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ON ARMENDARIZ RING AND ITS GENERALIZATIONS

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Abstract

The paper deals with Armendariz rings, and their relationships with well-known rings. Then we treat generalizations of Armendariz rings, such as McCoy ring, weak Armendariz rings, π-Armendariz rings and their links.

1. Introduction

Throughout this paper, all rings are associative with identity unless otherwise stated. Rege and Chhawchharia [13] introduced the notion of Armendariz ring. A ring R is called *Armendariz*, if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i and j. (The converse is always true.) The term Armendariz ring is chosen because Armendariz [3] had noted that a reduced ring (i.e., a ring without nonzero nilpotent

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elements) satisfies this condition. Some properties of Armendariz rings have been studied in Rege and Chhawchharia [13], Anderson and Camillo [1], Kim and Lee [8], and Hirano [4]. Generalizations of Armendariz rings have been investigated in Antoine [2], Huh et al. [6], and Liu and Zhao [11].

In this paper, we explore relationships between several classes of rings, provide examples confirming these relationships and prove some statements about them. Some of the examples are well known, others are not. However, our aim is to bring them together to represent whole picture and make some conclusions.

The organization of the paper is as follows. First, we consider the relationships between Armendariz rings and some other classes of rings, in Section 2, then we treat generalizations of Armendariz rings, such as McCoy ring, weak Armendariz and π-Armendariz rings, in Section 3.

For any ring R and for any positive integer n, the ring of all $n \times n$ upper triangular matrices over R is denoted by $T_n(R)$. The set of all nilpotent elements in R and the prime radical (i.e., the intersection of all prime ideals) are denoted by $N(R)$ and $P(R)$, respectively.

2. The Relationships between Armendariz Ring and Some Other Rings

2.1. von Neumann regular rings

Recall that a ring R is said to be *von Neumann regular*, if $a \in aRa$ for any $a \in R$. Every Boolean ring is von Neumann regular. Reduced rings are Armendariz, but the converse is not true. Anderson and Camillo [1] proved that a von Neumann regular ring is Armendariz, if and only if it is reduced.

Proposition 2.1. *A commutative von Neumann regular ring is Armendariz.*

Proof. Let R be a commutative von Neumann regular ring and a be an element of R. Suppose that $a^2 = 0$. By the hypothesis, $a = aba = a^2b = 0$, hence R is reduced. Therefore, it is Armendariz. □

A field is an example of Armendariz ring.

2.2. Baer and *p.p.*-rings

A ring is said to be *abelian*, if every its idempotent is central [8]. Kim and Lee proposed without proving that Armendariz rings are abelian. By the following lemma, we prove it.

Lemma 2.1. *An Armendariz ring is abelian*.

Proof. Let e be an idempotent element of R. We put $f(x) = e - ea(1-e)x$ and $g(x) = (1-e) + ea(1-e)x$, then $f(x)g(x) = 0$. Hence, $ea(1-e) = 0$, since R is Armendariz, and then $ea = eae$. Since $(1-e)$ is idempotent and from the similar observation we have, $(1-e)ae = 0$, and hence, $ae = eae$. So e is central, therefore R is abelian. □

Here is an example from [8] showing that abelian ring need not to be Armendariz.

Example 2.1. Let $\mathbb{Z}$ be the ring of integers and let

$$R = \left\{\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mid a - b \equiv c \equiv 0 \ (\text{mod } 2)\right\}.$$

The only idempotents of R are $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so R is abelian. Let $f(x) = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}x$, $g(x) = \begin{pmatrix} 0 & 2 \\ 0 & -2 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}x \in R[x]$. Then $f(x)g(x) = 0$ but $\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \neq 0$, hence, R is not an Armnedariz ring.

By Kaplansky [7], a ring R is called a *right p.p.-ring*, if the right annihilator of each element of R is generated by an idempotent. A ring R is called *Baer*, if the right annihilator of every nonempty subset of R is generated by an idempotent. We denote the right annihilator over a ring R by $r_R(-)$. It is easy to see that a Baer ring is right *p.p.*-ring, a domain is Baer. Any Baer ring has nonzero central nilpotent element and so a commutative Baer ring is reduced. A commutative Baer ring is Armendariz. However, $Mat_2(\mathbb{Z}_2)$ is an example of Baer ring [9], that is not Armendariz. Thus, a noncommutative Baer ring needs not to be Armendariz.

Proposition 2.2. *An abelian right p.p.-ring is Armendariz*.

Proof. Indeed, let e be an idempotent elements of R. Suppose that $r^2 = 0$. Since

$r \in r_R(r) = eR$, there exists $r' \in R$ such that $r = er'$, and then $er = e^2r' = er'$. So $r = er = re = 0$, since e is central. Then R is reduced, hence R is Armendariz. □

Thus, we conclude that an Armendariz ring is abelian and an abelian, right *p.p.*-ring is Armendariz. A Boolean ring is a *p.p.*-ring.

Here is an example of ring that is commutative, Boolean, von Neumann regular, *p.p.*-ring and reduced, Armendariz, but is not Baer (see [9]). We use the so-called Dorroh extension.

Example 2.2. Let $S_0 = \mathbb{Z}_2, S_1 = \mathbb{Z}_2 * \mathbb{Z}_2, S_3 = S_2 * \mathbb{Z}_2, \ldots, S_n = S_{n-1} * \mathbb{Z}_2, \ldots,$ where the operation on S_n is defined as follows: for $(a, \bar{b}), (c, \bar{d}) \in S_n$ with $a, c \in S_{n-1}$ and $(a, \bar{b}) + (c, \bar{d}) = (a + c, \overline{b + d})$ and $(a, \bar{b}), (c, \bar{d}) = (ac + bc + da, \overline{bd})$, where $n = 1, 2, \ldots$. It is clear, there is the ring-monomorphism $f : S_{n-1} \to S_n$ defined by $f(x) = (x, 0)$. Now construct the direct product $\prod_{n=1}^{\infty} S_n$ with $S_1 \subset S_2 \subset \ldots$ and consider $R = \left\langle \bigoplus_{n=1}^{\infty} S_n, 1_s \right\rangle$. The last is a $\mathbb{Z}_2$-subalgebra of $\prod_{n=1}^{\infty} S_n$, generated by $\bigoplus_{n=1}^{\infty} S_n$ and 1_s, where $S = \prod_{n=1}^{\infty} S_n$.

Then the ring R is the required example.

2.3. Semi-commutative rings

In this section, we study relationships between Armendariz rings and semi-commutative rings. Reduced rings can be included in the class of Armendariz rings and the class of semi-commutative rings, and both of them are abelian. So, it is natural to explore the relationships between them.

A ring is said to be *semi-commutative*, if it satisfies the following condition: whenever elements a, b in R satisfy $ab = 0$, then $acb = 0$ for every element c of R.

Recall that a ring R is called 2-*primal*, if the prime radical of R coincides with the set of all nilpotent elements of it $(P(R) = N(R))$.

Lemma 2.2. *A ring R is* 2-*primal*, *if and only if* $R/P(R)$ *is reduced*.

Proof. Let R be a 2-primal ring. Then it is easy to see that $R/N(R)$ is reduced, hence, $R/P(R)$ is reduced. The "only if" part is obvious. □

Proposition 2.3. *A semi-commutative ring is* 2*-primal*.

Proof. Suppose that R is a semi-commutative ring. Then $(a + P(R))^2 = P(R)$, where $a \in R$. Also so, $a^2 \in P(R)$. Therefore, $aRa \subset P(R)$, since R is semi-commutative. Then $aRa = (aR)(Ra) \subset P(R) = \bigcap I_i$, where $\{I_i\}$ is the set of all prime ideals of R. We get $a \in I_i$ for each i, and it implies that $a \in \bigcap I_i = P(R)$. Hence, $R/P(R)$ is reduced, then R is 2-primal with the help of the previous lemma. □

Proposition 2.4. *A semi-commutative ring is abelian*.

Proof. Let e be an idempotent element of a semi-commutative ring R. Then $e^2 - e = 0$. Since $ea(e-1) = 0$ for each element a of R, we get $ea = eae.$ Likewise, we can prove that $ae = eae.$ So, e is central, then R is abelian. □

Semi-commutative rings are abelian, but the converse is not true. Following the example 2.1, R is abelian. $\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & -2 \end{pmatrix} = 0,$ but $\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & -2 \end{pmatrix} \neq 0,$ so R is not semi-commutative.

Concerning some kinds of rings related to the semi-commutative rings, we make the following observations:

1. Division rings are semi-commutative, in particular, fields are semi-commutative by simple computations.

$$\text{field} \Rightarrow \text{reduced} \Rightarrow \text{semi-commutative} \Rightarrow \text{abelian}.$$

2. Recall that a ring R is called *symmetric*, if $abc = 0 \Rightarrow bac = 0,$ for all a, b, $c \in R$, reversible, if $ab = 0 \Rightarrow ba = 0$, for all $a, b \in R$. The following implications hold by a simple computation:

commutative reduced $\Rightarrow$ symmetric $\Rightarrow$ reversible $\Rightarrow$ semi-commutative $\Rightarrow$ abelian.

3. Von Neumann regular Armendariz rings are semi-commutative, since reduced rings are semi-commutative. Finite Armendariz rings are semi-commutative [5].

2.4. Quasi-Armendariz rings

Following Hirano, a ring R is said to be *quasi-Armendariz*, if whenever two polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)R[x]g(x) = 0$, we have $a_i R b_j = 0$ for every i and j. For a ring R, put $rAnn_R(2^R) = \{r_R(U) | U \subseteq R\}$ and $rAnn_R(id(R)) = \{r_R(U) | U$ is an ideal of $R\}$. According to Hirano, a ring R is Armendariz, if $rAnn_R(2^R) \to rAnn_{R[X]}(2^{R[X]})$ defined as $A \to AR[x]$ is bijective. Hirano proved that a ring R is quasi-Armendariz, if $rAnn_R(id(R)) \to rAnn_{R[X]}(idR[x]))$ defined as $A \to AR[x]$ is bijective.

First, we recall that for semi-commutative, in particular, reduced, commutative, symmetric, reversible rings to be Armendariz and quasi-Armendariz is equivalent.

Proposition 2.5. *Reduced rings are quasi-Armendariz.*

Proof. First, we notice that a reduced ring is Armendariz. Then the proof is obvious, if we take into account the fact that, for reduced rings to be Armendariz and quasi-Armendariz is equivalent. □

Corollary 2.1. *If R is a reduced ring, then for any positive integer n, the ring $T_n(R)$ is quasi-Armendariz.*

Proof. This corollary follows from [4] and the previous proposition. □

Given a ring R and a bimodule ${}_R M_R$. The *trivial extension* of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the multiplication defined as:

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2).$$

Corollary 2.2. *Let R be a reduced ring and ${}_R M_R$ be a bimodule. Then the trivial extension of R by M, $T(R, M)$ is a quasi-Armendariz ring.*

Proof. Note that $T(R, M)$ is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used, hence, $T(R, M)$ is quasi-Armendariz. □

Corollary 2.3. *For any positive integer n the ring* $\mathbb{Z}_n$ *is quasi-Armendariz.*

Proof. It is derived from the commutativity of $\mathbb{Z}_n$. □

For a natural number *n*, which is not square free, the ring $\mathbb{Z}_n$ is not reduced, but it is quasi-Armendariz [13].

Let *R* be a commutative ring, $h : R \to R$ be a ring homomorphism and *M* be an *R*-module. Consider a ring structure on $R \oplus M$, which was denoted in [13] by $R(+)_h(M)$, where the product is defined by:

$$(a, m)(b, n) = (ab, h(a)n + bm).$$

Proposition 2.6. *If K is a field and V is a vector space over K, then the ring* $K(+)V$ *is quasi-Armendariz.*

Proof. Since $K(+)V$ is commutative Armendariz, therefore it is quasi-Armendariz. □

Here are some conclusions concerning quasi-Armendariz rings:

1. If *R* is Armendariz and von Neumann regular ring, then it is quasi-Armendariz by Proposition 2.5 (a reduced ring is quasi Armendariz) and [1].

2. A commutative von Neumann regular ring is quasi-Armendariz, in particular, a field is quasi-Armendariz.

3. A semi-commutative quasi-Armendariz ring is abelian.

4. An abelian right *p.p.*-ring is quasi-Armendariz, in particular, an abelian Baer ring is quasi-Armendariz.

5. A commutative Baer ring is quasi-Armendariz [9].

2.4.1. Gaussian rings

For $f(x) \in R[x]$, the content $c(f)$ of $f(x)$ is the ideal of *R*, generated by the coefficients of $f(x)$. A commutative ring *R* with identity is *Gaussian*, if $c(fg) = c(f)c(g)$, for all $f(x), g(x) \in R[x]$. Gaussian rings are Armendariz, but the converse is false. Any integral domain is Armendariz, but it is not necessarily Gaussian [1]. The homomorphic image of a Gaussian ring is Gaussian, the homomorphic image of an Armendariz ring need not to be Armendariz. For example, $\mathbb{Z}(+)\mathbb{Z}/8\mathbb{Z}$ is Armendariz, but its homomorphic image $\mathbb{Z}_8(+)\mathbb{Z}_8$ is not Armendariz [13]. Anderson and Camillo noted that the exact relationship between

Gaussian and Armendariz rings is as follows. A commutative ring is Gaussian, if and only if every its homomorphic image is Armendariz. A field is Gaussian, hence it is Armendariz.

Lemma 2.3. *Gaussian rings are quasi-Armendariz*.

Proof. The proof follows from the fact that a Gaussian ring is Armendariz and, for commutative rings, to be Armendariz and quasi-Armendariz is equivalent.

$$\text{Field} \Rightarrow \text{Gaussian} \Rightarrow \text{Quasi-Armendariz.} \qquad \square$$

3. Generalizations of Armendariz rings

3.1. McCoy rings

Recall that a ring R is a *left McCoy*, if whenever $g(x)$ is a right zero-divisor in $R[x]$, there exists a non-zero element c in R such that $cg(x) = 0$. *Right McCoy* ring is defined dually. A ring is said to be *McCoy ring*, if it is both left and right McCoy. Armendariz rings are McCoy [13]. The converse is not true. Commutative rings are McCoy [15], but there are examples of commutative non-Armendariz rings, for instance, $\mathbb{Z}_8(+)\mathbb{Z}_8$ [13]. Both semi-commutative and McCoy rings are generalizations of commutative rings, but semi-commutative rings are not McCoy [12]. Semi-commutative rings are right McCoy [4].

Here is an example of a noncommutative McCoy ring that is not Armendariz.

Example 3.1. Let R be a reduced ring and let

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & . & . & . & a_{1n} \\ 0 & a & a_{23} & . & . & . & a_{2n} \\ 0 & 0 & a & . & . & . & a_{3n} \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ 0 & 0 & 0 & . & . & . & a \end{pmatrix} : a,\, a_{ij} \in R \right\}.$$

Then R_n is McCoy for any $n \geq 1$ [10], but it is not Armendariz for $n \neq 1, 3$ [8].

3.2. Weak Armendariz rings and π-Armendariz rings

In this section, we concern the structure of weak Armendariz and π-Armendariz rings which are generalizations of Armendariz rings. A ring is *weak Armendariz*, if whenever the product of two polynomials is zero, then all the products of their coefficients are nilpotent. A π-Armendariz ring satisfies a stronger condition that, if the product of two polynomials has coefficients in the set of nilpotent elements, then the product of the coefficients of the polynomials is also nilpotent. Following [11], a ring R is said to be *weak Armendariz*, if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j \in N(R)$ for each i and j. Following [2] and [6], a ring R is called π-*Armendariz*, if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)g(x) \in N(R[x])$, then $a_ib_j \in N(R)$ for each i and j.

π-Armendariz ring is a generalization of 2-primal [6] and Armendariz rings, and it is a particular case of the weak Armendariz rings

$$\text{Armendariz} \Rightarrow \pi\text{-Armendariz} \Rightarrow \text{weak Armendariz}.$$

Theorem 3.1. *Let R be a reduced ring. Then the following hold*:

1. $T_n(R)$ *is quasi-Armendariz.*

2. $T_n(R)$ *is weak Armendariz.*

Proof. (1) Follows from Corollary 2.1.

(2) Let $I = \{A \in T_n(R) |$ each diagonal entry of A is zero$\}$. Obviously, I is nilpotent ideal of $T_n(R)$ and $T_n(R)/I \cong R \oplus R \oplus \cdots \oplus R.$ It is well-known fact that the direct sum of Armendariz rings is Armendariz, so, $T_n(R)/I$ is Armendariz, hence it is π-Armendariz. Suppose that $f(x), g(x) \in T_n(R)[x]$ such that $f(x)g(x) \in N(T_n(R)[x])$, $\overline{f(x)}\,\overline{g(x)} \in N(T_n(R)/I)$, where $\overline{f(x)} = f(x) + I$, $\overline{g(x)} = g(x) + I$. Then $\overline{a}_i\overline{b}_j \in N(T_n(R)/I)$ for each i, j, since $T_n(R)/I$ is π-Armendariz and I is nilpotent. Clearly, $(\overline{a}_i\overline{b}_j)^n = I$ implies that $(a_ib_j)^n \in I$. Therefore, there exists $m \in N$ such that $((a_ib_j)^n)^m = (a_ib_j)^{nm} = 0$. Hence, $T_n(R)$ is π-Armendariz ring. □

Corollary 3.1. *A ring R is* π-*Armendariz*, *if and only if* $T_n(R)$ *is* π-*Armendariz for any n*.

Proof. It is known that any subring of π-Armendariz rings is π-Armendariz. Thus, if $T_n(R)$ is a weak Armendariz ring, so is R. The converse follows from [6]. □

For weak Armendariz, π-Armendariz and quasi-Armendariz rings, their trivial extension $T(R, R)$ is weak Armendariz, π-Armendariz and quasi-Armendariz. However, for Armendariz rings, the ring $T(R, R)$ may not be Armendariz [8].

Let R be a reduced ring. Then

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & . & . & . & a_{1n} \\ 0 & a & a_{23} & . & . & . & a_{2n} \\ 0 & 0 & a & . & . & . & a_{3n} \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ 0 & 0 & 0 & . & . & . & a \end{pmatrix} : a, a_{ij} \in R \right\}$$

is McCoy [10], quasi-Armendariz and π-Armendariz by the previous argument, but it is not Armendariz for $n \neq 1, 3$.

Proposition 3.1. *Let R be a ring and $n \geq 2$ be a natural number. If $R[x]/(x^n)$ is reduced, then R is π-Armendariz, where (x^n) is the ideal of $R[x]$ generated by x^n.*

Proof. Suppose that $R[x]/(x^n)$ is reduced. Then it is Armendariz, so R is reduced by [1], R is π-Armendariz. □

We make the following conclusions:

1. Antoine proved that, if $N(R)$ is a nil ideal of R, then R is π-Armendariz. For semi-commutative ring R the set $N(R)$ is an ideal [11], so semi-commutative rings are π-Armendariz, in particular, commutative rings are π-Armendariz.

Reduced $\Rightarrow$ semi-commutative $\Rightarrow$ 2-primal $\Rightarrow$ π-Armendariz.

2. Field $\Rightarrow$ Gaussian $\Rightarrow$ Armendariz $\Rightarrow$ π-Armendariz.

3. Semi-commutative quasi-Armendariz rings are π-Armendariz, in particular, commutative quasi-Armendariz rings are π-Armendariz.

References

[1] D. D. Anderson and V. Camillo, Armendariz rings and Gaussian rings, Comm. Algebra 26(7) (1998), 2265-2272.

[2] R. Antoine, Nilpotent elements and Armendariz rings, J. Algebra 319(8) (2008), 3128-3140.

[3] E. P. Armendariz, A note on extensions of Baer and *p.p.*-rings, J. Austral. Math. Soc. 18 (1974), 470-473.

[4] Y. Hirano, On annihilator ideals of a polynomial ring over a noncommutative ring, J. Pure Appl. Algebra 168(1) (2002), 45-52.

[5] C. Huh, Y. Lee and A. Smoktunowicz, Armendariz rings and semicommutative rings, Comm. Algebra 30(2) (2002), 751-761.

[6] C. Huh, C. I. Lee, K. S. Park and S. J. Ryu, On π-Armendariz rings, Bull. Korean Math. Soc. 44(4) (2007), 641-649.

[7] I. Kaplansky, Rings of operations, Mathematics Lecture Notes Series, Benjamin, New York, 1965.

[8] N. K. Kim and Y. Lee, Armendariz rings and reduced rings, J. Algebra 223(2) (2000), 477-488.

[9] Y. Lee, N. K. Kim and C. Y. Hong, Counter example on Baer rings, Comm. Algebra 25(2) (1997), 497-507.

[10] Z. Lei, J. Chen and Z. Ying, A question on McCoy rings, Bull. Austral. Math. Soc. 76 (2007), 137-141.

[11] Z. Liu and R. Zhao, On weak Armendariz rings, Comm. Algebra 34 (2006), 2607-2616.

[12] P. P. Nielsen, Semi-commutativity and the McCoy condition, J. Algebra 298(1) (2006), 134-141.

[13] M. B. Rege and S. Chhawchharia, Armendariz rings, Proc. Japan Acad. Ser. A Math. Sci. 73(1) (1997), 14-17.

[14] B. de law Rosa and G. A. P. Heyman, A note on radicals and the Dorroh extension, Arch. Math. 42 (1984), 516.

[15] W. R. Scott, Divisors of zero in polynomial rings, Amer. Math. Monthly 61 (1954), 336.