Far East Journal of Mathematical Education
Volume 3, Number 3, 2009, Pages 287-293
Published Online: December 28, 2009
This paper is available online at http://www.pphmj.com

SOME SUMMATION LIMIT PROBLEMS

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Abstract

Students may be asked to deal with some summation limits that could not be solved directly by the integral method. In this short paper, we consider them and present some useful solving methods.

1. Introduction

Students in studying calculus may be asked to consider a type of summation limit as follows:

$$\lim_{n\to\infty}\sum_{j=k_1n+1}^{k_2n}\left[f\left(\left(\frac{j^{\sigma}}{n^{1+\sigma}}\right)^{\frac{1}{m}}\right)-\sum_{i=0}^{m-1}\frac{1}{i!}f^{(i)}(0)\left(\frac{j^{\sigma}}{n^{1+\sigma}}\right)^{\frac{i}{m}}\right], \tag{1.1}$$

where $f^{(m)}(0)$ (the mth derivative of f at 0) exists for some positive integer m, and k_1 and k_2 are two nonnegative integers with $k_1 < k_2$. The following are such types of exercises:

$$\lim_{n\to\infty}\sum_{j=1}^{3n}\sin\left(\sin\frac{j}{n^2}\right),\ \lim_{n\to\infty}\sum_{j=n+1}^{2n}\tan\left(\arcsin\frac{j^4}{n^5}\right),$$

2000 Mathematics Subject Classification: 97C30, 97D40, 97D50.

Keywords and phrases: summation limits, Stolz theorem, differential, Taylor's formula.

Received September 7, 2009

$$\lim_{n\to\infty}\sum_{j=1}^{7n}\left[\sin\left(\frac{j^6}{n^7}\right)^{\frac{1}{5}}-\left(\frac{j^6}{n^7}\right)^{\frac{1}{5}}+\frac{1}{3!}\left(\frac{j^6}{n^7}\right)^{\frac{3}{5}}\right]. \tag{1.2}$$

Since for the function $f(x)$ of (1.1) and for the corresponding integer m, we have

$$f\left(\left(\frac{j^\sigma}{n^{1+\sigma}}\right)^{\frac{1}{m}}\right)-\sum_{i=0}^{m-1}\frac{1}{i!}f^{(i)}(0)\left(\frac{j^\sigma}{n^{1+\sigma}}\right)^{\frac{i}{m}}\sim\frac{f^{(m)}(0)}{m!}\frac{j^\sigma}{n^{1+\sigma}}\,(k_1n+1\le j\le k_2n)$$

as $n\to\infty$,

we may ask that whether one could deal with (1.1) by considering the limit $\lim\limits_{n\to\infty}\sum\limits_{j=k_1n+1}^{k_2n}\frac{j^\sigma}{n^{1+\sigma}}$? By virtue of the differential method [1-3], we claim that the answer of the question is positive, and three theorems below present the existence of (1.1) and show us how to compute it.

2. Main Results

In case $m=1$.

Theorem 1. *Let* $f'(x_0)$ *exist,* k_1, k_2 *be two integers with* $k_2>k_1\ge 0$, *and* $\sigma>0$ *be a fixed number. Then the limit*

$$\lim_{n\to\infty}\sum_{j=k_1n+1}^{k_2n}\left[f\left(x_0+\frac{j^\sigma}{n^{1+\sigma}}\right)-f(x_0)\right]$$

exists and satisfies

$$\lim_{n\to\infty}\sum_{j=k_1n+1}^{k_2n}\left[f\left(x_0+\frac{j^\sigma}{n^{1+\sigma}}\right)-f(x_0)\right]$$

$$=f'(x_0)\lim_{n\to\infty}\sum_{j=k_1n+1}^{k_2n}\frac{j^\sigma}{n^{1+\sigma}}=f'(x_0)\frac{k_2^{1+\sigma}-k_1^{1+\sigma}}{1+\sigma}. \tag{2.1}$$

In case $m \geq 1$.

Theorem 2. *Let* $m \geq 1$, $f^{(m)}(x_0)$ *exist, and* k_1, k_2, σ *be given as in Theorem* 1. *Then the limit*

$$\lim_{n\to\infty} \sum_{j=k_1n+1}^{k_2n} \left[f\left(x_0 + \left(\frac{j^\sigma}{n^{1+\sigma}} \right)^{\frac{1}{m}} \right) - \sum_{\mu=0}^{m-1} \frac{1}{\mu!} f^{(\mu)}(x_0) \left(\frac{j^\sigma}{n^{1+\sigma}} \right)^{\frac{\mu}{m}} \right]$$

exists and satisfies

$$\lim_{n\to\infty} \sum_{j=k_1n+1}^{k_2n} \left[f\left(x_0 + \left(\frac{j^\sigma}{n^{1+\sigma}} \right)^{\frac{1}{m}} \right) - \sum_{\mu=0}^{m-1} \frac{1}{\mu!} f^{(\mu)}(x_0) \left(\frac{j^\sigma}{n^{1+\sigma}} \right)^{\frac{\mu}{m}} \right]$$

$$= \frac{f^{(m)}(x_0)}{m!} \lim_{n\to\infty} \sum_{j=k_1n+1}^{k_2n} \frac{j^\sigma}{n^{1+\sigma}} = \frac{f^{(m)}(x_0)}{m!} \cdot \frac{k_2^{1+\sigma} - k_1^{1+\sigma}}{1+\sigma}. \tag{2.2}$$

Clearly, Theorem 1 is a corollary of Theorem 2. Moreover, Theorem 2 is the special case of the following general result.

Theorem 3. *Let* $m \geq 1$, $f^{(m)}(x_0)$ *exist and* k_1, k_2 *be given as in Theorem* 2. *Let* $\{a_n\}, \{b_n\}$ $(n \geq 1)$ *be two strictly increasing and positive sequences such that*

$$\begin{cases} \text{(a)} \quad \lim_{n\to\infty} a_n = \infty, \ \lim_{n\to\infty} \dfrac{a_n - a_{n-1}}{a_{n-1} - a_{n-2}} = 1, \\ \text{(b)} \quad \exists M_{k_2} > 0 \ \textit{s.t.} \ \dfrac{b_{k_2n}}{a_n} \leq \dfrac{M_{k_2}}{n} \ (n \geq 1), \\ \text{(c)} \quad \forall k = 1, 2, \ldots, k_2, \exists c_k \ \textit{s.t.} \ \lim_{n\to\infty} \dfrac{b_{kn}}{a_n - a_{n-1}} = c_k \\ \qquad (\textit{if } k = 0, \textit{ we may set } b_0 = c_0 = 0). \end{cases} \tag{2.3}$$

Then the limit

$$\lim_{n\to\infty} \sum_{j=k_1n+1}^{k_2n} \left[f\left(x_0 + \left(\frac{b_j}{a_n} \right)^{\frac{1}{m}} \right) - \sum_{\mu=0}^{m-1} \frac{f^{(\mu)}(x_0)}{\mu!} \left(\frac{b_j}{a_n} \right)^{\frac{\mu}{m}} \right]$$

exists and satisfies

$$\lim_{n\to\infty}\sum_{j=k_1n+1}^{k_2n}\left[f\left(x_0+\left(\frac{b_j}{a_n}\right)^{\frac{1}{m}}\right)-\sum_{\mu=0}^{m-1}\frac{f^{(\mu)}(x_0)}{\mu!}\left(\frac{b_j}{a_n}\right)^{\frac{\mu}{m}}\right]$$

$$=\frac{f^{(m)}(x_0)}{m!}\lim_{n\to\infty}\sum_{j=k_1n+1}^{k_2n}\frac{b_j}{a_n}=\frac{f^{(m)}(x_0)}{m!}(k_2c_{k_2}-k_1c_{k_1}). \tag{2.4}$$

$(If\ k_1=0,\ then\ k_1c_{k_1}=0).$

Proof of Theorem 3. By assumptions, it is easy to see that

$$\begin{cases}\text{(a)}\quad \forall n\ge 1,\ \forall j=k_1n+1,\ k_1n+2,\ \ldots,\ k_2n,\ 0\le\dfrac{b_j}{a_n}\le\dfrac{M_{k_2}}{n},\\[2ex] \text{(b)}\quad \forall k=1,2,\ldots,k_2,\ n>1,\ \dfrac{kb_{k(n-1)}}{a_n-a_{n-1}}\le\dfrac{\sum_{j=k(n-1)+1}^{kn}b_j}{a_n-a_{n-1}}\le\dfrac{kb_{kn}}{a_n-a_{n-1}},\\[2ex] \text{(c)}\quad \forall k=1,2,\ldots,k_2,\ \lim\limits_{n\to\infty}\dfrac{kb_{k(n-1)}}{a_n-a_{n-1}}=kc_k=\lim\limits_{n\to\infty}\dfrac{kb_{kn}}{a_n-a_{n-1}}.\end{cases} \tag{2.5}$$

By Stolz's Theorem [1-3], both (b) and (c) of (2.5) imply that

$$\lim_{n\to\infty}\frac{\sum_{j=1}^{kn}b_j}{a_n}=\lim_{n\to\infty}\frac{\sum_{j=k(n-1)+1}^{kn}b_j}{a_n-a_{n-1}}=kc_k\ (1\le k\le k_2)\ \text{and}$$

$$\lim_{n\to\infty}\sum_{j=k_1n+1}^{k_2n}\frac{b_j}{a_n}=k_2c_{k_2}-k_1c_{k_1}. \tag{2.6}$$

(As $c_0=0$ by (2.3) (c), the second equality of (2.6) is also true for $k_1=0$).

Since $f^{(m)}(x_0)$ exists, by Taylor's formula with Peano's remainder [1-3], we have $f(x_0+x)=\sum_{\mu=0}^{m}\frac{f^{(\mu)}(x_0)}{\mu!}x^\mu+o(x^m)(x\to 0)$. This implies that for each $\varepsilon>0$, there exists $\delta>0$ such that

$$\left|f(x_0+x)-\sum_{\mu=0}^{m}\frac{f^{(\mu)}(x_0)}{\mu!}x^\mu\right|<\varepsilon|x|^m\quad\text{for}\quad |x|<\delta. \tag{2.7}$$

Let $x = \left(\frac{b_j}{a_n}\right)^{\frac{1}{m}}$. Since $n > \frac{M_{k_2}}{\delta^m}$ implies $\left(\frac{M_{k_2}}{n}\right)^{\frac{1}{m}} < \delta$. By (2.5) (a), we have

$$\forall n > \frac{M_{k_2}}{\delta^m}, \quad \forall j = k_1 n + 1, \ldots, k_2 n, \quad 0 \leq \left(\frac{b_j}{a_n}\right)^{\frac{1}{m}} < \delta.$$

Associating this with (2.7), we obtain

$$\forall n > \frac{M_{k_2}}{\delta^m}, \quad \forall k_1 n + 1 \leq j \leq k_2 n,$$

$$-\varepsilon \frac{b_j}{a_n} < f\left(x_0 + \left(\frac{b_j}{a_n}\right)^{\frac{1}{m}}\right) - \sum_{\mu=0}^{m} \frac{f^{(\mu)}(x_0)}{\mu!}\left(\frac{b_j}{a_n}\right)^{\frac{\mu}{m}} < \varepsilon \frac{b_j}{a_n}.$$

Taking sum from $j = k_1 n + 1$ to $k_2 n$, it follows that for all $n > \frac{M_{k_2}}{\delta^m}$,

$$\left| \sum_{j=k_1 n+1}^{k_2 n} \left[f\left(x_0 + \left(\frac{b_j}{a_n}\right)^{\frac{1}{m}}\right) - \sum_{\mu=0}^{m-1} \frac{f^{(\mu)}(x_0)}{\mu!}\left(\frac{b_j}{a_n}\right)^{\frac{\mu}{m}} \right] - \frac{f^{(m)}(x_0)}{m!} \sum_{j=k_1 n+1}^{k_2 n} \frac{b_j}{a_n} \right|$$

$$< \varepsilon \sum_{j=k_1 n+1}^{k_2 n} \frac{b_j}{a_n}.$$

Combining this with (2.6), we conclude that Theorem 3 is true. □

Proof of Theorem 2. Assume $a_n = n^{1+\sigma}$, $b_n = n^{\sigma}$ and $\sigma > 0$. Then both sequences $\{a_n\}$ and $\{b_n\}$ are strictly increasing and positive. Since $\lim_{n\to\infty} a_n = +\infty$,

$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{a_{n-1} - a_{n-2}} = 1$, $\frac{b_{k_2 n}}{a_n} = \frac{(k_2 n)^{\sigma}}{n^{1+\sigma}} = \frac{k_2^{\sigma}}{n}$ and

$$\lim_{n\to\infty} \frac{b_{kn}}{a_n - a_{n-1}} = \lim_{n\to\infty} \frac{(kn)^{\sigma}}{n^{1+\sigma} - (n-1)^{1+\sigma}} = \frac{k^{\sigma}}{1+\sigma},$$

we can see that $M_{k_2} = k_2^{\sigma}$ and $c_k = \frac{k^{\sigma}}{1+\sigma}$ for each $k = 1, 2, \ldots, k_2$. And so, by Theorem 3, Theorem 2 is true. □

3. Some Applications

(i) For the limits of (1.2), let $f(x) = \sin(\sin x)$ or $\tan(\arcsin x)$. Then $f(0) = 0$, $f'(0) = 1$, and from Theorem 1, we have

(a) $\lim_{n\to\infty} \sum_{j=1}^{3n} \sin\left(\sin \frac{j}{n^2}\right) = \frac{3^{1+1} - 0^{1+1}}{1+1} = \frac{9}{2}$ (because $k_1 = 0$, $k_2 = 3$ and $\sigma = 1$).

(b) $\lim_{n\to\infty} \sum_{j=n+1}^{2n} \tan\left(\arcsin \frac{j^4}{n^5}\right) = \frac{2^{1+4} - 1^{1+4}}{1+4} = \frac{31}{5}$ (because $k_1 = 1$, $k_2 = 2$ and $\sigma = 4$).

(c) Let $g(x) = \sin x$. Then $\sin x - x + \frac{1}{3!}x^3 = \frac{1}{5!}x^5 + o(x^5)(x \to 0)$. Since $m = 5$, $\sigma = 6$, $k_1 = 0$, $k_2 = 7$, $\frac{g^{(5)}(0)}{5!} = \frac{1}{5!}$, by Theorem 2, we have

$$\lim_{n\to\infty} \sum_{j=1}^{7n} \left[\sin\left(\frac{j^6}{n^7}\right)^{\frac{1}{5}} - \left(\frac{j^6}{n^7}\right)^{\frac{1}{5}} + \frac{1}{3!}\left(\frac{j^6}{n^7}\right)^{\frac{3}{5}}\right] = \frac{\sin^{(5)}(0)}{5!} \cdot \frac{7^{6+1} - 0^{6+1}}{1+6} = \frac{7^6}{5!}.$$

(ii) As an another application of Theorem 3, we present the following example.

Let $f(x)$ be rth differentiable at x_0, $P_l(x) = d_l x^l + \cdots + d_0$ and $P_m(x) = e_m x^m + \cdots + e_0$ be lth and mth positive coefficient polynomials with $l > m \geq 1$. If $a_n = P_l(n)$ and $b_n = P_m(n)$, then both $\{a_n\}$ and $\{b_n\}$ $(n \geq 1)$ are strictly increasing and positive sequences with $\lim_{n\to\infty} a_n = +\infty$ and $\lim_{n\to\infty} \frac{a_n - a_{n-1}}{a_{n-1} - a_{n-2}} = 1$. For any integer $k \geq 1$, we have

$$\frac{b_{kn}}{a_n - a_{n-1}} \sim \frac{e_m k^m}{l d_l n^{l-m-1}} \to c_k = \begin{cases} 0, & \text{if } l > m+1, \\ \frac{e_m k^m}{l d_l}, & \text{if } l = m+1, \end{cases} \quad (n \to \infty).$$

Moreover, since $\frac{b_{k_2 n}}{a_n} \sim \frac{e_m k_2^m}{d_l n^{l-m}}$ $(n \to \infty)$ and $\frac{e_m k_2^m}{d_l n^{l-m}} \leq \frac{\left(\frac{e_m k_2^m}{d_l}\right)}{n}$, there exists $M_{k_2} > 0$ such that $\frac{b_{k_2 n}}{a_n} \leq \frac{M_{k_2}}{n}$ for all $n \geq 1$. And so, by Theorem 3, we obtain

$$\lim_{n\to\infty}\sum_{j=k_1n+1}^{k_2n}\left[f\left(x_0+\left(\frac{P_m(j)}{P_l(n)}\right)^{\frac{1}{r}}\right)-\sum_{\mu=0}^{r-1}\frac{f^{(\mu)}(x_0)}{\mu!}\left(\frac{P_m(j)}{P_l(n)}\right)^{\frac{\mu}{r}}\right]$$

$$=\begin{cases}0\ (l>m+1),\\ \dfrac{f^{(r)}(x_0)e_m(k_2^{m+1}-k_1^{m+1})}{r!\,ld_l}\ (l=m+1).\end{cases}$$

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