2)$ is zero. In this case, *TM* is also locally flat [8, 10].

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