JP Journal of Algebra, Number Theory and Applications
Volume 15, Number 2, 2009, Pages 113-125
Published Online: January 4, 2010
This paper is available online at http://www.pphmj.com

THE MAXIMUM IDEMPOTENT SEPARATING CONGRUENCE ON EVENTUALLY REGULAR SEMIGROUPS

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Abstract

The aim of this paper is to characterize an idempotent separating congruence on an eventually regular semigroup and the maximum idempotent separating congruence on its and on eventually regular E-semigroups which are analogous to the characterization of an eventually regular semigroup considered by Luo and Li [3] and to characterization of an eventually regular E-semigroup considered by Siripitukdet and Sattayaporn [5].

1. Introduction

Let S be a semigroup and $E(S)$ denote the set of all idempotents of S. For $a \in S$, $V(a) := \{x \in S \mid a = axa, x = xax\}$ is the set of all inverses of element a and $W(a) := \{x \in S \mid x = xax\}$ is the set of all weak inverses of element a. An element a in a semigroup S is called *E-inversive* [6] if there exists $x \in S$ such that ax is an idempotent of S. A semigroup S is called *E-inversive* if for all $a \in S$, a is E-

2000 Mathematics Subject Classification: 20M10.

Keywords and phrases: eventually regular semigroup, idempotent separating congruence.

Received October 7, 2009

inversive. A semigroup S is called an *E-semigroup* if $E(S)$ forms a subsemigroup of S. A semigroup S is regular if and only if $V(a) \neq \varnothing$ for each $a \in S$ and a regular semigroup S is an orthodox if $E(S)$ forms a subsemigroup of S. A semigroup S is an eventually regular [1] if every $a \in S$, there exists a positive integer n such that a^n is regular. For each $a \in S$, a^n is a-regular, we mean that n is the smallest positive integer for which a^n is regular. The class of eventually regular semigroups generalizes regular semigroups and finite semigroups.

Edwards [1] described basic properties and some results of an eventually regular semigroup. He proved that the maximum idempotent separating congruence exists. A congruence ρ on a semigroup S is an idempotent separating congruence on S if every ρ-class contains at most one idempotent, i.e., for all e, $f \in E(S)$ and $e\rho f$ implies $e = f$. Weipoltshammer [6] introduced the maximum idempotent separating congruence on an E-inversive E-semigroup which Luo and Li [3] generalized for eventually regular semigroups. In this paper, we investigated the maximum idempotent separating congruence on an eventually regular E-semigroup which are analogous to characterization of E-inversive E-semigroups considered by [5] and we instead for all $e \in E(S)$ as in [3] by for all $x \in H$ as in [4].

A subset H of a semigroup S is full [7] if $E(S) \subseteq H$. A subsemigroup H of a semigroup S is called *weakly self-conjugate* if for all $a \in S$, $x \in H$, $a' \in W(a)$, we have axa', $a'xa \in H$, for any subsets H and B of a semigroup S, let

$$H_{\omega_B} := \{a \in S \mid ba \in H \text{ for some } b \in B\}.$$

If $B = H$, then H_{ω_H} will be denoted by H_ω and it is called the *closure of H*. If H is a subsemigroup of a semigroup S, then $H \subseteq H_\omega$. H is called a *closed subsemigroup* [7] of S if $H = H_\omega$.

A subset H of an eventually regular semigroup S is called *weakly self-conjugate* if for all $a \in S$, $(a^n)' \in W(a^n)$, where a^n is a-regular $aHa^{n-1}(a^n)' \subseteq H$ and $a^{n-1}(a^n)'Ha \subseteq H$.

Example 1. Let $S := \{a, b, c, d, e\}$ defined by the multiplication as below:

$\circ$	a	b	c	d	e
a	a	a	a	d	d
b	a	b	c	d	d
c	a	c	b	d	d
d	d	d	d	a	a
e	d	e	e	a	a.

Then S is an eventually regular semigroup under usual multiplication. Now, we have $E(S) = \{a, b\}$ and $a \in W(e^2)$, where e^2 is e-regular.

Let $H = \{a, b, c\}$. Then $E(S) \subseteq H$ and H satisfies a weakly self-conjugate subset of S.

For any nonempty subset H of a semigroup S, we define a relation τ on S as follows:

$$\tau := \{(a, b) \in S \times S \,|\, \forall a^{n-1}(a^n)' \in W(a)\, \exists b^{m-1}(b^m)' \in W(b) \text{ such that } axa^{n-1}(a^n)' = bxb^{m-1}(b^m)',\ a^{n-1}(a^n)'xa = b^{m-1}(b^m)'xb,\ \forall x \in H, \text{ where } (a^n)' \in W(a^n),\ a^n \text{ is } a\text{-regular and } (b^m)' \in W(b^m),\ b^m \text{ is } b\text{-regular and } \forall\, b^{m-1}(b^m)' \in W(b)\, \exists a^{n-1}(a^n)' \in W(a) \text{ such that } axa^{n-1}(a^n)' = bxb^{m-1}(b^m)',\ a^{n-1}(a^n)'xa = b^{m-1}(b^m)'xb,\ \forall x \in H, \text{ where } (a^n)' \in W(a^n),\ a^n \text{ is } a\text{-regular and } (b^m)' \in W(b^m),\ b^m \text{ is } b\text{-regular}\}.$$

Note that τ may be an empty set. If S is an eventually regular semigroup, then $(a, a) \in \tau$, for all $a \in S$, so τ is not an empty set.

For basic concepts in semigroup theory, see [6] and [3].

The following results are used in this research.

Lemma 1.1. *Let S be a semigroup and $a \in S$. If a^n is a-regular, then $W(a^n) \neq \varnothing$.*

Proof. Let $n \in \mathbb{N}$ and a^n be a-regular, there exists $x \in S$ such that $a^n = a^n x a^n$ and $a^n x \in E(S)$. Then

$$a^n x = (a^n x)(a^n x),$$

$$(xa^n x)a^n(xa^n x) = x[(a^n x)(a^n x)](a^n x)$$

$$= x(a^n x)(a^n x) = xa^n x.$$

Thus $xa^n x \in W(a^n)$. □

Proposition 1.2. *Let S be any semigroup and* $\mathbb{N}$ *be the positive integer. Then*

(1) *for all* $(a^n)' \in W(a^n)$, $(a^n)'a^n$, $a^n(a^n)' \in E(S)$.

(2) *for all* $n \in \mathbb{N}$, $n > 1$, $(a^n)' \in W(a^n)$, $a^{n-1}(a^n)'a$, $a(a^n)'a^{n-1} \in E(S)$.

Proof. (1) Let $n \in \mathbb{N}$ and $(a^n)' \in W(a^n)$. Then $(a^n)' = (a^n)'a^n(a^n)'$ and $(a^n)'a^n = ((a^n)'a^n)((a^n)'a^n) = ((a^n)'a^n)^2$ and $a^n(a^n)' = (a^n)(a^n)'(a^n)(a^n)' = ((a^n)(a^n)')^2$.

(2) Let $n > 1$ and $(a^n)' \in W(a^n)$. Thus

$$(a^{n-1}(a^n)'a)(a^{n-1}(a^n)'a) = a^{n-1}(a^n)'(aa^{n-1})(a^n)'a$$

$$= a^{n-1}(a^n)'a^n(a^n)'a = a^{n-1}(a^n)'a.$$

$$(a(a^n)'a^{n-1})(a(a^n)'a^{n-1}) = a(a^n)'(a^{n-1}a)(a^n)'a^{n-1} = a(a^n)'a^{n-1}.$$ □

Proposition 1.3. *If S is an E-semigroup,* $a \in S$, a^n *is a-regular and* e, $f \in E(S)$, $(a^n)' \in W(a^n)$, *then*

(1) $e(a^n)'$, $(a^n)'f$, $f(a^n)'e \in W(a^n)$,

(2) $a^n e(a^n)'$, $(a^n)'ea^n \in E(S)$,

(3) $ae(a^{n-1})(a^n)'$, $a^{n-1}(a^n)'ea \in E(S)$,

(4) $a^{n-1}(a^n)', (a^n)' a^{n-1} \in W(a)$,

(5) $fa^{n-1}(a^n)', a^{n-1}(a^n)' e, fa^{n-1}(a^n)' e \in W(a)$,

(6) $f(a^n)' a^{n-1}, (a^n)' a^{n-1}e, f(a^n)' a^{n-1}e \in W(a)$.

Proof. (1) and (2) by [4].

(3) Note that

$$[aea^{n-1}(a^n)'][aea^{n-1}(a^n)'] = aea^{n-1}(a^n)' aea^{n-1}(a^n)' a^n (a^n)'$$

$$= a[ea^{n-1}(a^n)' a][ea^{n-1}(a^n)' a](a^{n-1})(a^n)'$$

$$= aea^{n-1}(a^n)' a(a^{n-1})(a^n)'$$

$$= aea^{n-1}[(a^n)' a^n (a^n)'] = aea^{n-1}(a^n)'$$

and

$$(a^{n-1}(a^n)' ea)(a^{n-1}(a^n)' ea) = a^{n-1}[(a^n)' a^n (a^n)'] ea^n (a^n)' ea$$

$$= a^{n-1}(a^n)'[a^n (a^n)' e][a^n (a^n)' e]a$$

$$= a^{n-1}(a^n)' ea.$$

(4) If $(a^n)' \in W(a^n)$, then $a^{n-1}(a^n)', (a^n)' a^{n-1} \in W(a)$. That is,

$$(a^{n-1}(a^n)')a(a^{n-1}(a^n)') = a^{n-1}(a^n)' a^n (a^n)' = a^{n-1}(a^n)'$$

and

$$((a^n)' a^{n-1})a(a^n)' a^{n-1} = (a^n)' a^n (a^n)' a^{n-1} = (a^n)' a^{n-1}.$$

(5) and (6) follow from (4) and by [4]. □

Proposition 1.4. *If S is a semigroup, $((ac)^n)' \in W((ac)^n)$ and $(ac)^n$ is ac-regular, then $c(ac)^{n-1}((ac)^n)' a, c((ac)^n)'(ac)^{n-1} a \in E(S)$.*

Proof. Let $((ac)^n)' \in W((ac)^n)$. Then

$$\begin{aligned}(c(ac)^{n-1}((ac)^n)'a)(c(ac)^{n-1}((ac)^n)'a) &= c(ac)^{n-1}((ac)^n)'ac(ac)^{n-1}((ac)^n)'a\\ &= c(ac)^{n-1}[((ac)^n)'(ac)^n((ac)^n)']a\\ &= c(ac)^{n-1}((ac)^n)'a.\end{aligned}$$

Similarly, we have $c((ac)^n)'(ac)^{n-1}a \in E(S)$. □

Proposition 1.5. *If S is an E-semigroup, a^n is a-regular and b^m is b-regular, then $W(b^m)W(a^n) \subseteq W(a^n b^m)$. If S is commutative, then $W(a^n b^n) = W((ab)^n)$.*

Proof. Let $(a^n)' \in W(a^n)$, $(b^m)' \in W(b^m)$. Then

$$\begin{aligned}(b^m)'(a^n)'a^n b^m (b^m)'(a^n)' &= (b^m)'b^m(b^m)'(a^n)'a^n b^m (b^m)'(a^n)'a^n(a^n)'\\ &= (b^m)'b^m(b^m)'(a^n)'a^n(a^n)'\\ &= (b^m)'(a^n)'.\end{aligned}$$

Therefore, $(b^m)'(a^n)' \in W(a^n b^m)$ and so $W(b^m)W(a^n) \subseteq W(a^n b^m)$.

If S is commutative semigroup, then $W(a^n b^n) = W((ab)^n)$. □

The following lemma, Edwards [1] investigated the maximum idempotent separating congruence on an eventually regular semigroup which used by Green's relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$.

Lemma 1.6 [1]**.** *The following are equivalent for a congruence ρ on an eventually regular semigroup S.*

(1) $\rho \subseteq \mu$,

(2) *for all* $e \in E(S)$ *and for all* $b \in S$, $e\rho b$ *implies* $\mathcal{H}_e \leq \mathcal{H}_b$,

(3) *for all* $a \in Reg(S)$ *and for all* $b \in S$, $a\rho b$ *implies* $\mathcal{H}_a \leq \mathcal{H}_b$,

(4) ρ *is idempotent separating congruence on S*,

where Edwards [1] *defined a relation* μ *as follows*:

$$(a, b) \in \mu \Leftrightarrow \text{ if } x \in Reg(S), \text{ then each of } x\mathcal{R}xa, \ x\mathcal{R}xb \text{ implies } xa\mathcal{H}xb \text{ and each of } x\mathcal{L}ax, \ x\mathcal{L}bx \text{ implies } ax\mathcal{H}bx$$

and μ *is the maximum idempotent separating congruence on an eventually regular semigroup*.

Luo and Li [3] gave the maximum idempotent separating congruence on an eventually regular semigroup as follows.

Theorem 1.7 [3]**.** *Let S be an eventually regular semigroup and* ρ *be a congruence on S*. *Then the following are equivalent*:

(1) $\rho \subseteq \mu$.

(2) ρ *is idempotent separating congruence*, *where*

$$\mu := \left\{(a, b) \in S \times S \,|\left(\begin{matrix}(\forall a' \in W(a))(\exists b' \in W(b)), \ (aa' = bb, \ a'a = b'b)\\ (\forall b' \in W(b))(\exists a' \in W(a)), \ (aa' = bb, \ a'a = b'b)\end{matrix}\right)\right\}.$$

Moreover, μ *is the maximum idempotent separating congruence on S*.

2. Main Results

The next theorem is analogous to the result for eventually regular semigroups as Theorem 2.4 in [3].

Theorem 2.1. *Let S be an eventually regular semigroup and* ρ *be a congruence on S*. *Then* ρ *is an idempotent separating congruence on S if and only if* $\rho \subseteq \mu^*$ *where*

$$\mu^* := \{(a, b) \in S \times S \,|\, \forall a^{n-1}(a^n)' \in W(a) \exists b^{m-1}(b^m)' \in W(b), \text{ where } (a^n)' \in W(a^n), \ a^n \text{ is } a\text{-regular and } (b^m)' \in W(b^m), \ b^m \text{ is } b\text{-regular, such that } a^{n-1}(a^n)'a = b^{m-1}(b^m)'b \text{ and } aa^{n-1}(a^n)' = bb^{m-1}(b^m)' \text{ and } \forall b^{m-1}(b^m)' \in W(b) \exists a^{n-1}(a^n)' \in W(a), \text{ where } (a^n)' \in W(a^n), \ a^n \text{ is } a\text{-regular and } (b^m)' \in W(b^m), \ b^m \text{ is } b\text{-regular, such that } a^{n-1}(a^n)'a = b^{m-1}(b^m)'b, \ aa^{n-1}(a^n)' = bb^{m-1}(b^m)'\}.$$

Moreover, μ^* *is the maximum idempotent separating congruence on an eventually regular semigroup S.*

Proof. $(\Rightarrow)$ Let ρ be an idempotent separating congruence on S and $a, b \in S$ with $(a, b) \in \rho$. Let $a^{n-1}(a^n)' \in W(a)$, where $(a^n)' \in W(a^n)$, a^n is a-regular, we have $a^{n-1}(a^n)' a \rho a^{n-1}(a^n)' b$. Let m be the smallest positive integer such that $(a^{n-1}(a^n)' b)^m$ is $a^n(a^n)' b$-regular. Since ρ is a congruence and $a^{n-1}(a^n)' a \in E(S)$, by Howie [2], $a^{n-1}(a^n)' a \rho (a^{n-1}(a^n)' b)^m$ and so $a^{n-1}(a^n)' a \mathcal{H} (a^{n-1}(a^n)' b)^m$ by Lemma 1.6 (3), that is, $(a^{n-1}(a^n)' b)^m \in H(a^{n-1}(a^n)' a)$. Let $(c^m)'$ is weak inverse of $(a^{n-1}(a^n)' b)^m$ in $H(a^{n-1}(a^n)' a)$. Consider

$$\begin{aligned} &((c^m)'(a^{n-1}(a^n)' b)^{m-1} a^{n-1}(a^n)')b((c^m)'(a^{n-1}(a^n)' b)^{m-1} a^{n-1}(a^n)') \\ &= ((c^m)'(a^{n-1}(a^n)' b)^m (c^m)')(a^{n-1}(a^n)' b)^{m-1} a^{n-1}(a^n)' \\ &= (c^m)'(a^{n-1}(a^n)' b)^{m-1} a^{n-1}(a^n)'. \end{aligned}$$

We have $(c^m)'(a^{n-1}(a^n)' b)^{m-1} a^{n-1}(a^n)' \in W(b)$ and so, we choose $b^{m-1}(b^m)' = (c^m)'(a^{n-1}(a^n)' b)^{m-1} a^{n-1}(a^n)'$. Thus $b^{m-1}(b^m)' b = (c^m)'(a^{n-1}(a^n)' b)^{m-1} a^{n-1}(a^n)' b$ $= (c^m)'(a^{n-1}(a^n)' b)^m = (a^{n-1}(a^n)' a)$ because $\mathcal{H}$-class contains at most one idempotent.

On the other hand, $b(b^{m-1}(b^m)' = b(c^m)'(a^{n-1}(a^n)' b)^{m-1} a^{n-1}(a^n)'$ and $a^{n-1}(a^n)' a \rho (a^{n-1}(a^n)' b)^m$.

Thus

$$\begin{aligned} bb^{m-1}(b^m)' &= b(c^m)'(a^{n-1}(a^n)' b)^{m-1} a^{n-1}(a^n)' \\ &= b(c^m)'(a^{n-1}(a^n)' b)^{m-1} a^{n-1}(a^n)' a a^{n-1}(a^n)' \end{aligned}$$

and

$$\begin{aligned} &(b(c^m)'(a^{n-1}(a^n)' b)^{m-1} a^{n-1}(a^n)' a a^{n-1}(a^n)')\rho \\ &= (b(c^m)'(a^{n-1}(a^n)' b)^{m-1}(a^{n-1}(a^n)' b) a^{n-1}(a^n)')\rho \end{aligned}$$

and

$$b(c^m)'(a^{n-1}(a^n)'b)^{m-1}(a^{n-1}(a^n)'b)a^{n-1}(a^n)' = b(c^m)'(a^{n-1}(a^n)'b)^m a^{n-1}(a^n)'$$

but $b(c^m)'(a^{n-1}(a^n)'b)^m = a^{n-1}(a^n)'a,$ we have

$$bb^{m-1}(b^m)'\rho b(c^m)'(a^{n-1}(a^n)'b)^m a^{n-1}(a^n)'\rho ba^{n-1}(a^n)'aa^{n-1}(a^n)' = ba^{n-1}(a^n)'$$

and $ba^{n-1}(a^n)'\rho aa^{n-1}(a^n)'$ and ρ is an idempotent separating congruence on *S*, so $bb^{m-1}(b^m)' = aa^{n-1}(a^n)'$.

We can show that for all $b^{m-1}(b^m)' \in W(b)$ there exists $a^{n-1}(a^n)' \in W(a)$ such that $b^{m-1}(b^m)'b = a^{n-1}(a^n)'a$, and $bb^{m-1}(b^m)' = aa^{n-1}(a^n)'$, hence $\rho \subseteq \mu^*$.

$(\Leftarrow)$ Suppose that ρ is a congruence on *S* and $\rho \subseteq \mu^*$. To show that ρ is an idempotent separating congruence on *S*.

Let $e, f \in E(S)$ with $(e, f) \in \rho$ and let $e \in W(e)$. Then there exists $f' \in W(f)$ such that $e = ee = ff' = f'f$ and so $e = ef = fe$. For all $f \in W(f)$ there exists $e' \in W(e)$ such that $f = ff = e'e = ee'$ and $ef = f = fe$, hence $e = f$.

Therefore ρ is an idempotent separating congruence on *S*. An analog to the proof as Theorem 2.6 in [3], we get that μ^* is the maximum idempotent separating congruence on *S*. □

Note that, every eventually regular semigroup is an *E*-inversive semigroup. In 2002, Weipoltshammer [6] gave some characterizations for idempotent separating congruence on an *E*-inversive *E*-semigroup, whose $E(S)$ was a rectangular band, and $\mu := \{(a, b) \in S \times S \mid \forall a' \in W(a) \exists b' \in W(b)$ such that $aea' = beb'$, $a'ea = b'eb$, for all $e \in E(S)$ and $\forall b' \in W(b) \exists a' \in W(a)$ such that $aea' = beb'$, $a'ea = b'eb$, for all $e \in E(S)\}$. Luo and Li [3] gave μ for this kinds of congruences by mean defined above on an eventually orthodox semigroup. Now, we give a similar description for the maximum idempotent separating congruence on an eventually regular *E*-semigroup as Siripitukdet and Sattayaporn [5] did for *E*-inversive *E*-semigroups by using weakly self-conjugate subsemigroup *H* of *S*.

Theorem 2.2. *Let S be an eventually regular E-semigroup and H be full and weakly self-conjugate subsemigroup of S. Then a relation*

$$\tau := \{(a, b) \in S \times S \mid \forall a^{n-1}(a^n)' \in W(a) \exists b^{m-1}(b^m)' \in W(b) \text{ such that } axa^{n-1}(a^n)' = bxb^{m-1}(b^m)',\ a^{n-1}(a^n)' xa = b^{m-1}(b^m)' xb,\ \forall x \in H, \text{ where } (a^n)' \in W(a^n), a^n \text{ is a-regular and } (b^m)' \in W(b^m),\ b^m \text{ is b-regular and } \forall b^{m-1}(b^m)' \in W(b) \exists a^{n-1}(a^n)' \in W(a) \text{ such that } axa^{n-1}(a^n)' = bxb^{m-1}(b^m)',\ a^{n-1}(a^n)' xa = b^{m-1}(b^m)' xb,\ \forall x \in H, \text{ where } (a^n)' \in W(a^n),\ a^n \text{ is a-regular and } (b^m)' \in W(b^m),\ b^m \text{ is b-regular}\}$$

is an idempotent separating congruence on S.

If $E(S) = H_{\omega_E}$*, then* τ *is the maximum idempotent separating congruence on S.*

Proof. Obviously, τ is reflexive and symmetric.

To show that τ is a transitive, let $a, b, c \in S$, $x \in H$ and $(a^n)' \in W(a^n)$, $(b^m)' \in W(b^m)$, where a^n is a-regular, b^m is b-regular be such that $a\tau b$ and $b\tau c$. Let $a^{n-1}(a^n)' \in W(a)$. Then there exists $(b^{m-1})(b^m)' \in W(b)$ such that $axa^{n-1}(a^n)' = bxb^{m-1}(b^m)'$, $a^{n-1}(a^n)' xa = b^{m-1}(b^m)' xb$, for all $x \in H$. Since $b\tau c$ and $b^{m-1}(b^m)' \in W(b)$, there exists $c^{k-1}(c^k)' \in W(c)$, where $(c^k)' \in W(c^k)$, c^k is c-regular, we have $bxb^{m-1}(b^m)' = cxc^{k-1}(c^k)'$, $b^{m-1}(b^m)' xb = c^{k-1}(c^k)' xc$, for all $x \in H$. Since m is the smallest positive integer, we have $axa^{n-1}(a^n)' = cxc^{k-1}(c^k)'$, $a^{n-1}(a^n)' xa = c^{k-1}(c^k)' xc$, for all $x \in H$. Similarly, we can show that for all $(c^{k-1})(c^k)' \in W(c)$, where $(c^k)' \in W(c^k)$, c^k is c-regular, there exists $a^{n-1}(a^n)' \in W(a)$, where $(a^n)' \in W(a^n)$, a^n is a-regular such that $axa^{n-1}(a^n)' = cxc^{k-1}(c^k)'$, $a^{n-1}(a^n)' xa = c^{k-1}(c^k)' xc$, for all $x \in H$. Hence $a\tau c$ and τ is a transitive. We shall show that τ is a compatible, let a, $b \in S$ with $a\tau b$ and let $(a^n)' \in W(a^n)$, where a^n is a-regular and $(b^m)' \in W(b^m)$, where b^m is b-regular.

For $a^{n-1}(a^n)' \in W(a)$, there exists $b^{m-1}(b^m)' \in W(b)$ such that $axa^{n-1}(a^n)' = bxb^{m-1}(b^m)'$, $a^{n-1}(a^n)'xa = b^{m-1}(b^m)'xb$, for all $x \in H$.

Let $c^{k-1}(c^k)' \in W(c)$, where $(c^k)' \in W(c^k)$ and c^k is c-regular. By $c^{k-1}(c^k)'a^{n-1}(a^n)' \in W(ac)$ and $c^{k-1}(c^k)'b^{m-1}(b^m)' \in W(bc)$, for all $x \in H$,

$$\begin{aligned}(ac)x(c^{k-1}(c^k)')a^{n-1}(a^n)' &= a(cxc^{k-1}(c^k)')a^{n-1}(a^n)' \\ &= b(cxc^{k-1}(c^k)')b^{m-1}(b^m)' \\ &= (bc)x(c^{kn-1}(c^n)'b^{m-1}(b^m)') \quad (\because cxc^{k-1}(c^k)' \in H)\end{aligned}$$

and

$$\begin{aligned}(c^{k-1}(c^k)'a^{n-1}(a^n)')xac &= c^{k-1}(c^k)'(a^{n-1}(a^n)'xa)c \\ &= c^{k-1}(c^k)'(b^{m-1}(b^m)'xb)c \\ &= (c^{k-1}(c^k)'b^{m-1}(b^m)')x(bc) \quad (\because b^{m-1}(b^m)'xb \in H).\end{aligned}$$

Similarly, we can show that the second part holds. Hence $ac\tau bc$, so τ is a right compatible. Similarly, we can show that τ is a left compatible. Therefore τ is a congruence on S. Let e, f be elements in S such that $e\tau f$. Since $e \in W(e)$, there exists $f' \in W(f)$ such that $exe = f'xf = fxf'$, for all $x \in H$. Since H is full, $e \in H$ and $e = eee = f'ef = fef'$, we have $ef = (f'ef)f = f'ef = e$. Now $f \in W(f)$. There exists $e' \in W(e)$ such that $fxf = e'xe = exe'$, for all $x \in H$. Since $f \in H$, $f = fff = e'fe = efe'$, it follows that $ef = e(efe') = efe' = f$. Therefore, $e = f$ and so τ is an idempotent separating congruence on S.

Suppose that $E(S) = H_{\omega_E}$. To show that τ is the maximum idempotent separating congruence on S, let ρ be an arbitrary idempotent separating congruence on S. Let $a, b \in S$ be such that $a\rho b$. By Theorem 2.1, $\rho \subseteq \mu^*$. Then, for all $a^{n-1}(a^n)' \in W(a)$, there exists $b^{m-1}(b^m)' \in W(b)$ such that

$$a^{n-1}(a^n)'a = b^{m-1}(b^m)'b,$$

where a^n is a-regular, b^m is b-regular. Consider

$$a^{n-1}(a^n)' = (a^{n-1}(a^n)'a(a^{n-1}(a^n)')\rho b^{m-1}(b^m)'ba^{n-1}(a^n)'$$

and

$$b^{m-1}(b^m)'ba^{n-1}(a^n)' = b^{m-1}(b^m)'ba^{n-1}(a^n)'bb^{m-1}(b^m)'$$

and

$$[b^{m-1}(b^m)'ba^{n-1}(a^n)'bb^{m-1}(b^m)']\rho[b^{m-1}(b^m)'a(a^{n-1})(a^n)'bb^{m-1}(b^m)']$$

and

$$b^{m-1}(b^m)'a(a^{n-1})(a^n)'bb^{m-1}(b^m)' = b^{m-1}(b^m)'.$$

Hence $a^{n-1}(a^n)'\rho b^{m-1}(b^m)'$.

For any $x \in H$ and $a\rho b$, $a^{n-1}(a^n)'\rho b^{m-1}(b^m)'$, we have

$$a^{n-1}(a^n)'xa\rho b^{m-1}(b^m)'xb \text{ and } axa^{n-1}(a^n)'\rho bxb^{m-1}(b^m)'.$$

Since H is a weakly self conjugate, we have $a^{n-1}(a^n)'xa$, $b^{m-1}(b^m)'xb \in H$. Since $E(S) = H_{\omega_E}$, we get

$$a^{n-1}(a^n)'xa = a^{n-1}(a^n)'aa^{n-1}(a^n)'xa \in H,$$

so $a^{n-1}(a^n)'xa \in H_{\omega_E} = E(S)$.

Similarly, $b^{m-1}(b^m)'xb = b^{m-1}(b^m)'bb^{m-1}(b^m)'xb \in H$. So, $b^{m-1}(b^m)'xb \in H_{\omega_E} = E(S)$. Since ρ is an idempotent separating congruence, we get $a^{n-1}(a^n)'xa = b^{m-1}(b^m)'xb$. The proof of the second part is similarly. Therefore, $\rho \subseteq \tau$. Hence τ is the maximum idempotent separating congruence on S. □

Corollary 2.3. *Let S be an eventually regular E-semigroup. Then a relation*

$$\zeta' := \{(a, b) \in S \times S \mid \forall a^{n-1}(a^n)' \in W(a) \exists b^{m-1}(b^m)' \in W(b) \text{ such that } aea^{n-1}(a^n)' = beb^{m-1}(b^m)', a^{n-1}(a^n)'ea = b^{m-1}(b^m)'eb \ \forall e \in E(S), \text{ where } (a^n)' \in W(a^n), a^n \text{ is } a\text{-regular}, (b^m)' \in W(b^m), b^m \text{ is } b\text{-regular and } \forall b^{m-1}(b^m)' \in W(b) \exists a^{n-1}(a^n)' \in W(a) \text{ such that } aea^{n-1}(a^n)' = beb^{m-1}(b^m)', a^{n-1}(a^n)'ea = b^{m-1}(b^m)'eb, \ \forall e \in E(S), \text{ where } (a^n)' \in W(a^n), a^n \text{ is } a\text{-regular}, (b^m)' \in W(b^m), b^m \text{ is } b\text{-regular}\}$$

is the maximum idempotent separating congruence on S.

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