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ASYMPTOTICS FOR OPTION PRICING IN STOCHASTIC VOLATILITY ENVIRONMENT

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Abstract

In the classical Black-Scholes model, the risk asset is taken in a standard Brownian environment, where the risk is quantified by a constant volatility parameter. It has been proposed by many authors that the volatilities should be modeled by a stochastic process to obtain a more realistic model. For example, see Fouque et al. [7, 8], Cotton et al. [5], and Kallianpur and Karandikar [15]. Precedent is singular perturbation analysis for financial markets with stochastic volatility, which is a function of fast mean-reverting Ornstein-Uhlenbeck process driven by a standard Brownian motion.

Here we consider the European call option in a fractional Black-Scholes model in a financial market that has two instruments: a risk-less asset and a risky asset. A risky asset process X is governed by a standard Brownian motion W, whereas stochastic volatility is a function of fast mean-reverting Ornstein-Uhlenbeck process Y which is influenced by a fractional Brownian motion B_H with Hurst parameter $H \in (1/2, 1)$. We are interested in three parameters describing Y: (i) the effective volatility

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$\bar{\sigma}$ which is obtained by the average with respect to the long-run distribution of Y, (ii) the rate of mean reversion α which is characterized in terms of $1/\varepsilon$ with a small parameter ε, and (iii) the variance ν_H^2 of the long-run distribution of Y which is dependent on Hurst parameter H.

Our aim is to obtain asymptotics of the price of a European call option as $\varepsilon \to 0.$ We can derive the pricing partial differential equation in terms of ε, and obtain that the corrected Black-Scholes price is given by sum of the classical Black-Scholes price with constant volatility and the corrected term. Our theorem is an extension of the results in Fouque et al. [7] and Kallianpur and Karandikar [15] to a fractional Black-Scholes model with uncorrelated W and B_H.

1. Stochastic Volatility

The simplest financial derivative is a European call option, which can be priced by the classical Black-Scholes formula and the risk is quantified by a constant volatility parameter. A natural generalization is to model the volatility by a stochastic process. In reality, the volatility process cannot be directly observed. However through empirical studies of the stock price returns, one has observed that the estimated volatility fluctuates randomly around a mean level. The process is said to be *mean-reverting*.

We are motivated by Fouque et al. [7] and interested in more realistic market models, particularly ones in which volatility is uncertain. Andersson [1], Cotton et al. [5], Fouque et al. [7, 8], Jonsson and Sircar [14] and Kallianpur and Karandikar [15] write the canonical class of stochastic volatility models as a positive function of a simple ergodic Ito process, a mean-reverting Ornstein-Uhlenbeck process:

$$dX(t) = \mu X(t)dt + \sigma(t)X(t)dW(t), \tag{1.1}$$

$$\sigma(t) = f(Y(t)), \tag{1.2}$$

$$dY(t) = \alpha(m - Y(t))dt + \beta(\rho dW(t) + \sqrt{1-\rho^2}\,dB(t)). \tag{1.3}$$

Here $(W(t))$ and $(B(t))$ are independent standard Brownian motions with $-1 < \rho < 1$ the instantaneous correlation coefficient between asset price $(X(t))$ and volatility shocks. The factor $(Y(t))$ is called the *volatility-driving process* and f is some positive suitably regular function whose specification is unimportant for the principal asymptotic approximation.

Recall that α measures the characteristic speed of mean-reversion of $(Y(t))$ and $\nu^2 = \beta^2/(2\alpha)$ is the variance of the long-run distribution, measuring the typical size of the fluctuation of volatility-driving process.

Our main references are Fouque et al. [7] and Kallianpur and Karandikar [15]; the authors introduce the scaling

$$\alpha = 1/\varepsilon, \quad \beta = (\sqrt{2}\nu)/\sqrt{\varepsilon},$$

where $0 < \varepsilon \ll 1$ and $\nu = O(1)$ (fixed), to model fast mean-reversion (clustering) in market volatility, derive the pricing partial differential equation in terms of ε, and finally, obtain a corrected Black-Scholes price formula by singular perturbations.

Let $(W(t))$ be a standard Brownian motion and $(B_H(t))$ be a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. Then, our purpose is to obtain a corrected Black-Scholes price formula in a fractional Brownian environment, where $(W(t))$ and $(B_H(t))$ are uncorrelated, influencing risky asset process $(X(t))$ and volatility-driving process $(Y(t))$, respectively. Namely, our model corresponds to (1.1)-(1.3) with $\{W(t), B(t); \rho\}$ replaced by $\{W(t), B_H(t); \rho = 0\}$.

It remains to be proved that a corrected Black-Scholes price formula can be derived in the case where the sources of fluctuations $\{W(t), B(t); \rho\}$ in (1.1)-(1.3) are replaced by the following:

(i) $\{W(t), B_H(t); \rho \neq 0\}$,

(ii) $\{W_H(t), B(t); \rho = 0 \text{ or } \rho \neq 0\}$,

(iii) $\{W_H(t), B_H(t); \rho = 0 \text{ or } \rho \neq 0\}$,

with another fractional Brownian motion $W_H(t)$. However, if we appeal to the *fractional Wick-Ito calculus*, then the stochastic integral $\sigma(t)X(t)dW_H(t)$ in equation (1.1) is defined in the sense of the Wick product such that $\sigma(t) \lozenge X(t) \lozenge dW_H(t)$. Further, we shall need the explicit form of the *Malliavin* ϕ-*derivative*, such that $D_s^\phi \eta(t)$ of a random process $\eta(t)$, which appears in the fractional Ito formula. Furthermore, we shall need the explicit form of the fractional Girsanov formula for risk-neutral measure and the risk premium factor (market price). These provide us with more difficulty in calculations.

A simple model is the case where the risky asset $X(t)$ is under a fractional Brownian motion and the volatility $\sigma(t)$ is a rapidly varying deterministic function. Namely, let $W_H(t)$ be a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ and ε be a small parameter such that $0 < \varepsilon \ll 1$. Let us consider the ε-dependent market $(A(t), X^{\varepsilon}(t)),\ 0 \leq t \leq T,$ such that

$$dA(t) = rA(t)dt, \qquad A(0) = 1,$$

$$dX^{\varepsilon}(t) = \mu X^{\varepsilon}(t)dt + \sigma^{\varepsilon}(t)X^{\varepsilon}(t)dW_H(t), \qquad X^{\varepsilon}(0) = x \in \mathbb{R}_+,$$

$$\sigma^{\varepsilon}(t) = Y^{\varepsilon}(t).$$

Here $Y^{\varepsilon}(t)$ is the solution of the following ordinary differential equation:

$$\varepsilon dY^{\varepsilon}(t) = a[b - Y^{\varepsilon}(t)]dt, \qquad 0 \leq t \leq T,$$

$$Y^{\varepsilon}(0) = y_0 \in \mathbb{R}_+,$$

with constants $a > 0$ and $b > 0$. Then we have

$$Y^{\varepsilon}(t) = b + (y_0 - b)\exp\left(-\frac{a}{\varepsilon}t\right),$$

$$Y^{\varepsilon}(t) \to b \quad \text{as} \quad \varepsilon \to 0 \quad \text{for} \quad t > 0.$$

The constant b is also asymptotically stable value in the sense that $Y^{\varepsilon}(t) \to b$ as $t \to \infty$. Let $C^{\varepsilon}(t, X)$ (resp. $C^0(t, X)$) be the European call option price as obtained by the fractional Black-Scholes formula with the deterministic volatility function $\sigma^{\varepsilon}(t)$ (resp. the constant volatility b). Then, Narita [19] shows that as $\varepsilon \to 0$

$$C^{\varepsilon}(t, X) = C^0(t, X) + O(\varepsilon^H) \quad \text{for} \quad t > 0,$$

where the constants appearing in the big-oh notation $O(\cdot)$ are independent of ε.

2. Fractional Black-Scholes Model

Fractional Brownian motion (fBm) has been applied to describe the behavior to prices of assets and volatilities in stock markets. The long-range dependence self-similarity properties make this process a suitable model to describe these quantities; we shall give some details of fBm in Section 3.

Let us consider the fractional Black-Scholes (fractional BS) model with T the time of maturity, where the price of a risk-less asset (a bank account or bond) $A(t)$ at time $t \in [0, T]$ and the price of a risky asset (a stock) $X(t)$ at time $t \in [0, T]$ are given by the following equations:

$$dA(t) = rA(t)\,dt, \qquad A(0) = 1.$$

Here r represents the constant risk-less interest rate and hence $A(t) = e^{rt}$,

$$dX(t) = \mu X(t)\,dt + \sigma(t)X(t)\,dW(t), \tag{2.1}$$

$$\sigma(t) = f(Y(t)), \tag{2.2}$$

$$dY(t) = \alpha(m - Y(t))\,dt + \beta\,dB_H(t), \tag{2.3}$$

with constants $\mu(> r)$, $m > 0$, $\alpha > 0$ and $\beta > 0$.

Assumption 2.1. We assume the following:

(i) $(W(t))$ is a one-dimensional standard Brownian motion.

(ii) $(B_H(t))$ is a one-dimensional fractional Brownian motion (fBm) with Hurst parameter H. Throughout this paper, let H be fixed and it is assumed that $1/2 < H < 1$.

(iii) $(W(t))$ and $(B_H(t))$ are independent.

(iv) $f : \mathbb{R} \to \mathbb{R}$, is continuous.

The process $(Y(t))$ is a mean-reverting fractional Ornstein-Uhlenbeck (fractional OU) process. Examples of functions f are $f(y) = e^y$ (Scott model), $f(y) = |y|$ (Stein-Stein model).

For the future discussion, we shall introduce the terminology of mathematical finance in the following.

A *European call option* (a European option to buy) is a contract that gives the right (but not the obligation) to buy at time T (the maturity) a stock at price K (the strike or exercise price), which is fixed when the contract is signed.

If $X(T) \geq K$, the option enables its owner to buy the asset at price K and then sell it immediately at price $X(T)$; the *payoff*, that is, the difference $X(T) - K$ between the two prices is realized gain. If $X(T) < K$, the gain is zero. For example, we can express the payoff $F(\omega)$ at time T of a European call option in the fractional BS model by

$$F(\omega) = (X(T) - K)^{+} := \max\{X(T) - K,\ 0\}$$

as given in the classical Black-Scholes (classical BS) model.

More generally, we introduce the following concepts:

The process $u(t)$ (resp. $v(t)$) denotes the amount of the risk-less asset $A(t)$ (resp. the risky asset $X(t)$) that is held at time t. Then, $\theta(t) = (u(t), v(t))$ is called *portfolio*. Consequently, the *value*, or *wealth*, of the portfolio at time t is

$$V^{\theta}(t) := V^{\theta}(t,\ \omega) = u(t)A(t) + v(t)X(t).$$

A portfolio $\theta(t)$ is *self-financing* if

$$dV^{\theta}(t) = u(t)dA(t) + v(t)dX(t), \qquad 0 \leq t \leq T.$$

Namely, an investment strategy is said to be *self-financing* if no extra funds are added or withdrawn from the initial investment. The cost of acquiring more units of one security in the portfolio is completely financed by the sale of some units of another security within the same portfolio.

A portfolio $\theta(t)$ is said to provide an *arbitrage opportunity* if, with $V^{\theta}(0) \leq 0$, we have $V^{\theta}(T) \geq 0$ a.s. and

$$\text{Probab}\,(V^{\theta}(T) > 0) > 0.$$

Thus, an arbitrage opportunity means a self-financing trading strategy requiring no initial investment, having no probability of negative value at expiration, and yet having a possibility of a positive payoff. One of the fundamental concepts in the theory of option pricing is the absence of arbitrage opportunities, which is called *arbitrage-free*. We will allow ourselves to use *no-arbitrage* in place of arbitrage-free when convenient.

Let $F = F(\omega)$ be a *European contingent T-claim* (or just a *T-claim*); that is, F is a lower bounded random variable denoting the payoff. For F, if there exist an initial investment $z \in \mathbb{R}$ and $\theta(t)$ such that

$$F(\omega) = V^{\theta, z}(T, \omega) \qquad \text{(a.s.)},$$

then the financial market is said to be *complete*.

Suppose now that the T-claim $F(\omega)$ is *attainable* in the sense that there exists a portfolio $\theta(t) = (u(t), v(t))$ such that the value process equals F a.s. at the terminal time T, i.e.,

$$V(T, \omega) = F(\omega) \qquad \text{(a.s.)}.$$

If such a $\theta(t)$ exists, we call it a *replicating* or *hedging portfolio* for F.

Remark 2.2. Under Assumption 2.1, let us mention the following results on the model (2.1)-(2.3) that are proved in Hu [11]:

(i) The market is *incomplete* and martingale measures are not unique.

(ii) Set $\gamma(t) = (r - \mu)/\sigma(t)$ and

$$\frac{dQ}{dP} = \exp\left(\int_0^T \gamma(t) dW(t) - \frac{1}{2} | \gamma(t) |^2 dt\right),$$

where P is the probability measure in the underlying probability space. Then, Q is the *minimal martingale measure* associated with P.

(iii) The risk minimizing hedging price of a European call option is given by

$$\tilde{V} = e^{-rT} E_Q[(X(T) - K)^+],$$

where E_Q stands for the mathematical expectation with respect to Q.

If $\mathcal{G}_t$ denotes the filtration generated by fBm, it holds that

$$\begin{aligned} \tilde{V} &= e^{-rT} E_Q[E_Q((X(T) - K)^+ \mid \mathcal{G}_T)] \\ &= e^{-rT} E_Q[C_{BS}(X(0), \sigma)], \end{aligned}$$

where $\sigma = \sqrt{\int_0^T \sigma(s)^2 ds}$ and C_{BS} denotes the classical BS formula.

Remark 2.3. For the model (2.1)-(2.3), we shall introduce the scaling

$$\alpha = \frac{1}{\varepsilon}, \qquad \beta = \frac{\nu_H}{\sqrt{H\Gamma(2H)}}\left(\frac{1}{\varepsilon}\right)^H,$$

where $0 < \varepsilon \ll 1$ and $\nu_H = O(1)$ (fixed), to model fast mean-reversion in market volatility; ν_H is the parameter appearing in the long-run distribution $N(m, \nu_H^2)$ of $(Y(t))$ and $\Gamma(\cdot)$ is the Gamma function (Lemma 3.11 and Assumption 5.1). We shall derive the pricing partial differential equation in terms of ε (Lemma 4.1 and Lemma 5.2), and finally, obtain a corrected Black-Scholes price formula by singular perturbations (Theorem 6.1).

It is helpful to one in comparing the fractional BS formula with the classical BS formula. In the following, let $A(t)$ be the price at time t for the risk-less asset and $X(t)$ be the price at time t for the risky asset, characterized by a triple (r, μ, σ) of positive constants.

• **The fractional BS model** (see Hu and Øksendal [13] and Necula [20])

Let $W_H(t)$ be the fBm with Hurst parameter $H \in (1/2, 1)$.

(i) $dA(t) = rA(t)dt, \qquad A(0) = 1.$

(ii) $dX(t) = \mu X(t)dt + \sigma X(t)dW_H(t), \qquad X(0) = x > 0.$

Theorem 2.4 (Factional BS formula)**.** *The European call price at* $t \in [0, T]$ *with strike price K and maturity T is given by*

$$C(t, X(t)) = X(t)N(d_1) - Ke^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{\log\left(\frac{X(t)}{K}\right) + r(T-t) + \frac{\sigma^2}{2}(T^{2H} - t^{2H})}{\sigma\sqrt{T^{2H} - t^{2H}}},$$

$$d_2 = \frac{\log\left(\frac{X(t)}{K}\right) + r(T-t) - \frac{\sigma^2}{2}(T^{2H} - t^{2H})}{\sigma\sqrt{T^{2H} - t^{2H}}} \quad \left(= d_1 - \sigma\sqrt{T^{2H} - t^{2H}}\right),$$

and $N(\cdot)$ *is the cumulative probability of the standard normal distribution, i.e.*,

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left(-\frac{y^2}{2}\right) dy.$$

Theorem 2.5 (Factional BS equation)**.** *The price of a derivative on the stock price with bounded payoff* $h(X(T))$ *is given by* $C(t, X(t))$, *where* $C(t, X)$ *is the solution of the following partial differential equation*:

$$\frac{\partial C}{\partial t} + (H\sigma^2 t^{2H-1}) X^2 \frac{\partial^2 C}{\partial X^2} + rX \frac{\partial C}{\partial X} - rC = 0,$$

$$C(T, X) = h(X).$$

• **The classical BS model** (see Fouque et al. [7] and the references therein)

Let $W(t)$ be the standard Brownian motion.

(i) $A(t) = rA(t)dt, \qquad A(0) = 1.$

(ii) $dX(t) = \mu X(t)dt + \sigma X(t)dW(t), \qquad X(0) = x > 0.$

Theorem 2.6 (Classical BS formula)**.** *The European call price at* $t \in [0, T]$ *with strike price K and maturity T is given by*

$$C(t, X(t)) = X(t)N(d_1) - Ke^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{\log\left(\frac{X(t)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\log\left(\frac{X(t)}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \quad (= d_1 - \sigma\sqrt{T-t}),$$

and $N(\cdot)$ *is the cumulative probability of the standard normal distribution*.

Theorem 2.7 (Classical BS equation)**.** *The price of a derivative on the stock price with bounded payoff* $h(X(T))$ *is given by* $C(t, X(t))$, *where* $C(t, X)$ *is the* *solution of the following partial differential equation*:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 C}{\partial X^2} + rX\frac{\partial C}{\partial X} - rC = 0,$$

$$C(T, X) = h(X).$$

3. Fractional Ornstein-Uhlenbeck Process

We shall make preparations for our discussion and introduce the fractional stochastic calculus, omitting some details.

Definition 3.1. A one-dimensional *fractional Brownian motion* (fBm) with Hurst parameter $H \in (0, 1)$ is a Gaussian stochastic process with $B_H(0) = 0$, such that

$$E[B_H(t)] = 0, \qquad E[B_H(t)B_H(s)] = \frac{1}{2}\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\}$$

for all $s, t \in \mathbb{R}$. Here $E[\cdot]$ denotes the mathematical expectation with respect to the probability law μ_H for $B_H(\cdot)$.

Remark 3.2. FBm has the following properties:

(i) B_H is self-similar with self-similar index H, that is, for every $c > 0$, the process $\{B_H(ct); t \in \mathbb{R}\}$ is identical in distribution to $\{c^H B_H(t); t \in \mathbb{R}\}$.

(ii) B_H has stationary increments.

(iii) If $H = 1/2$, then B_H has independent increments.

(iv) If $H > 1/2$, then B_H has long-range dependence.

(v) If $H \neq 1/2$, then B_H is non-Markovian.

(vi) If $H \neq 1/2$, then B_H is not a semimartingale.

(vii) The covariance between future and past increments is positive if $H > 1/2$ and negative if $H < 1/2$.

If $H = 1/2$, then the fBm $B_{1/2}$ is one-dimensional standard Brownian motion.

Definition 3.3. Throughout this paper, let H be a fixed parameter and it is assumed that $1/2 < H < 1$. For given $H \in (1/2, 1)$, define $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ by

$$\phi(s, t) := H(2H - 1)| s - t |^{2H-2}, \qquad s, t \in \mathbb{R}.$$

Then we notice that

$$\int_0^t \int_0^t \phi(s, r) dsdr = t^{2H}.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable such that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(s) f(t) \phi(s, t) dsdt < \infty.$$

Then the stochastic integral with respect to the fBm $B_H(\cdot)$ is well defined. It follows from Gripenberg and Norros [9] and Nualart [21] that for any deterministic integrand $f, g \in L^2(\mathbb{R}, \mathbb{R}) \cap L^1(\mathbb{R}, \mathbb{R})$

$$E\left[\left(\int_{\mathbb{R}} f(s) dB_H(s) \int_{\mathbb{R}} g(t) dB_H(t)\right)\right] = \int_{\mathbb{R}} \int_{\mathbb{R}} f(s) g(t) \phi(s, t) dsdt.$$

The stochastic integral with respect to the fBm $B_H(\cdot)$ is extended to the case where the integrands are stochastic functions. We now follow from Duncan et al. [6], and Hu and Øksendal [13].

We will assume that Ω is the space $\mathcal{S}'(\mathbb{R})$ of tempered distributions on $\mathbb{R}$, which is the dual of the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing functions on $\mathbb{R}$. If $\omega \in \mathcal{S}'(\mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R})$, we let $\langle \omega, f \rangle = \omega(f)$ denote the action of ω applied to *f*. It can be extended to all $f : \mathbb{R} \to \mathbb{R}$ such that

$$| f |_\phi^2 := \int_{\mathbb{R}} \int_{\mathbb{R}} f(s) f(t) \phi(s, t) dsdt < \infty.$$

The space of all such functions *f* is denoted by $L_\phi^2(\mathbb{R})$.

Defining the inner product

$$\langle f, g \rangle_\phi := \int_{\mathbb{R}} \int_{\mathbb{R}} f(s) g(t) \phi(t, s) dsdt, \qquad f, g \in L_\phi^2(\mathbb{R}),$$

we notice that $(L_\phi^2(\mathbb{R}), \langle \cdot, \cdot \rangle_\phi)$ is a Hilbert space.

Remark 3.4 (Gaussian property)**.** Let $f, g \in L^2_\phi(\mathbb{R})$. Then, the stochastic integrals $\int_0^\infty f(s)dB_H(s)$ and $\int_0^\infty g(s)dB_H(s)$ are Gaussian random variables with mean 0 and variance $|f|^2_\phi, |g|^2_\phi$. In particular,

$$E\left(\int_0^\infty f(s)dB_H(s)\int_0^\infty g(t)dB_H(t)\right)$$

$$= \int_0^\infty \int_0^\infty f(s)g(t)\phi(s, t)dsdt = \langle f, g\rangle_\phi.$$

For $F : \Omega = \mathcal{S}'(\mathbb{R}) \to \mathbb{R}$, we denote by $D_t^\phi F$ the *Malliavin* ϕ*-derivative* of F at t; we shall cite a familiar notion in Definition 3.7 below.

According to Hu [12, Proposition 6.25], we define $\mathcal{L}_\phi^{\infty, \varepsilon}$ to be the set of processes $g(t, \omega) : \mathbb{R} \times \Omega \to \mathbb{R}$ such that $D_s^\phi g(s)$ exists for almost all $s \in \mathbb{R}$ and

$$|g|_{\mathcal{L}_\phi^{\infty, \varepsilon}} := E\left[\int_{\mathbb{R}}\int_{\mathbb{R}} g(s)g(t)\phi(s, t)dsdt + \int_{\mathbb{R}}\int_{\mathbb{R}} (D_s^\phi g(t))^2 dsdt\right] < \infty.$$

Then, by the method of the *Wick product* $\Diamond$ (the Wick calculus in white noise analysis), for $\sigma(t, \omega) \in \mathcal{L}_\phi^{\infty, \varepsilon}$, we can define $\int_{\mathbb{R}} \sigma(t, \omega)dB_H(t)$, that is, the *fractional Ito-integral* of the process $\sigma(t, \omega)$ with respect to $B_H(t)$.

The Wick product is used instead of the ordinary product in the Riemann sums, e.g.,

$$\int_a^b \sigma(t, \omega)dB_H(t) := \lim_{|\Delta| \to 0} \sum_{k=0}^{n-1} \sigma(t_k, \omega) \Diamond (B_H(t_{k+1}) - B_H(t_k)),$$

where

$$\Delta : a = t_0 < t_1 < \cdots < t_n = b, \qquad |\Delta| = \max_{0 \le k \le n-1}(t_{k+1} - t_k).$$

Remark 3.5 (Expectation of an integral of $f \in \mathcal{L}_\phi^{\infty, \varepsilon}$)**.** An importance of this fractional Ito-integral is

$$E\left[\int_{\mathbb{R}} \sigma(t, \omega)\, dB_H(t)\right] = 0,$$

where $E[\cdot]$ denotes the mathematical expectation with respect to the probability law μ_H for $B_H(\cdot)$; see Biagini et al. [2], Duncan et al. [6, pp. 588-592], Hu [12, pp. 56-58, 67-69], and Hu and Øksendal [13], and in particular the references therein.

Compare the fractional Ito-integral with the *fractional pathwise integral* defined by

$$\int_a^b \sigma(t, \omega)\,\delta B_H(t) := \lim_{|\Delta|\to 0} \sum_{k=0}^{n-1} \sigma(t_k, \omega)\,(B_H(t_{k+1}) - B_H(t_k)).$$

Then, these integrals do not have expectation zero. Moreover, the financial market based on $B_H(t)$ could have an arbitrage opportunity if we use the fractional pathwise integral. However, we will get no-arbitrage if we use the fractional Ito-integral; see Hu and Øksendal [13].

Remark 3.6 (Wick product)**.** The Wick product has the following properties; see Holden et al. [10, Section 2.4 and Chapter 3]:

(i) In Wick product $F \lozenge G$, commutative law, associative law and distributive law hold.

(ii) If at least one of F and G is deterministic, e.g., $F = a_0 \in \mathbb{R}$, then the Wick product coincides with the ordinary product in the deterministic case, that is,

$$F \lozenge G = F \cdot G, \quad \text{in particular,} \quad \text{if } F = 0, \text{ then } F \lozenge G = 0.$$

(iii) When applied to ordinary stochastic differential equations, derivative product rule holds as in the case of ordinary calculus:

$$(U \lozenge V)' = U' \lozenge V + U \lozenge V'.$$

(iv) Wick product is easier to handle with use such that if Y is in Hida space of stochastic distribution, then

$$\int_{\mathbb{R}} Y(t)\, dB_H(t) = \int_{\mathbb{R}} Y(t) \lozenge \mathcal{N}_H(t)\, dt,$$

where $\mathcal{N}_H(t)$ is the fractional white noise, that is, $\mathcal{N}_H(t) = dB_H(t)/dt$; see Hu and Øksendal [13, Definition 3.11] and Holden et al. [10, Section 2.5].

We shall need the stochastic gradient according to Biagini et al. [3, p. 281], Duncan et al. [6], Hu [12, p. 51, p. 99], Hu and Øksendal [13, Definition 4.3] and Nualart [21, Chapter 1].

Definition 3.7 (The Malliavin ϕ-derivative)**.** Let Φ be given by

$$(\Phi g)(t) := \int_0^\infty \phi(t, u) g(u) du, \qquad g \in L_\phi^2(\mathbb{R}).$$

Then the ϕ-*derivative* of a random variable $F \in L^p(\Omega)$ $(p \geq 1)$ in the direction of $\Phi g \in L_\phi^2$, where $g \in L_\phi^2(\mathbb{R})$, is defined as

$$D_{\Phi g} F(\omega) := \lim_{\delta \to 0} \frac{1}{\delta} \left\{ F\left(\omega + \delta \int_0^\cdot (\Phi g)(u) du \right) - F(\omega) \right\},$$

if the limit exists in $L^p(\Omega)$. Further, if there exists a process $(D_s^\phi F;\, s \geq 0)$ such that $D_{\Phi g} F = \int_0^\infty D_s^\phi F g(s) ds$ (a.s.) for all $g \in L_\phi^2$, then F is said to be ϕ-*differentiable*; $D_s^\phi F$ is an analogue of the Malliavin ϕ-derivative of F at s.

Without rigor, we note as follows: Let $F : \Omega = \mathbb{S}'(\mathbb{R}) \to \mathbb{R}$. Then

$$D_s^\phi F = \int_{\mathbb{R}} \phi(s, t) D_t F dt, \qquad D_t F(\omega) = \frac{dF}{d\omega}(t, \omega).$$

Here $D_t F(\omega)$ is the *stochastic gradient* (or *Hida*/*Malliavin derivative*) of F at t. Note that – in spite of the notation – $D_t F$ is not a derivative with respect to t but a (kind of) derivative with respect to $\omega \in \Omega$, such that

$$D_t \left(\int_{\mathbb{R}} f(s) dB_H(s) \right) = f(t) \quad \text{for} \quad \text{a.e. } t \quad \text{if} \quad f \in L_\phi^2(\mathbb{R}).$$

Remark 3.8 (Rules for differentiation)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be smooth and $F : \Omega \to \mathbb{R}$ be ϕ-differentiable. Then $f(F)$ is also ϕ-differentiable, satisfying

$$D_s^\phi f(F) = f'(F) D_s^\phi F.$$

Further, the following equations hold: If $f \in L_\phi^2(\mathbb{R})$, then

$$D_s^\phi\left(\int_0^\infty f(u)dB_H(u)\right) = \int_0^\infty \phi(u, s) f(u)du = (\Phi f)(s),$$

where $f, g \in L_\phi^2(\mathbb{R})$. Let $f \in L_\phi^2(\mathbb{R})$. Let $T > 0$ be arbitrary and fixed. Then

$$D_s^\phi\left(\int_0^t f(u)dB_H(u)\right) = D_s^\phi\left(\int_0^\infty \phi(u, s)\chi_{[0, t]}(u) f(u)du\right)$$

$$= \int_0^t \phi(u, s) f(u)du \qquad \text{for any} \qquad s, t \in [0, T].$$

In particular,

$$D_s^\phi(B_H(s)) = \int_0^s \phi(u, s)du = Hs^{2H-1}.$$

For further details concerning stochastic integrals with respect to fBm, we are referred to Biagini et al. [3], Hu [12], Mishura [16] and Nualart [21], and in particular the references therein. Moreover, we shall use the following theorem:

Theorem 3.9 (Fractional Ito formula)**.** *Consider the fractional SDE*:

$$dX(t) = \mu(t, \omega)dt + \sigma(t, \omega)dB_H(t), \qquad \mu, \sigma \in \mathcal{L}_\phi^{\infty, \varepsilon},$$

where $\mathcal{L}_\phi^{\infty, \varepsilon}$ *is the set of processes as given after Remark* 3.4, *and the stochastic integral means the fractional Ito integral*. *If* $f \in C^{1, 2}(\mathbb{R}_+ \times \mathbb{R})$, *then we have*

$$f(t, X(t)) = f(0, X(0)) + \int_0^t \frac{\partial f}{\partial s}(s, X(s))ds$$

$$+ \int_0^t \frac{\partial f}{\partial X}(s, X(s))\mu(s)ds + \int_0^t \frac{\partial f}{\partial X}(s, X(s))\sigma(s)dB_H(s)$$

$$+ \int_0^t \frac{\partial^2 f}{\partial X^2}(s, X(s))\sigma(s)D_s^\phi X(s)ds.$$

Here $D_s^\phi X(s)$ *is the Malliavin* ϕ-*derivative of* $F := X(s)$ *at* s *in the sense of Definition* 3.7.

Under preparations above, we consider the process of equation (2.3) with the initial state $Y(0) = y_0 \in \mathbb{R}$ and the time interval $[0, \infty)$, that is,

$$dY(t) = \alpha(m - Y(t))dt + \beta dB_H(t), \qquad t \geq 0,$$

$$Y(0) = y_0 \in \mathbb{R}, \tag{3.1}$$

with constants $m > 0$, $\alpha > 0$ and $\beta > 0$. This is a linear fractional stochastic differential equation. It follows from Biagini et al. [2, 3] and Narita [17, 18] that the solution cannot explode and hence the solution is pathwise unique. For the details of stochastic differential equations in fractional Brownian environment, see Holden et al. [10], Mishura [16] and Nualart [21].

We shall need the following estimates on the integrals:

Lemma 3.10. *Let* $\phi(s, r)$ *be the function as given in Definition* 3.3. *Define* $A(x)$ *and* $B(x)$ *by*

$$A(x) = \int_0^x e^{\xi}\xi^{2H-1}d\xi, \qquad B(x) = \int_0^x e^{-\xi}\xi^{2H-1}d\xi, \qquad \left(\frac{1}{2} < H < 1\right).$$

Then, for any constant $\alpha > 0$, *the following hold*:

(i)
$$\int_0^t\int_0^t \exp(\alpha(s + r))\phi(s, r)dsdr$$

$$= H\left(\frac{1}{\alpha}\right)^{2H}\{A(\alpha t) + \exp(2\alpha t)B(\alpha t)\} \quad \textit{for} \quad t \geq 0. \tag{3.2}$$

(ii)
$$\int_0^t\int_0^t \exp(-\alpha(s + r))\phi(s, r)dsdr$$

$$= H\left(\frac{1}{\alpha}\right)^{2H}\{B(\alpha t) + \exp(-2\alpha t)A(\alpha t)\} \quad \textit{for} \quad t \geq 0. \tag{3.3}$$

(iii)
$$\int_0^s \exp(\alpha r)\phi(s, r)dr$$

$$= H \exp(\alpha s)\left\{\exp(-\alpha s)s^{2H-1} + \left(\frac{1}{\alpha}\right)^{2H-1}B(\alpha s)\right\}. \tag{3.4}$$

Proof. We shall show equation (3.3). Put

$$\begin{cases} s + r = u, \\ s - r = v, \end{cases}$$

and hence

$$\begin{cases} s = (u + v)/2, \\ r = (u - v)/2. \end{cases}$$

Consider the domain $D = \{(s, r) : 0 \le s \le t, 0 \le r \le t\}$. Then, D is transformed to the following domain D':

$$D' = \{(u, v) : 0 \le u + v \le 2t, \quad 0 \le u - v \le 2t\} = D_1' \cup D_2',$$

$$D_1' = \{(u, v) : 0 \le u \le t, \quad -u \le v \le u\},$$

$$D_2' = \{(u, v) : t \le u \le 2t, \quad u - 2t \le v \le -u + 2t\}.$$

The Jacobian $J = J(u, v)$ of the coordinate transformation above is given by

$$J = \begin{vmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial r}{\partial u} & \frac{\partial r}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}.$$

Thus we have

$$\begin{aligned} I(t) &:= \int\int_D \exp(-\alpha(s + r))H(2H - 1)| s - r |^{2H-2} dsdr \\ &= \int\int_{D'} \exp(-\alpha u)H(2H - 1)| v |^{2H-2} | J | dsdr \\ &= I_1(t) + I_2(t), \end{aligned}$$

$$I_1(t) := \frac{1}{2}\int\int_{D_1'} \exp(-\alpha u)H(2H - 1)| v |^{2H-2} dudv,$$

$$I_2(t) := \frac{1}{2}\int\int_{D_2'} \exp(-\alpha u)H(2H - 1)| v |^{2H-2} dudv.$$

Step 1.

$$I_1(t) = \frac{1}{2}\int_0^t \exp(-\alpha u)\,du\int_{-u}^{u} H(2H-1)|\,v\,|^{2H-2}dv$$

$$= \int_0^t \exp(-\alpha u)\,du\int_0^{u} H(2H-1)v^{2H-2}dv$$

$$= H\int_0^t \exp(-\alpha u)u^{2H-1}du \qquad \left(\frac{1}{2} < H < 1\right)$$

$$= H\left(\frac{1}{\alpha}\right)^{2H} B(\alpha t), \qquad B(x) = \int_0^x e^{-\xi}\xi^{2H-1}d\xi.$$

Step 2.

$$I_2(t) = \frac{1}{2}\int_t^{2t} \exp(-\alpha u)\,du\int_{-(2t-u)}^{2t-u} H(2H-1)|\,v\,|^{2H-2}dv$$

$$= \int_t^{2t} \exp(-\alpha u)\,du\int_0^{2t-u} H(2H-1)v^{2H-2}dv$$

$$= H\int_t^{2t} \exp(-\alpha u)(2t-u)^{2H-1}du \qquad \left(\frac{1}{2} < H < 1\right)$$

$$= H\exp(-2\alpha t)\int_0^t \exp(\alpha z)\,z^{2H-1}dz \qquad (2t-u=z)$$

$$= H\left(\frac{1}{\alpha}\right)^{2H} \exp(-2\alpha t)\,A(\alpha t), \qquad A(x) = \int_0^x e^{\xi}\xi^{2H-1}d\xi.$$

By Steps 1-2, we obtain equation (3.3). The same argument as taken in the preceding leads us to equation (3.2). Finally, we shall show equation (3.4). Integration by parts implies that

$$\int_0^s \exp(\alpha r)\phi(s,\, r)\,dr$$

$$= H(2H-1)\int_0^s |\,s-r\,|^{2H-2}\exp(\alpha r)\,dr$$

$$= H(2H-1)\int_0^s (s-r)^{2H-2}\exp(\alpha r)\,dr$$

$$= H(2H-1)\int_0^s \exp(\alpha r)\left\{-\frac{1}{2H-1}(s-r)^{2H-1}\right\}' dr \qquad \left(\frac{1}{2} < H < 1\right)$$

$$= H\left\{s^{2H-1} + \alpha\int_0^s \exp(\alpha r)(s-r)^{2H-1}dr\right\}$$

$$= H\left\{s^{2H-1} + \alpha\exp(\alpha s)\int_0^s \exp(-\alpha z)z^{2H-1}dz\right\} \qquad (s-r=z)$$

$$= H\left\{s^{2H-1} + \alpha\exp(\alpha s)\left(\frac{1}{\alpha}\right)^{2H}\int_0^{\alpha s}\exp(-\xi)\xi^{2H-1}d\xi\right\} \qquad (\alpha z = \xi)$$

$$= H\left\{s^{2H-1} + \left(\frac{1}{\alpha}\right)^{2H-1}\exp(\alpha s)B(\alpha s)\right\}, \qquad B(x) = \int_0^x e^{-\xi}\xi^{2H-1}d\xi,$$

which yields equation (3.4). This completes the proof of Lemma 3.10. □

Lemma 3.11. *Let* $Y(t)$ *be the solution of* (3.1). *Then*

$$Y(t) = m + e^{-\alpha t}(y_0 - m) + \beta e^{-\alpha t}\int_0^t e^{\alpha s}dB_H(s). \tag{3.5}$$

Hence $Y(t)$ *is a Gaussian stochastic process and has the long-run distribution which is the normal distribution* $N(m, \nu_H^2)$ *with mean m and variance* ν_H^2, *such that*

$$\nu_H^2 = \beta^2 H\left(\frac{1}{\alpha}\right)^{2H}\Gamma(2H), \tag{3.6}$$

where $\Gamma(\cdot)$ *is the Gamma function;* $\Gamma(x) = \int_0^\infty e^{-\xi}\xi^{x-1}d\xi.$

Proof. Remark 3.6 implies that

$$\frac{dY(t)}{dt} = \alpha(m - Y(t)) + \beta\mathcal{N}_H(t), \qquad Y(0) = y_0$$

with the fractional white noise $\mathcal{N}_H(t) = dB_H(t)/dt$. This a linear differential equation of the first order, and hence

$$Y(t) = e^{-\alpha t}\left[\int_0^t e^{\alpha s}\{(\alpha m) + \beta \mathcal{N}_H(s)\}ds + y_0\right]$$

$$= e^{-\alpha t}\left[\int_0^t e^{\alpha s}(\alpha m)ds + \int_0^t e^{\alpha s}\beta \mathcal{N}_H(s)ds + y_0\right]$$

$$= e^{-\alpha t}m[e^{\alpha t} - 1] + \beta e^{-\alpha t}\int_0^t e^{\alpha s}dB_H(s) + e^{-\alpha t}y_0,$$

which yields equation (3.5). Recall (ii) of Remark 3.6 and equation (3.2) of Lemma 3.10. Then we notice that the fractional Ito integral $\int_0^t f(s)dB_H(s)$ with deterministic integrand $f(s) = e^{\alpha s}$ is well-defined; see the explanation after Definition 3.3 and Gripenberg and Norros [9]. Take the mathematical expectation on equation (3.5). Then, by Remark 3.5, we have

$$\hat{M}(t) := E[Y(t)] = m + (y_0 - m)e^{-\alpha t}.$$

Moreover, by Remark 3.4 and equation (3.2) of Lemma 3.10, we get

$$\hat{V}(t) := V[Y(t)] = \beta^2 e^{-2\alpha t}E\left[\left(\int_0^t e^{\alpha s}dB_H(s)\right)^2\right]$$

$$= \beta^2 e^{-2\alpha t}\int_0^t\int_0^t e^{\alpha(s+r)}\phi(s, r)dsdr$$

$$= \beta^2 e^{-2\alpha t}H\left(\frac{1}{\alpha}\right)^{2H}\{A(\alpha t) + e^{2\alpha t}B(\alpha t)\}$$

$$= \beta^2 H\left(\frac{1}{\alpha}\right)^{2H}\{e^{-2\alpha t}A(\alpha t) + B(\alpha t)\},$$

where

$$A(x) = \int_0^x e^{\xi}\xi^{2H-1}d\xi, \qquad B(x) = \int_0^x e^{-\xi}\xi^{2H-1}d\xi, \qquad \left(\frac{1}{2} < H < 1\right).$$

Observe the expression (3.5). Then, by Remark 3.4, we obtain that $Y(t)$ is a Gaussian process and hence has the normal distribution $N(\hat{M}(t), \hat{V}(t))$ with mean $\hat{M}(t)$ and variance $\hat{V}(t)$. Notice that

$$\lim_{x\to\infty} e^{-2x}A(x) = \lim_{x\to\infty} \frac{A(x)}{e^{2x}} = \lim_{x\to\infty} \frac{e^x x^{2H-1}}{2e^{2x}} = \lim_{x\to\infty} \frac{x^{2H-1}}{2e^x} = 0,$$

$$\lim_{x\to\infty} B(x) = \int_0^\infty e^{-\xi}\xi^{2H-1}d\xi = \Gamma(2H).$$

Then we obtain that as $t \to \infty$

$$\hat{M}(t) \to m, \qquad \hat{V}(t) \to \beta^2 H\left(\frac{1}{\alpha}\right)^{2H}\Gamma(2H).$$

By characteristic of the normal distribution, the limit distribution as $t \to \infty$ is also the normal distribution, which yields equation (3.6). This completes the proof of Lemma 3.11. □

Lemma 3.12. *Let* $Y(t)$ *be the solution of equation* (3.1). *If* $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$, *then we have*

$$\begin{aligned} f(t, Y(t)) = {} & f(0, Y(0)) + \int_0^t \frac{\partial f}{\partial s}(s, Y(s))ds \\ & + \int_0^t \frac{\partial f}{\partial y}(s, Y(s))\alpha(m - Y(s))ds + \int_0^t \frac{\partial f}{\partial y}(s, Y(s))\beta dB_H(s) \\ & + \int_0^t \frac{\partial^2 f}{\partial y^2}(s, Y(s)) \\ & \times H\beta^2\left\{\exp(-\alpha s)s^{2H-1} + \left(\frac{1}{\alpha}\right)^{2H-1}B(\alpha s)\right\}ds, \end{aligned} \tag{3.7}$$

where

$$B(x) = \int_0^x e^{-\xi}\xi^{2H-1}d\xi.$$

Proof. Apply the fractional Ito formula (Theorem 3.9) to $Y(t)$. Then we have

$$f(t, Y(t)) = f(0, Y(0)) + \int_0^t \frac{\partial f}{\partial s}(s, Y(s))ds$$

$$+ \int_0^t \frac{\partial f}{\partial y}(s, Y(s))\alpha(m - Y(s))ds + \int_0^t \frac{\partial f}{\partial y}(s, Y(s))\beta dB_H(s)$$

$$+ \int_0^t \frac{\partial^2 f}{\partial y^2}(s, Y(s))\beta D_s^{\phi}Y(s)ds.$$

Here $D_s^{\phi}Y(s)$ is the Malliavin ϕ-derivative of $F := Y(s)$ at s in the sense of Definition 3.7. Lemma 3.11 implies that $Y(t)$ has the explicit form (3.5). Therefore, by Remark 3.8 and equation (3.4) of Lemma 3.10, we can calculate the Malliavin ϕ-derivative $D_s^{\phi}Y(s)$ as follows:

$$D_s^{\phi}Y(s) = \beta \exp(-\alpha s) D_s^{\phi}\left(\int_0^s \exp(\alpha r) dB_H(r)\right)$$

$$= \beta \exp(-\alpha s)\int_0^s \exp(\alpha r)\phi(s, r)dr$$

$$= \beta \exp(-\alpha s) H \exp(\alpha s)\left\{\exp(-\alpha s)s^{2H-1} + \left(\frac{1}{\alpha}\right)^{2H-1} B(\alpha s)\right\},$$

where

$$B(x) = \int_0^x e^{-\xi}\xi^{2H-1}d\xi.$$

This completes the proof of Lemma 3.12. □

4. Asset Pricing

In the following sections, we take the same argument as given by Fouque et al. [7]; we also owe the proof to the introductory method in Andersson [1].

Under Assumption 2.1, we consider the model (2.1)-(2.3). In this case, there is one risky asset X and two random sources W and B_H. Namely, there are two sources of randomness instead of one as in the classical BS model. When constructing a portfolio, the derivatives cannot be perfectly hedged with just the underlying asset. Instead we also need a benchmark derivative called *G*. A risk-less portfolio Π is formed, containing the quantity $-\Delta_X$ of the underlying asset *X*, the quantity $-\Delta_G$ of another traded asset *G* (Benchmark option) and the priced derivative *P*. The total value of the portfolio is

$$\Pi = P - \Delta_X X - \Delta_G G. \tag{4.1}$$

The differential of the portfolio value is needed to construct a risk-less and no-arbitrage, satisfying

$$d\Pi = dP - \Delta_X dX - \Delta_G dG. \tag{4.2}$$

Notice that the model (2.1)-(2.3) can be written by the vector form

$$d\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \mu X \\ \alpha(m - Y) \end{pmatrix} dt + \begin{pmatrix} f(Y)X & 0 \\ 0 & \beta \end{pmatrix} d\begin{pmatrix} W \\ B_H \end{pmatrix}.$$

Then, the classical Ito formula and the fractional one (Lemma 3.12) are applied to *dP* and *dG* as follows:

$$\begin{aligned} d\Pi = &\Bigg[\frac{\partial P}{\partial t} + \frac{1}{2}\frac{\partial^2 P}{\partial X^2} f(y)^2 X^2 + \frac{\partial^2 P}{\partial y^2} H\beta^2 \\ &\times \left\{\exp(-\alpha t)t^{2H-1} + \left(\frac{1}{\alpha}\right)^{2H-1} B(\alpha t)\right\}\Bigg] dt \\ &+ \frac{\partial P}{\partial X} dX + \frac{\partial P}{\partial y} dY - \Delta_X dX \\ &- \Delta_G \Bigg\{\Bigg[\frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2} f(y)^2 X^2 + \frac{\partial^2 G}{\partial y^2} H\beta^2 \\ &\times \left\{\exp(-\alpha t)t^{2H-1} + \left(\frac{1}{\alpha}\right)^{2H-1} B(\alpha t)\right\}\Bigg] dt \\ &+ \frac{\partial G}{\partial X} dX + \frac{\partial G}{\partial y} dY\Bigg\}. \end{aligned} \tag{4.3}$$

Collecting the dX and dY terms, we have

$$d\Pi = \left[\frac{\partial P}{\partial X} - \Delta_G \frac{\partial G}{\partial X} - \Delta_X\right] dX + \left[\frac{\partial P}{\partial y} - \Delta_G \frac{\partial G}{\partial y}\right] dY$$

$$+ \left[\frac{\partial P}{\partial t} + \frac{1}{2}\frac{\partial^2 P}{\partial X^2} f(y)^2 X^2\right.$$

$$\left. + \frac{\partial^2 P}{\partial y^2} H\beta^2 \left\{\exp(-\alpha t) t^{2H-1} + \left(\frac{1}{\alpha}\right)^{2H-1} B(\alpha t)\right\}\right] dt$$

$$- \Delta_G \left[\frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2} f(y)^2 X^2\right.$$

$$\left. + \frac{\partial^2 G}{\partial y^2} H\beta^2 \left\{\exp(-\alpha t) t^{2H-1} + \left(\frac{1}{\alpha}\right)^{2H-1} B(\alpha t)\right\}\right] dt. \tag{4.4}$$

We want this portfolio to be risk-less by eliminating the coefficients in front of dX and dY, and hence

$$\frac{\partial P}{\partial X} - \Delta_G \frac{\partial G}{\partial X} - \Delta_X = 0, \tag{4.5}$$

$$\frac{\partial P}{\partial y} - \Delta_G \frac{\partial G}{\partial y} = 0. \tag{4.6}$$

From the equations above, Δ_X and Δ_G are solved as follows:

$$\Delta_G = \left(\frac{\partial P}{\partial y}\right)\left(\frac{\partial G}{\partial y}\right)^{-1}, \tag{4.7}$$

$$\Delta_X = \frac{\partial P}{\partial X} - \frac{\partial G}{\partial X}\left(\frac{\partial P}{\partial y}\right)\left(\frac{\partial G}{\partial y}\right)^{-1}. \tag{4.8}$$

Thus, if the portfolio is well-balanced according to (4.7) and (4.8), the risk is eliminated. Moreover, we want $\Pi(t)$ to be risk-less with instantaneous interest rate r, such that

$$d\Pi(t) = r\Pi(t)\,dt.$$

Namely,

$$\begin{aligned} d\Pi &= \left[\frac{\partial P}{\partial t} + \frac{1}{2}\frac{\partial^2 P}{\partial X^2} f(y)^2 X^2 \right. \\ &\quad \left. + \frac{\partial^2 P}{\partial y^2} H\beta^2 \left\{ \exp(-\alpha t) t^{2H-1} + \left(\frac{1}{\alpha}\right)^{2H-1} B(\alpha t) \right\} \right] dt \\ &\quad - \Delta_G \left[\frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2} f(y)^2 X^2 \right. \\ &\quad \left. + \frac{\partial^2 G}{\partial y^2} H\beta^2 \left\{ \exp(-\alpha t) t^{2H-1} + \left(\frac{1}{\alpha}\right)^{2H-1} B(\alpha t) \right\} \right] dt \\ &= r\Pi dt, \end{aligned} \tag{4.9}$$

where $\Pi = P - \Delta_X X - \Delta_G G$. Substitute Δ_G and Δ_X, which are given by equations (4.7) and (4.8), into the equation (4.9), and then multiply both sides by $\partial G / \partial y$. Then we obtain

$$\begin{aligned} &\left[\frac{\partial P}{\partial t} + \frac{1}{2}\frac{\partial^2 P}{\partial X^2} f(y)^2 X^2 \right. \\ &\quad + \frac{\partial^2 P}{\partial y^2} H\beta^2 \left\{ \exp(-\alpha t) t^{2H-1} + \left(\frac{1}{\alpha}\right)^{2H-1} B(\alpha t) \right\} \\ &\quad \left. - rP + rX \frac{\partial P}{\partial X} \right] \left(\frac{\partial P}{\partial y} \right)^{-1} \\ &= \left[\frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2} f(y)^2 X^2 \right. \\ &\quad + \frac{\partial^2 G}{\partial y^2} H\beta^2 \left\{ \exp(-\alpha t) t^{2H-1} + \left(\frac{1}{\alpha}\right)^{2H-1} B(\alpha t) \right\} \\ &\quad \left. - rG + rX \frac{\partial G}{\partial X} \right] \left(\frac{\partial G}{\partial y} \right)^{-1}. \end{aligned} \tag{4.10}$$

The left hand side in equation (4.10) does only depend on P and the right hand side does only depend on G. Both sides are thus equal to some function $k(t, X, y)$.

Lemma 4.1. *The equation governing P can be written as*

$$\begin{aligned}&\frac{\partial P}{\partial t}+\frac{1}{2}\frac{\partial^2 P}{\partial X^2}f(y)^2X^2\\&+\frac{\partial^2 P}{\partial y^2}H\beta^2\left\{\exp(-\alpha t)t^{2H-1}+\left(\frac{1}{\alpha}\right)^{2H-1}B(\alpha t)\right\}\\&-rP+rX\frac{\partial P}{\partial X}\\=\ &k(t,\ X,\ y)\frac{\partial P}{\partial y}.\end{aligned}\tag{4.11}$$

The terminal condition for P is the contract function $h(X)$, *i.e.*, $P(T,\ X,\ y)=h(X(T))$; *for example*, $h(X(T))=(X(T)-K)^+$ *with T and K, the time of maturity and the strike price, respectively.*

Proof. The argument in the preceding and equation (4.10) leads us to equation (4.11). This completes the proof of Lemma 4.1. □

The function k cannot be determined by arbitrage theory alone. However, it is completely determined in terms of the traded benchmark asset G. One can say that the market knows the function k.

For a moment, recall the model (1.1)-(1.3), where the asset process $(X(t))$ and the volatility driving process $(Y(t))$ are governed by a standard Brownian motion $(W(t))$ and another standard Brownian motion $(\hat{B}(t))$, respectively, such that

$$\hat{B}(t)=\rho W(t)+\sqrt{1-\rho^2}B(t).\tag{4.12}$$

Here $(W(t))$ and $(B(t))$ are independent Brownian motions with $-1<\rho<1$ the instantaneous correlation coefficient between asset price and volatility shocks. Then, according to Fouque et al. [7], the function k is taken by

$$k(t,\ X,\ y)=\alpha(m-y)-\beta\Lambda(t,\ X,\ y),\tag{4.13}$$

where

$$\Lambda(t,\ X,\ y)=\rho\left(\frac{\mu-r}{f(y)}\right)+\gamma(t,\ X,\ y)\sqrt{1-\rho^2}\tag{4.14}$$

with an arbitrary function $\gamma(t,\ X,\ y)$.

Under Assumption 2.1, our model (2.1)-(2.3) corresponds to the case of (1.1)-(1.3) where

$$B(t) = B_H(t), \qquad \rho = 0, \qquad \text{and hence} \qquad \hat{B}(t) = B_H(t).$$

Hereafter, in our model (2.1)-(2.3), it is convenient to assume that

$$k(t, X, y) = \alpha(m - y) - \alpha^{\frac{1}{2}-H}\beta\gamma(t, X, y), \tag{4.15}$$

appealing to the Hurst parameter $H > 1/2$. We notice that equation (4.15) is equal to equation (4.13), if $H = 1/2$ is formally substituted into equation (4.15). Further, for simplicity, we assume that the function γ depends only on the variable y. Therefore, we assume the following:

Assumption 4.2.

$$k(t, X, y) = \alpha(m - y) - \alpha^{\frac{1}{2}-H}\beta\gamma(y). \tag{4.16}$$

5. The Pricing PDE in Terms of ε

Here we consider the model (2.1)-(2.3). Lemma 3.11 implies that the fractional OU process $(Y(t))$ has the long-run distribution $N(m, \nu_H^2)$ as $t \to \infty$, that is, the density

$$n(y) = \frac{1}{\sqrt{2\pi\nu_H^2}} \exp\left(-\frac{(y - m)^2}{2\nu_H^2}\right), \tag{5.1}$$

where

$$\nu_H^2 = \beta^2 H \left(\frac{1}{\alpha}\right)^{2H} \Gamma(2H), \tag{5.2}$$

and $\Gamma(\cdot)$ is the Gamma function; $\Gamma(x) = \int_0^\infty e^{-\xi}\xi^{x-1}d\xi$.

We take the same argument as taken by Fouque et al. [7] and assume the following:

Assumption 5.1.

(i) The rate of mean reversion α or its inverse, the typical correlation time of $(Y(t))$, is characterized by a small parameter ε, such that

$$\varepsilon = \frac{1}{\alpha}.$$

(ii) Let ν_H^2 be given by equation (5.2), which controls the long-run size of the volatility fluctuations. Then we assume that this quantity remains fixed as we consider smaller and smaller values of ε, such that

$$\beta = \left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\left(\frac{1}{\alpha}\right)^{-H} = \left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\frac{1}{\varepsilon^H}.$$

Under Assumptions 4.2 and 5.1, we observe the multiplier of the second derivative $\partial^2 P/\partial y^2$ in the partial differential equation (4.11), that is,

$$H\beta^2 \exp(-\alpha t)t^{2H-1} + H\beta^2\left(\frac{1}{\alpha}\right)^{2H-1} B(\alpha t), \qquad B(x) = \int_0^x e^{-\xi}\xi^{2H-1}d\xi.$$

Step 1.

$$H\beta^2 \exp(-\alpha t) = \left(\frac{\nu_H^2}{\Gamma(2H)}\right)\left(\frac{1}{\alpha}\right)^{-2H} \exp(-\alpha t)$$

$$= \left(\frac{\nu_H^2}{\Gamma(2H)}\right)\frac{\alpha^{2H}}{\exp(\alpha t)} \qquad \left(\alpha = \frac{1}{\varepsilon}\right).$$

The L'Hospital rule for limits of indeterminate form yields that for $t > 0$,

$$\lim_{\varepsilon\to 0} H\beta^2 \exp(-\alpha t) = \left(\frac{\nu_H^2}{\Gamma(2H)}\right)\lim_{\alpha\to\infty}\frac{\alpha^{2H}}{\exp(\alpha t)}\left[= \left(\frac{\nu_H^2}{\Gamma(2H)}\right)\lim_{\alpha\to\infty}\frac{\{\alpha^{2H}\}'}{\{\exp(\alpha t)\}'}\right]$$

$$= \left(\frac{\nu_H^2}{\Gamma(2H)}\right)\lim_{\alpha\to\infty}\frac{2H\alpha^{2H-1}}{\exp(\alpha t)t} \qquad \left(\alpha = \frac{1}{\varepsilon}\right)$$

$$= \left(\frac{\nu_H^2}{\Gamma(2H)}\right)\lim_{\alpha\to\infty}\frac{2H(2H-1)\alpha^{2H-2}}{\exp(\alpha t)t^2} \qquad \left(\frac{1}{2} < H < 1\right)$$

$$= \left(\frac{\nu_H^2}{\Gamma(2H)}\right)\lim_{\alpha\to\infty}\frac{2H(2H-1)}{\alpha^{2(1-H)}\exp(\alpha t)t^2} = 0. \tag{5.3}$$

Step 2.

$$H\beta^2\left(\frac{1}{\alpha}\right)^{2H-1} B(\alpha t) = \left(\frac{\nu_H^2}{\Gamma(2H)}\right)\left(\frac{1}{\alpha}\right)^{-1} B(\alpha t)$$

$$= \left(\nu_H^2 \frac{1}{\varepsilon}\right)\left(\frac{B\left(\frac{t}{\varepsilon}\right)}{\Gamma(2H)}\right) \qquad \left(\alpha = \frac{1}{\varepsilon}\right)$$

$$= \left(\nu_H^2 \frac{1}{\varepsilon}\right) + \left(\nu_H^2 \frac{1}{\Gamma(2H)}\right)\left(\frac{B\left(\frac{t}{\varepsilon}\right) - \Gamma(2H)}{\varepsilon}\right).$$

We notice that

$$B(x) = \int_0^x e^{-\xi}\xi^{2H-1}d\xi \to \Gamma(2H) \qquad \text{as} \qquad x \to \infty.$$

Then the L'Hospital rule for limits of indeterminate form yields that for $t > 0$,

$$\lim_{\varepsilon\to 0} \frac{B\left(\frac{t}{\varepsilon}\right) - \Gamma(2H)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{\left\{B\left(\frac{t}{\varepsilon}\right) - \Gamma(2H)\right\}'}{\{\varepsilon\}'}$$

$$= \lim_{\varepsilon\to 0}\left(\frac{-t}{\varepsilon^2}\right)\exp\left(-\frac{t}{\varepsilon}\right)\left(\frac{t}{\varepsilon}\right)^{2H-1}$$

$$= (-t^{2H}) \lim_{\varepsilon\to 0} \exp\left(-\frac{t}{\varepsilon}\right)\left(\frac{1}{\varepsilon}\right)^{2H+1}$$

$$= (-t^{2H}) \lim_{\alpha\to\infty} \frac{\alpha^{2H+1}}{\exp(\alpha t)}\left[= (-t^{2H}) \lim_{\alpha\to\infty} \frac{\{\alpha^{2H+1}\}'}{\{\exp(\alpha t)\}'}\right] \qquad \left(\alpha = \frac{1}{\varepsilon}\right)$$

$$= (-t^{2H}) \lim_{\alpha\to\infty} \frac{(2H+1)\alpha^{2H}}{\exp(\alpha t)t}$$

$$= (-t^{2H}) \lim_{\alpha\to\infty} \frac{(2H+1)2H\alpha^{2H-1}}{\exp(\alpha t)t^2} \qquad \left(\frac{1}{2} < H < 1\right)$$

$$= (-t^{2H}) \lim_{\alpha\to\infty} \frac{(2H+1)2H(2H-1)\alpha^{2H-2}}{\exp(\alpha t)t^3} = 0.$$

Thus, we have that for $t > 0$,

$$H\beta^2\left(\frac{1}{\alpha}\right)^{2H-1}B(\alpha t) = \left(\nu_H^2 \frac{1}{\varepsilon}\right) + o(1) \tag{5.4}$$

for ε small enough; $\varepsilon = \dfrac{1}{\alpha}$.

Further, let $k(t, X, y)$ be the function as given in equation (4.16). Then, under Assumption 5.1, we have the following:

Step 3.

$$\begin{aligned} k(t, X, y) &= \alpha(m - y) - \alpha^{\frac{1}{2}-H}\beta\gamma(y) \\ &= \frac{1}{\varepsilon}(m - y) - \left(\frac{1}{\varepsilon}\right)^{\frac{1}{2}-H}\left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\frac{1}{\varepsilon^H}\gamma(y) \\ &= \frac{1}{\varepsilon}(m - y) - \left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\frac{1}{\sqrt{\varepsilon}}\gamma(y). \end{aligned} \tag{5.5}$$

Lemma 5.2. *Under Assumptions* 4.2 *and* 5.1, *for* ε *small enough*, *the pricing PDE* (4.11) *can be written in terms of* ε *as follows*:

$$\begin{aligned} &\frac{\partial P^\varepsilon}{\partial t} + \frac{1}{2}\frac{\partial^2 P^\varepsilon}{\partial X^2} f(y)^2 X^2 \\ &+ \frac{\nu_H^2}{\varepsilon}\frac{\partial^2 P^\varepsilon}{\partial y^2} + r\left(X\frac{\partial P^\varepsilon}{\partial X} - P^\varepsilon\right) \\ &+ \left\{\frac{1}{\varepsilon}(m - y) - \left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\frac{1}{\sqrt{\varepsilon}}\gamma(y)\right\}\frac{\partial P^\varepsilon}{\partial y} = 0, \end{aligned} \tag{5.6}$$

for $t < T$ *with the terminal condition* $P^\varepsilon(T, X, y) = h(X)$, *where* $h(X)$ *stands for the nonnegative payoff function*.

Proof. Observe the multiplier of $\partial^2 P/\partial y^2$ in PDE (4.11). Then the equations (5.3), (5.4) and (5.5) yield equation (5.6). This completes the proof of Lemma 5.2. □

6. Asymptotic Solution

We shall solve PDE (5.6) by singular perturbation analysis. We write PDE (5.6) with the notation as follows:

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)P^{\varepsilon} = 0, \tag{6.1}$$

where we define

$$\mathcal{L}_0 = \nu_H^2 \frac{\partial^2}{\partial y^2} + (m - y)\frac{\partial}{\partial y}, \tag{6.2}$$

$$\mathcal{L}_1 = -\left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\gamma(y)\frac{\partial}{\partial y}, \tag{6.3}$$

$$\mathcal{L}_2 = \mathcal{L}_{BS}(f(y)) = \frac{\partial}{\partial t} + \frac{1}{2}f(y)^2 X^2 \frac{\partial^2}{\partial X^2} + r\left(X\frac{\partial}{\partial X} - 1\right). \tag{6.4}$$

Here $\mathcal{L}_{BS}(\sigma)$ is the classical Black-Scholes operator with the deterministic volatility parameter σ, that is,

$$\mathcal{L}_{BS}(\sigma) = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2}{\partial X^2} + r\left(X\frac{\partial}{\partial X} - 1\right). \tag{6.5}$$

We look for an expansion

$$P^{\varepsilon} = P_0 + \sqrt{\varepsilon}P_1 + \varepsilon P_2 + \varepsilon\sqrt{\varepsilon}P_3 + \cdots \tag{6.6}$$

for small ε, where $P_0, P_1, \ldots$ are functions of (T, X, y) to be determined by the terminal conditions

$$P_0(T, X, y) = h(X), \qquad P_i(T, X, y) = 0 \qquad \text{for} \quad i \geq 1.$$

Substituting equation (6.6) into equation (6.1) and collecting powers of ε, we have

$$\begin{aligned}
&\frac{1}{\varepsilon}\mathcal{L}_0 P_0 + \frac{1}{\sqrt{\varepsilon}}(\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0)\\
&+ (\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0)\\
&+ \sqrt{\varepsilon}(\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1)\\
&+ \cdots\\
&= 0.
\end{aligned} \tag{6.7}$$

For equation (6.7), step by step, the terms of order $1/\varepsilon$, $1/\sqrt{\varepsilon}$, $\ldots$ will be studied.

Term of order $1/\varepsilon$. At order $1/\varepsilon$, we have

$$\mathcal{L}_0 P_0 = 0. \tag{6.8}$$

The operator $\mathcal{L}_0$ contains partial derivatives with respect to *y* but no derivatives with respect to *X*. Hence P_0 must be a constant with respect to the variable *y*, which implies that

$$P_0 = P_0(t, X) \tag{6.9}$$

with terminal condition $P_0(T, X) = h(X)$.

Term of order $1/\sqrt{\varepsilon}$. At order $1/\sqrt{\varepsilon}$, we have

$$\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0 = 0. \tag{6.10}$$

Notice that P_0 only depends on *t* and *X* and that the operator $\mathcal{L}_1$ involves the derivative with respect to *y*. Then we have that $\mathcal{L}_1 P_0 = 0$, and hence equation (6.10) is reduced to $\mathcal{L}_0 P_1 = 0$. The operator $\mathcal{L}_0$ involves derivatives with respect to *y*. Thus, P_1 must be a constant with respect to *y*, which implies that $P_1 = P_1(t, X)$, with the terminal condition $P_1(T, X) = 0$.

Thus, we note that the term $P_0 + \sqrt{\varepsilon} P_1$ in equation (6.6) will not depend on *y*.

Zeroth order-term. At order 1, we have

$$\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0 = 0. \tag{6.11}$$

The discussion above implies that P_0 and P_1 only depend on (t, X) and that $\mathcal{L}_1$ and $\mathcal{L}_0$ involve derivatives with respect to *y*. Thus, $\mathcal{L}_1 P_1 = 0$, and hence equation (6.11) is reduced to

$$\mathcal{L}_0 P_2 + \mathcal{L}_2 P_0 = 0. \tag{6.12}$$

Here, P_0 only depends on *t* and *X*. When regarding *X* as fixed, $\mathcal{L}_2 P_0$ only depends on *y*. Hence the equation (6.12) is a Poisson equation for P_2 with respect to $\mathcal{L}_0$, that is, $\mathcal{L}_0 P_2 = -\mathcal{L}_2 P_0$.

In the following, we let $\langle \cdot \rangle$ denote the averaging with respect to the invariant distribution $N(m, \nu_H^2)$ of the fractional OU process $(Y(t))$ (see Lemma 3.11 and equation (5.1)):

$$\langle g \rangle = \int_{-\infty}^{\infty} g(y) n(y) dy = \frac{1}{\sqrt{2\pi\nu_H^2}} \int_{-\infty}^{\infty} g(y) \exp\left(-\frac{(y-m)^2}{2\nu_H^2} \right) dy.$$

Notice that this averaged quantity does not depend on ε.

In order to have a solution to the Poisson equation (6.12), $\mathcal{L}_2 P_0$ must be in the orthogonal complement of the null space of $\mathcal{L}_0^*$, where $\mathcal{L}_0^*$ is the adjoint operator of $\mathcal{L}_0$, such that

$$\mathcal{L}_0^* p = -\frac{\partial}{\partial y}((m-y)p) + \nu_H^2 \frac{\partial^2 p}{\partial y^2} \tag{6.13}$$

for $p \in C^2(\mathbb{R})$. This solvability condition is equivalent to saying that $\mathcal{L}_0 P_2$ has mean zero with respect to the invariant distribution, which yields

$$E[\mathcal{L}_2 P_0] = \langle \mathcal{L}_2 P_0 \rangle = \int_{-\infty}^{\infty} \mathcal{L}_2 P_0 n(y) dy = 0, \tag{6.14}$$

where $n(y)$ in equation (5.1) solves $\mathcal{L}_0^* n = 0$. Namely, the solvability condition above implies that $\langle \mathcal{L}_2 P_0 \rangle = 0$. Since P_0 does not depend on y, the solvability condition is reduced to

$$\mathcal{L}_{BS}(\bar{\sigma}) P_0 = 0, \tag{6.15}$$

where $\langle \mathcal{L}_2 \rangle = \mathcal{L}_{BS}(\bar{\sigma})$ and $\bar{\sigma}$ is the effective constant volatility defined by

$$\bar{\sigma}^2 = \langle f^2 \rangle = \int_{-\infty}^{\infty} f(y)^2 n(y) dy, \tag{6.16}$$

which is the average with respect to the invariant distribution of the process Y.

Therefore, P_0 is the solution of Black-Scholes equations with terminal condition $P_0(T, X) = h(X)$ and the effective constant volatility $\bar{\sigma} = \sqrt{\bar{\sigma}^2}$ as given by equation (6.16).

Observe the equation (6.12), again, that is,

$$\mathcal{L}_0 P_2 = -\mathcal{L}_2 P_0. \tag{6.17}$$

Apply the same solvability condition as given in equation (6.14). Then, since $\langle \mathcal{L}_2 P_0 \rangle = 0,$ we see that $\mathcal{L}_2 P_0$ in the right hand side of equation (6.17) can be written as

$$\mathcal{L}_2 P_0 = \mathcal{L}_2 P_0 - \langle \mathcal{L}_2 P_0 \rangle = \frac{1}{2}(f(y)^2 - \overline{\sigma}^2) X^2 \frac{\partial^2 P_0}{\partial X^2}. \tag{6.18}$$

Thus, equation (6.17) is given by

$$\mathcal{L}_0 P_2 = -\frac{1}{2}(f(y)^2 - \overline{\sigma}^2) X^2 \frac{\partial^2 P_0}{\partial X^2}. \tag{6.19}$$

The solution of the Poisson equation in equation (6.19) is given by

$$\begin{aligned} P_2 &= -\frac{1}{2}\mathcal{L}_0^{-1}(f(y)^2 - \overline{\sigma}^2) X^2 \frac{\partial^2 P_0}{\partial X^2} \\ &= -\frac{1}{2}(\psi(y) + c(t, X)) X^2 \frac{\partial^2 P_0}{\partial X^2}, \end{aligned} \tag{6.20}$$

where $\psi(y)$ is the solution of the Poisson equation

$$\mathcal{L}_0 \psi = f(y)^2 - \overline{\sigma}^2 \tag{6.21}$$

and $c(t, X)$ is a constant that may depend on (t, X).

Term of order $\sqrt{\varepsilon}$. At order $\sqrt{\varepsilon}$, we have

$$\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 = 0. \tag{6.22}$$

This is a Poisson equation for P_3 with respect to $\mathcal{L}_0$, which is written by

$$\mathcal{L}_0 P_3 = -(\mathcal{L}_1 P_2 + \mathcal{L}_2 P_1). \tag{6.23}$$

Again, applying the same solvability condition as given in equation (6.14), we obtain

$$\langle \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 \rangle = 0. \tag{6.24}$$

Hence

$$\langle \mathcal{L}_2 P_1 \rangle = -\langle \mathcal{L}_1 P_2 \rangle, \tag{6.25}$$

where P_2 is already known by equation (6.20). In the following, we investigate $\langle \mathcal{L}_2 P_1 \rangle$.

Notice that P_1 does not depend on y and consider that $\langle \mathcal{L}_2 \rangle = \mathcal{L}_{BS}(\overline{\sigma})$. Then we have that the left hand side of equation (6.25) is equal to $\mathcal{L}_{BS}(\overline{\sigma}) P_1$. Observe the right hand side of equation (6.25) with P_2 replaced by equation (6.20). Then we find

$$\begin{aligned} -\langle \mathcal{L}_1 P_2 \rangle &= \frac{1}{2} \langle \mathcal{L}_1(\psi(y) + c(t, X)) \rangle X^2 \frac{\partial^2 P_0}{\partial X^2} \\ &= \frac{1}{2} \langle \mathcal{L}_1 \psi(y) \rangle X^2 \frac{\partial^2 P_0}{\partial X^2}, \end{aligned} \tag{6.26}$$

where ψ is a solution of the Poisson equation (6.21); here we used that $\mathcal{L}_1 c(t, X) = 0$ since $\mathcal{L}_1$ does only involve the variable y. Hence, by equations (6.25) and (6.26), we obtain

$$\mathcal{L}_{BS}(\overline{\sigma}) P_1 = \frac{1}{2} \langle \mathcal{L}_1 \psi(y) \rangle X^2 \frac{\partial^2 P_0}{\partial X^2}. \tag{6.27}$$

Let $\varphi(X)$ be a general function. Then we can compute as follows:

$$\begin{aligned} \langle \mathcal{L}_1 \psi(y) \varphi(X) \rangle &= \left\langle -\left(\frac{\nu_H}{\sqrt{H \Gamma(2H)}} \right) \gamma(y) \frac{\partial}{\partial y} (\psi(y) \varphi(X)) \right\rangle \\ &= -\left(\frac{\nu_H}{\sqrt{H \Gamma(2H)}} \right) \langle \gamma(y) \psi'(y) \rangle \varphi(X). \end{aligned} \tag{6.28}$$

In equation (6.28), set $\varphi(X) = X^2 \left(\frac{\partial^2 P_0}{\partial X^2} \right)$. Then, by equation (6.27), we obtain

$$\mathcal{L}_{BS}(\overline{\sigma}) P_1 = -\frac{1}{2} \left(\frac{\nu_H}{\sqrt{H \Gamma(2H)}} \right) \langle \gamma(y) \psi'(y) \rangle X^2 \frac{\partial^2 P_0}{\partial X^2}, \tag{6.29}$$

and the terminal condition is $P_1(T, X) = 0$. According to Fouque et al. [7], we denote the first correction by

$$\widetilde{P}_1(t, X) = \sqrt{\varepsilon} P_1(t, X), \tag{6.30}$$

which is a solution of

$$\mathcal{L}_{BS}(\overline{\sigma})\widetilde{P}_1 = H(t, X), \tag{6.31}$$

and the terminal condition $\widetilde{P}_1(T, X) = 0$. Here we define the source term $H(t, X)$ by

$$H(t, X) = V_2 X^2 \frac{\partial^2 P_0}{\partial X^2}; \tag{6.32}$$

V_2 is a small coefficient, given in terms of $\alpha = 1/\varepsilon$ by

$$V_2 = -\frac{1}{\sqrt{\alpha}}\frac{1}{2}\left(\frac{\nu_H}{\sqrt{H\Gamma(2H)}}\right)\langle\gamma\psi'\rangle. \tag{6.33}$$

The first correction satisfies the classical Black-Scholes equation (6.31) with a zero terminal condition and a small source term computed from derivative of the leading term $P_0(t, X)$.

Finally, we want to find the explicit form of the solution $\widetilde{P}_1$ to equation (6.31). Now, a straightforward calculation yields that for $n \geq 1$,

$$\mathcal{L}_{BS}(\overline{\sigma})\left(X^n \frac{\partial^n P_0}{\partial X^n}\right) = X^n \frac{\partial^n}{\partial X^n}(\mathcal{L}_{BS}(\overline{\sigma})P_0). \tag{6.34}$$

Further, it holds that

$$\mathcal{L}_{BS}(\overline{\sigma})(-(T-t)H(t, X)) = H - (T-t)\mathcal{L}_{BS}(\overline{\sigma})H(t, X). \tag{6.35}$$

Here, equations (6.32) and (6.34) yield

$$\begin{aligned}
\mathcal{L}_{BS}(\overline{\sigma})H(t, X) &= \mathcal{L}_{BS}(\overline{\sigma})\left(V_2 X^2 \frac{\partial^2 P_0}{\partial X^2}\right) \\
&= V_2\mathcal{L}_{BS}(\overline{\sigma})\left(X^2 \frac{\partial^2 P_0}{\partial X^2}\right) \\
&= V_2 X^2 \frac{\partial^2}{\partial X^2}(\mathcal{L}_{BS}(\overline{\sigma})P_0) \\
&= 0,
\end{aligned}$$

since $\mathcal{L}_{BS}(\overline{\sigma})P_0 = 0$ by equation (6.15). Therefore, equation (6.35) is reduced to

$$\mathcal{L}_{BS}(\overline{\sigma})(-(T-t)H(t, X)) = H.$$

Hence the solution of the Black-Scholes equation (6.31) is explicitly given by

$$\widetilde{P}_1(t, X) = -(T-t)H(t, X) = -(T-t)\left(V_2 X^2 \frac{\partial^2 P_0}{\partial X^2}\right). \tag{6.36}$$

Theorem 6.1. *Under Assumptions* 2.1, 4.2 *and* 5.1, *for* ε *small enough*, *the corrected Black-Scholes price is given by*

$$P \approx P_0 + \widetilde{P}_1 = P_0 - (T-t)\left(V_2 X^2 \frac{\partial^2 P_0}{\partial X^2}\right), \tag{6.37}$$

where P_0 *is the solution of the classical Black-Scholes equation with effective constant volatility* $\overline{\sigma}$ *as given by equation* (6.16), *i.e.*, $\mathcal{L}_{BS}(\overline{\sigma})P_0 = 0$ *with terminal condition* $P_0(T, X) = h(X)$ *for a payoff function* $h(X)$. *Further, the function* V_2 *is given by equation* (6.33) *with* ν_H *the parameter of the normal distribution* $N(m, \nu_H^2)$ *as given by equation* (5.1).

Proof. By equations (6.6), (6.15), (6.30), (6.31) and (6.36), we obtain equation (6.37), thereby completing the proof of Theorem 6.1. ☐

Remark 6.2. Recall the model (1.1)-(1.3), where the volatility driving process $(Y(t))$ is governed by $\hat{B}(t) = \rho W(t) + \sqrt{1-\rho^2}B(t)$. Here $(W(t))$ and $(B(t))$ are independent standard Brownian motions with $-1 < \rho < 1$ the instantaneous correlation coefficient between asset price and volatility shocks. Then, Fouque et al. [7] obtain the corrected price function of the form

$$P \approx P_0 + \widetilde{P}_1 = P_0 - (T-t)\left(V_2 X^2 \frac{\partial^2 P_0}{\partial X^2} + V_3 X^3 \frac{\partial^3 P_0}{\partial X^3}\right),$$

where P_0 and $\widetilde{P}_1$ satisfy

$$\mathcal{L}_{BS}(\overline{\sigma})P_0 = 0, \quad \widetilde{P}_1 = \sqrt{\varepsilon}P_1, \quad \mathcal{L}_{BS}(\overline{\sigma})\widetilde{P}_1 = H(t, X).$$

Here $H(t, X)$ is given by

$$H(t, X) = V_2 X^2 \frac{\partial^2 P_0}{\partial X^2} + V_3 X^3 \frac{\partial^3 P_0}{\partial X^3}$$

with the functions V_2 and V_3, such that

$$V_2 = \frac{\nu}{\sqrt{2\alpha}}(2\rho\langle f\psi'\rangle - \langle \gamma\psi'\rangle), \qquad V_3 = \frac{\rho\nu}{\sqrt{2\alpha}}\langle f\psi'\rangle,$$

$$\alpha = \frac{1}{\varepsilon}, \qquad \beta = \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}, \qquad \left(\nu^2 = \frac{\beta^2\varepsilon}{2} = \frac{\beta^2}{2\alpha}\right),$$

where ν is the parameter appearing in the invariant distribution $N(m, \nu^2)$ for the volatility driving process $(Y(t))$ in equation (1.3), ψ is the solution of the Poisson equation (6.21), and Λ is the function as given by equation (4.14).

In Theorem 6.1, if we replace H by $H = 1/2$ formally, then we obtain the same result as given by Fouque et al. [7, equation (5.43), p. 96] in the case of $\rho = 0$;

$$\nu_H^2 = \beta^2 H\left(\frac{1}{\alpha}\right)^{2H}\Gamma(2H) = \frac{\beta^2}{2\alpha} \qquad \text{if} \qquad H = \frac{1}{2}.$$

We also note that under suitable scaling, Theorem 6.1 corresponds to an extension of the result in Kallianpur and Karandikar [15, Theorem 13.7], where asymptotic analysis is given for models (1.1)-(1.3) with correlation coefficient $\rho = 0$.

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