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SPECTRAL PROPERTIES OF DEGENERATE NON-SELFADJOINT ELLIPTIC DIFFERENTIAL OPERATORS UNDER DIRICHLET BOUNDARY CONDITIONS

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Abstract

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$. In this article, we investigate the spectral properties of a non-selfadjoint elliptic differential operator $(Au)(x) = -\sum_{i, j=1}^{n} (\rho^{2\alpha}(x) a_{ij}(x) q(x) u'_{x_i}(x))'_{x_j}$ acting on Hilbert space $H = L^2(\Omega)$ with Dirichlet-type boundary conditions. Here $\rho(x) = \operatorname{dist}\{x, \partial\Omega\}$, $0 \le \alpha < 1$, $q(x) \in C^2(\overline{\Omega})$, $a_{ij}(x) \in C^2(\overline{\Omega})$, $a_{ij}(x) = a_{ji}(x)$ $(i, j = 1, 2, ..., n)$, $c|s|^2 \le \sum_{i, j=1}^{n} a_{ij}(x) s_i \overline{s_j}$ $(x \in \overline{\Omega}, s \in \mathbf{C}^n)$, $c > 0$. Moreover, assume that $q(x) \in \mathbf{C} \backslash \Phi$, $\forall x \in \overline{\Omega}$, where $\Phi = \{z \in \mathbf{C} : |\arg z| \le \varphi\}$, $\varphi \in (0, \pi)$.

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1. Introduction

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega \in C^\infty$. We introduce the weighted Sobolev space $\mathcal{H} = W^2_{2,\alpha}(\Omega)$ as the space of complex valued functions $u(x)$ defined on Ω with the finite norm:

$$|u|_+ = \left(\sum_{i=1}^{n} \int_\Omega \rho^{2\alpha}(x) |u'_{x_i}(x)|^2 dx + \int_\Omega |u(x)|^2 dx\right)^{1/2},$$

where $0 \le \alpha < 1$, and $\rho(x) = \operatorname{dist}\{x, \partial\Omega\}$. We denote by $\overset{\circ}{\mathcal{H}}$ the closure of $C_0^\infty(\Omega)$ in $\mathcal{H}$ with respect to the above norm, i.e., $\overset{\circ}{\mathcal{H}}$ is the closure of $C_0^\infty(\Omega)$ in $W^2_{2,\alpha}(\Omega)$. The notion $C_0^\infty(\Omega)$ stands for the space of infinitely differentiable functions with compact support in Ω. In this paper, we investigate the spectral properties, in particular, we estimate the resolvent of a non-selfadjoint elliptic differential operator of type

$$(Au)(x) = -\sum_{i,j=1}^{n} (\rho^{2\alpha}(x) a_{ij}(x) q(x) u'_{x_i}(x))'_{x_j} \quad \text{defined on} \quad H = L_2(\Omega).$$

For a closed extension of the operator A with respect to space $= W^2_{2,\alpha}(\Omega)$, we need to extend its domain to the closed domain

$$D(A) = \left\{ y \in \overset{\circ}{\mathcal{H}} \cap W^2_{2,loc}(\Omega) : \sum_{i,j=1}^{n} (\rho^{2\alpha} a_{ij} q y'_{x_i})'_{x_j} \in H \right\},$$

(see [6]) where the local space $W^2_{2,loc}(\Omega)$ is the class of the functions $u(x)$ $(x \in \Omega)$ in this form $W^2_{2,loc}(\Omega) = \left\{ u(x) : \sum_{i=0}^{2} \int_J |u^{(i)}(x)|^2 dx < \infty,\ J \subset \Omega, \text{ open} \right\}$. Here $\rho(x) = \operatorname{dist}\{x, \partial\Omega\}$, $0 \le \alpha < 1$, $q(x) \in C^2(\overline{\Omega})$, $a_{ij}(x) \in C^2(\overline{\Omega})$, $a_{ij}(x) = a_{ji}(x)$, and the functions $a_{ij}(x)$ satisfy the uniformly elliptic condition, i.e., there exists $c > 0$ such that $c|s|^2 \le \sum_{i,j=1}^{n} a_{ij}(x) s_i \overline{s_j}$, where $s = (s_1, \ldots, s_n) \in \mathbf{C}^n$, $x \in \Omega$.

Assume that $q(x) \in \mathbf{C} \backslash \Phi$, $\forall x \in \overline{\Omega}$, where $\Phi = \{z \in \mathbf{C} : |\arg z| \le \varphi\}$, $\varphi \in (0, \pi)$ (i.e., the value of $q(x)$ lies on the complex plane and outside of the closed angle Φ), and let $q(x) \in C^2(\overline{\Omega})$. Here, and in the sequel, the value of the functions $\arg z \in (-\pi, \pi]$, and $\| T \|$ denotes the norm of the bounded operator $T : H \to H$.

To get a feeling for the history of the subject under study, refer to our earlier papers [7, 8]. Indeed this paper was written in continuing on our earlier papers, the paper is sufficiently more general than our earlier papers, which here, we obtain the resolvent estimate of the operator *A*, that satisfies the special and general conditions. The paper consists of three sections: Section 1 is devoted to Introduction. In Section 2, we have Theorem 2.1 on the resolvent estimate of the differential operator *A*, acting on *H* in the certain case (i.e., in this case, we will study Theorem 2.1 under assumption (2.2)). In Section 3, we have Theorem 3.1 on the resolvent estimate of the differential operator *A*, acting on *H* in the general case (i.e., in this case, we will study Theorem 3.1 in contrast to Theorem 2.1. In other words, Theorem 3.1 does not include assumption (2.2) of Theorem 2.1).

2. The Resolvent Estimate of Degenerate Elliptic Differential Operators in Some Special Case

Theorem 2.1. *Let A and* Φ *be defined as in Section* 1. *Choose a closed sector* $S \subset \Phi$ *with its vertex at zero* (*for more explanation see* [6]), *such that* $S \cap R_+ = \varnothing$. *Let the complex function* $q(x)$ *satisfy the following conditions*:

$$q(x) \in C^1(\overline{\Omega}), \quad q(x) \in \mathbf{C} \backslash S, \quad (\forall x \in \overline{\Omega}), \tag{2.1}$$

$$|\arg\{q(x_1) q^{-1}(x_2)\}| \le \frac{\pi}{8}, \quad (\forall x_1, x_2 \in \overline{\Omega}). \tag{2.2}$$

Then, *for sufficiently large in modulus* $\lambda \in S$, *the inverse operator* $(A - \lambda I)^{-1}$ *exists and is continuous in H*, *and the following estimates are valid*:

$$\| (A - \lambda I)^{-1} \| \le M_S |\lambda|^{-1} \quad (\lambda \in S, |\lambda| > C_S), \tag{2.3}$$

$$\left\| \rho^{\alpha} \frac{\partial}{\partial x_i} (A - \lambda I)^{-1} \right\| \le M'_S |\lambda|^{-\frac{1}{2}} \quad (\lambda \in S, |\lambda| > C_S), \tag{2.4}$$

for $i = 1, \ldots, n,$ *where* $M_S, C_S > 0$ *are sufficiently large numbers depending on S. The symbol* $\|\cdot\|$ *stands for the norm of a bounded arbitrary operator T in H.*

Proof. Here, to establish Theorem 2.1, we will first prove the assertion of Theorem 2.1 together with estimate (2.3). So, as in Section 1, for a closed extension of the operator A (for more explanation see Chapter 6 of [6]), we need to extend its domain to the closed set

$$D(A) = \{v \in \overset{\circ}{\mathcal{H}} \cap W_{2,loc}^{2}(\Omega) : hu' \in H, \quad (hqv')' \in H\}.$$

Let the operator A, now satisfy (2.1) and (2.2). Then there exists a complex number $Z \in C$ (notice that we can take $Z = e^{i\gamma}$, for a fixed real $\gamma \in (-\pi, \pi]$), such that $|Z = e^{i\gamma}| = 1,$ and so

$$c' \leq \mathrm{Re}\{Zq(x)\}, \quad c'|\lambda| \leq -\mathrm{Re}\{Z\lambda\}, \quad c' > 0 \quad (\forall x \in \overline{\Omega}, \lambda \in \Phi). \tag{2.5}$$

In view of the uniformly elliptic condition, we have

$$c|s|^2 = c\sum_{i=1}^{n}|s_i|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)s_i\overline{s_j}, \quad (c > 0,\ s = (s_1, \ldots, s_n) \in \mathbf{C}^n,\ x \in \Omega),$$

taking $s_i = y'_{x_i}$ implies that $c\sum_{i=1}^{n}|y'_{x_i}(x)|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)y'_{x_i}(x)\overline{y'_{x_j}(x)}.$ From this, and according to $c' \leq \mathrm{Re}\{Zq(x)\}$ in (2.4), we then multiply these two positive relations with each other which implies that

$$c_1\sum_{i=1}^{n}|y'_{x_i}(x)|^2 \leq \mathrm{Re}Zq(x)\sum_{i,j=1}^{n} a_{ij}(x)y'_{x_i}(x)\overline{y'_{x_j}(x)} \quad \text{for} \quad y \in D(A).$$

Multiply both sides of the latter relation by the positive term $\rho^{2\alpha}(x),$ and then integrate from both sides, we will have

$$c_1\sum_{i=1}^{n}\int_{\Omega}\rho^{2\alpha}(x)|y'_{x_i}(x)|^2 dx \leq \mathrm{Re}Z\sum_{i,j=1}^{n}\int_{\Omega}\rho^{2\alpha}(x)a_{ij}(x)q(x)y'_{x_i}(x)\overline{y'_{x_j}(x)}dx.$$

Now, by applying the integration by parts, and using Dirichlet-type condition, then the right side of the latter relation without multiple $\mathrm{Re}Z$ becomes

$$\sum_{i,\,j=1}^{n}\int_{\Omega}\rho^{2\alpha}(x)a_{ij}(x)q(x)y'_{x_i}(x)\overline{y'_{x_j}(x)}dx$$

$$= -\sum_{i,\,j=1}^{n}\int_{\Omega}(\rho^{2\alpha}(x)a_{ij}(x)q(x)y'_{x_i}(x))'_{x_j}\,\bar{y}(x)dx$$

$$= \left(-\sum_{i,\,j=1}^{n}(\rho^{2\alpha}(x)a_{ij}(x)q(x)y'_{x_i}(x))'_{x_j},\ y(x)\right) = (Ay,\ y), \tag{2.6}$$

since $(Ay)(x) = -\sum_{i,\,j=1}^{n}(\rho^{2\alpha}(x)a_{ij}(x)q(x)u'_{x_i}(x))'_{x_j}$.

Here, the symbol $(\,'\,)$ denotes the inner product in H.

Notice that the above equality in (2.6) is obtained by the well-known theorem of the m-sectorial operators which are closed by extending its domain to the closed domain in $\mathcal{H}$. These operators are associated with the closed sectorial bilinear forms that are densely defined in $\mathcal{H}$ (for more explanation see the well-known Theorem 2.1, Chapter 6 of [6]). This is why we extend the domain of the operator A to the closed domain in space $\mathcal{H}$ above. Therefore,

$$c_1\sum_{i=1}^{n}\int_{\Omega}\rho^{2\alpha}(x)|\,y'_{x_i}(x)\,|^2dx \le \operatorname{Re} Z(Ay,\ y),$$

from (2.4), we have $c'|\,\lambda\,| \le -\operatorname{Re}\{Z\lambda\},\ \ c' > 0,\ \ \forall\lambda \in \Phi.$ Multiply this inequality by $\int_{\Omega}|\,y(x)\,|^2dx = (y,\ y) = \|\,y\,\|^2 > 0.$ It follows that

$$c'|\,\lambda\,|\int_{\Omega}|\,y(x)\,|^2dt \le -\operatorname{Re}\{Z\lambda\}(y,\ y).$$

From this and the above inequality, we will have

$$\begin{aligned} c_1\sum_{i=1}^{n}\int_{\Omega}\rho^{2\alpha}(x)|\,y'_{x_i}(x)\,|^2dx + c'|\,\lambda\,|\int_{\Omega}|\,y(x)\,|^2dx &\le \operatorname{Re}\{Z(Ay,\ y) - Z\lambda(y,\ y)\} \\ &= \operatorname{Re}\{Z((A-\lambda I)y,\ y)\} \\ &\le \|\,Z\,\|\|\,y\,\|\|\,(A-\lambda I)y\,\| \\ &= \|\,y\,\|\|\,(A-\lambda I)y\,\|, \end{aligned} \tag{2.7}$$

i.e.,

$$c_1 \sum_{i=1}^{n} \int_{\Omega} \rho^{2\alpha}(x) | y'_{x_i}(x) |^2 dx + c' | \lambda | \int_{\Omega} | y(x) |^2 dx \le \| y \| \| (A - \lambda I) y \|.$$

Since $c_1 \sum_{i=1}^{n} \int_{\Omega} \rho^{2\alpha}(x) | y'_{x_i}(x) |^2 dx$ is positive, we will have either $c' | \lambda | \| y(x) \|^2 = | \lambda | \int_{\Omega} | y(x) |^2 dx \le \| y \| \| (A - \lambda I) y \|$ or

$$| \lambda | \| y(x) \| \le M_S \| (A - \lambda I) y \|. \tag{2.8}$$

This inequality ensures that the operator $(A - \lambda I)$ is one to one, which implies that $\ker(A - \lambda I) = 0$. Therefore, the inverse operator $(A - \lambda I)^{-1}$ exists, and its continuity follows from the proof of the estimate (2.3) of Theorem 2.1. To prove (2.3), we set $v = (A - \lambda I)^{-1} f, \ \ f \in H$ in (2.7) implies that

$$| \lambda | \int_{\Omega} | (A - \lambda I)^{-1} f |^2 dx \le M_S \| (A - \lambda I)^{-1} f \| \| (A - \lambda I)(A - \lambda I)^{-1} f \|.$$

Since $(A - \lambda I)(A - \lambda I)^{-1} f = I(f) = f$,

$$| \lambda | \int_{\Omega} | (A - \lambda I)^{-1} f |^2 dx \le M_S \| (A - \lambda I)^{-1} f \| | f |.$$

So

$$| \lambda | \| (A - \lambda I)^{-1}(f) \|^2 \le M_S \| (A - \lambda I)^{-1}(f) \| | f |,$$

this implies that $| \lambda | \| (A - \lambda I)^{-1}(f) \| \le M_S | f |$. Since $\lambda \ne 0$, $\| (A - \lambda I)^{-1}(f) \| \le M_S | \lambda |^{-1} | f |$, i.e., $\| (A - \lambda I)^{-1} \| \le M_S | \lambda |^{-1}$. This estimate completes the proof of the assertion of Theorem 2.1 together with the estimate (2.3). Now, we start to prove the estimate (2.4) of Theorem 2.1. As in the above argument, we drop the positive term $c' | \lambda | \int_{\Omega} | y(x) |^2 dx$ from

$$c_1 \sum_{i=1}^{n} \int_{\Omega} \rho^{2\alpha}(x) | y'_{x_i}(x) |^2 dx + c' | \lambda | \int_{\Omega} | y(x) |^2 dx \le \| y \| \| (A - \lambda I) y \|.$$

It follows that

$$c_1 \sum_{i=1}^{n} \int_{\Omega} \rho^{2\alpha}(x) |\, y'_{x_i}(x)\,|^2 dx \le \|\, y \,\| \|\, (A - \lambda I)\, y \,\|.$$

Equivalently

$$c_1 \left\| \rho^{\alpha} \frac{\partial}{\partial x_i} (A - \lambda I)^{-1} f \right\|^2 \le \|\, y \,\| \|\, (A - \lambda I)\, y \,\|.$$

Set $y = (A - \lambda I)^{-1} f, \quad f \in H$ in the latter relation, and proceeding by similar calculation as in the proof of (2.3), we then obtain

$$c_1 \left\| \rho^{\alpha} \frac{\partial}{\partial x_i} (A - \lambda I)^{-1} f \right\|^2 \le \|\, (A - \lambda I)^{-1} f \,\| \|\, (A - \lambda I)(A - \lambda I)^{-1} f \,\|.$$

Since $(A - \lambda I)(A - \lambda I)^{-1} f = I(f) = f$,

$$c_1 \left\| \rho^{\alpha} \frac{\partial}{\partial x_i} (A - \lambda I)^{-1} f \right\|^2 \le \|\, (A - \lambda I)^{-1} \,\| \|\, f \,\|^2,$$

consequently, by (2.3) this implies that

$$c_1 \left\| \rho^{\alpha} \frac{\partial}{\partial x_i} (A - \lambda I)^{-1} f \right\|^2 \le M_S |\, \lambda \,|^{-1} \|\, f \,\|^2,$$

to this end, we will have

$$\left\| \rho^{\alpha} \frac{\partial}{\partial x_i} (A - \lambda I)^{-1} \right\| \le M'_S |\, \lambda \,|^{-\frac{1}{2}}.$$

Thus, here the proof of the estimate (2.4) is finished, i.e., this completes the proof of Theorem 2.1.

Now, let the condition (2.2) does not hold. Then we will have the following statement.

3. The Resolvent Estimate of Some Classes of Degenerate Elliptic Differential Operators

In this section, we will derive a new general theorem by dropping the assumption (2.2) from Theorem 2.1 in Section 2.

Theorem 3.1. *As in Section* 1, *let* Φ *be some closed sector with vertex at zero in the complex plane* (*for more explanation see* [6]), $S \subset \Phi$ *be some closed sector with the vertex at zero*, *so that* $S \cap R_+ = \varnothing$, *and let the complex function* $q(x)$ *satisfy*

$$q(x) \in C^1(\overline{\Omega}), \quad q(x) \in \mathbf{C} \backslash S, \quad (\forall x \in \overline{\Omega}). \tag{3.1}$$

Then, *for sufficiently large in modulus* $\lambda \in S$, *the inverse operator* $(A - \lambda I)^{-1}$ *exists and is continuous in H*, *and the following estimate holds*:

$$\| (A - \lambda I)^{-1} \| \leq M_S |\lambda|^{-1}, \quad (\lambda \in S, |\lambda| > C_S), \tag{3.2}$$

where $M_S, C_S > 0$ *are sufficiently large numbers depending on S*.

Proof. Let us assume that (2.3) does not satisfy. To prove the assertion of Theorem 3.1 together with (3.2), we construct the functions $\varphi_1(x), \ldots, \varphi_m(x)$, $q_1(x), \ldots, q_m(x)$, so that each one of the functions $q_1(x), \ldots, q_m(x)$ $(x \in \overline{\Omega})$, as the function $q(x)$ in Theorem 2.1 satisfies (2.2). Therefore, let

$$\varphi_1(x), \ldots, \varphi_m(x), \quad q_1(x), \ldots, q_m(x) \in C_0^{\infty}(\Omega),$$

satisfy

$$0 \leq \varphi_r(x), \quad r = 1, \ldots, m, \quad \varphi_1^2(x) + \cdots + \varphi_m^2(x) \equiv 1 \quad (x \in \overline{\Omega}),$$

$$\frac{d}{dt}\varphi_r(x) \in C_0^{\infty}(\Omega), \quad q_r(x) = q(x), \quad \forall x \in \operatorname{supp} \varphi_r,$$

$$q_r(x) \in \mathbf{C} \backslash \Phi, \quad (\forall x \in \overline{\Omega}), \quad r = 1, \ldots, m,$$

$$|\arg\{q_r(x_1) q_r^{-1}(x_2)\}| \leq \frac{\pi}{8}, \quad (\forall x_1, x_2 \in \operatorname{supp} \varphi_r), \quad r = 1, \ldots, m.$$

In view of Theorem 2.1, and by (2.3) and (2.4), set $A_r = A$ in the definition of the differential operator, implies that

$$A_r u(x) = -\sum_{i, j=1}^{n} (\rho^{2\alpha}(x) a_{ij}(x) q_r(x) u'_{x_i}(x))'_{x_j},$$

acting on H, where

$$D(A_r) = \left\{ u \in \overset{\circ}{\mathcal{H}} \cap W_{2,loc}^2(\Omega) : \sum_{i,j=1}^{n} (\rho^{2\alpha} a_{ij} q_r u'_{x_i})'_{x_j} \in H \right\}.$$

Due to the assertion of Theorem 2.1, for $0 \neq \lambda \in S$, the inverse operator $(A - \lambda I)^{-1}$ exists and is continuous in space $H = L^2(0, 1)$, and satisfies

$$\| (A_r - \lambda I)^{-1} \| \leq M_S |\lambda|^{-1},$$

$$\left\| \rho^{\alpha} \frac{\partial}{\partial x_i} (A_r - \lambda I)^{-1} \right\| \leq M'_S |\lambda|^{-\frac{1}{2}} \quad (\lambda \in S, |\lambda| > C_S),$$

$$(0 \neq \lambda \in S). \tag{3.3}$$

Let us introduce

$$G(\lambda) = \sum_{r=1}^{m} \varphi_r (A_r - \lambda I)^{-1} \varphi_r. \tag{3.4}$$

Here, φ_r is the multiplication operator in H by the function $\varphi_r(x)$. Consequently, easily we will have

$$(A - \lambda I) G(\lambda) = I + \rho^{2\alpha - 1}(x) \sum_{r=1}^{m} \beta_r(x) (A_r - \lambda I)^{-1} \varphi_r$$

$$+ \rho^{2\alpha}(x) \sum_{i=1}^{n} \sum_{r=1}^{m} \gamma_{i_r}(x) \frac{\partial}{\partial x_i} (A_r - \lambda I)^{-1} \varphi_r, \tag{3.5}$$

where $\beta_r, \gamma_{i_r} \in L_\infty(\Omega)$, $\operatorname{supp} \beta_r$ and $\operatorname{supp} \gamma_{i_r}$ are contained in $\operatorname{supp} \varphi_r$.

First, we prove that $G(\lambda) u \in D(P_r)$, $(\forall u \in \mathcal{H})$. Now, from the definition of $D(A_r)$, it follows that

$$\varphi_r (A_r - \lambda)^{-1} \varphi_r \in \overset{\circ}{\mathcal{H}} \cap W_{2,loc}^2(\Omega), \quad (r = 1, 2, ..., m).$$

Therefore, this implies that $G(\lambda)u \in D(A)$;

$$(A - \lambda I)G(\lambda)u$$

$$= A(G(\lambda)u) - \lambda G(\lambda)u$$

$$= -\sum_{i,\, j=1}^{n} \{\rho^{2\alpha}(x)a_{ij}(x)q_r(x)(G(\lambda)u)'_{x_i}\}'_{x_j} - \lambda G(\lambda)u$$

$$= -\sum_{i,\, j=1}^{n} \left\{\rho^{2\alpha}(x)a_{ij}(x)q_r(x)\left(\sum_{r=1}^{m} \varphi_r(A_r - \lambda I)^{-1}\varphi_r u\right)'_{x_i}\right\}'_{x_i} - \lambda G(\lambda)u$$

$$= -\sum_{i,\, j=1}^{n} \sum_{r=1}^{m} \{\rho^{2\alpha}(x)a_{ij}(x)q_r(x)(\varphi_r)'_{x_i}((A_r - \lambda I)^{-1})\varphi_r u\}'_{x_i} - \lambda G(\lambda)u$$

$$= -\sum_{i,\, j=1}^{n} \sum_{r=1}^{m} \{\rho^{2\alpha}(x)a_{ij}(x)q_r(x)((A_r - \lambda I)^{-1})\varphi_r u\}'_{x_i}\}'_{x_i}.$$

We write the second term as

$$L_1 + L_2 = -\sum_{i,\, j=1}^{n}\sum_{r=1}^{m} \{\rho^{2\alpha}a_{ij}q_r\varphi_r((A_r - \lambda I)^{-1}\varphi_r u)'_{x_i}\}'_{x_i},$$

where

$$L_1 = -\sum_{i,\, j=1}^{n}\sum_{r=1}^{m} \{\varphi_r\}'_{x_i}\{\rho^{2\alpha}a_{ij}q_r((A_r - \lambda I)^{-1}\varphi_r u)'_{x_i}\}$$

and

$$L_2 = -\sum_{i,\, j=1}^{n}\sum_{r=l}^{m} \{\varphi_r\} \cdot \{\rho^{2\alpha}a_{ij}q_r((A_r - \lambda I)^{-1}\varphi_r u)'_{x_i}\}'_{x_i}$$

$$= \sum_{r=1}^{m} \{\varphi_r\} \cdot \left(-\sum_{i,\, j=1}^{n} \{\rho^{2\alpha}a_{ij}q_r((A_r - \lambda I)^{-1}\varphi_r u)'_{x_i}\}'_{x_i}\right)$$

$$= \sum_{r=1}^{m} \{\varphi_r\} (A_r \{(A_r - \lambda I)^{-1} \varphi_r u\})$$

$$= \sum_{r=1}^{m} \{\varphi_r\} ((A_r - \lambda I) + \lambda I) \{(A_r - \lambda I)^{-1} \varphi_r u\}$$

$$+ \lambda I \sum_{r=1}^{m} \{\varphi_r\} \{(A_r - \lambda I)^{-1} \varphi_r u\}$$

$$= \sum_{r=1}^{m} \{\varphi_r\} I \varphi_r u + \lambda \sum_{r=1}^{m} \varphi_r (A_r - \lambda I)^{-1} \varphi_r$$

$$= \sum_{r=1}^{m} \varphi_r^2 u + \lambda G(\lambda) u = u + \lambda G(\lambda) u.$$

Thus, we have

$$(A - \lambda I) G(\lambda) u$$

$$= -\sum_{i,\, j=1}^{n} \sum_{r=1}^{m} \{\rho^{2\alpha}(x) a_{ij}(x) q_r(x) (\varphi_r)'_{x_i} ((A_r - \lambda I)^{-1} \varphi_r u)\}'_{x_i} + L_1 + L_2 - \lambda G(\lambda) u$$

$$= -\sum_{i,\, j=1}^{n} \sum_{r=1}^{m} \{\rho^{2\alpha}(x) a_{ij}(x) q_r(x) (\varphi_r)'_{x_i}\} ((A_r - \lambda I)^{-1} \varphi_r u)'_{x_i}$$

$$- \sum_{i,\, j=1}^{n} \sum_{r=1}^{m} \{\rho^{2\alpha}(x) a_{ij}(x) q_r(x) (\varphi_r)'_{x_i}\}'_{x_i} ((A_r - \lambda I)^{-1} \varphi_r u) + L_1 + L_2 - \lambda G(\lambda) u,$$

and since

$$-\sum_{i,\, j=1}^{n} \sum_{r=1}^{m} \{\rho^{2\alpha}(x) a_{ij}(x) q_r(x) (\varphi_r)'_{x_i}\} ((A_r - \lambda I)^{-1} \varphi_r u)'_{x_j}$$

$$= -\sum_{i,\, j=1}^{n} \sum_{r=1}^{m} \{\rho^{2\alpha}(x) a_{ij}(x) q_r(x) (\varphi_r)'_{x_i}\} ((A_r - \lambda I)^{-1} \varphi_r u)'_{x_i},$$

therefore, we have

$$(A - \lambda I)G(\lambda)u$$

$$= -\sum_{i,\, j=1}^{n}\sum_{r=1}^{m} \{\rho^{2\alpha}(x) a_{ij}(x) q_r(x)(\varphi_r)'_{x_i}\}((A_r - \lambda I)^{-1}\varphi_r u)'_{x_i}$$

$$- \sum_{i,\, j=1}^{n}\sum_{r=1}^{m} \{\rho^{2\alpha}(x) a_{ij}(x) q_r(x)(\varphi_r)'_{x_i}\}'_{x_i}((A_r - \lambda I)^{-1}\varphi_r u)$$

$$- \sum_{i,\, j=1}^{n}\sum_{r=1}^{m} \{\varphi_r\}'_{x_i}\{\rho^{2\alpha}(x) a_{ij}(x) q_r(x)((A_r - \lambda I)^{-1}\varphi_r u)'_{x_i}\} + u$$

$$+ \lambda G(\lambda)u - \lambda G(\lambda)u.$$

Then

$$(A - \lambda I)G(\lambda)u = -\sum_{i,\, j=1}^{n}\sum_{r=1}^{m} \{\rho^{2\alpha}(x) a_{ij}(x) q_r(x)(\varphi_r)'_{x_i}\}((A_r - \lambda I)^{-1}\varphi_r u)'_{x_i}$$

$$- \sum_{i,\, j=1}^{n}\sum_{r=1}^{m} \{\rho^{2\alpha}(x) a_{ij}(x) q_r(x)(\varphi_r)'_{x_i}\}'_{x_i}((A_r - \lambda I)^{-1}\varphi_r u)$$

$$- \sum_{i,\, j=1}^{n}\sum_{r=1}^{m} \{\varphi_r\}'_{x_i}\{\rho^{2\alpha}(x) a_{ij}(x) q_r(x)((A_r - \lambda I)^{-1}\varphi_r u)'_{x_i}\} + u.$$

By factoring out $\rho^{2\alpha}$, we will have

$$(A - \lambda I)G(\lambda)u = \rho^{2\alpha}(x) - \sum_{i,\, j=1}^{n}\sum_{r=1}^{m} \{\rho^{2\alpha}(x) a_{ij}(x) q_r(x)(\varphi_r)'_{x_i}$$

$$+ a_{ij}(x) q_r(x)\{\varphi_r\}'_{x_i}((A_r - \lambda I)^{-1}\varphi_r u)'_{x_i}\}$$

$$- \sum_{i,\, j=1}^{n}\sum_{r=1}^{m} \{\rho^{2\alpha}(x) a_{ij}(x) q_r(x)(\varphi_r)'_{x_i}\}'_{x_i}((A_r - \lambda I)^{-1}\varphi_r u) + u.$$

Since $i = 1, \ldots, n$ and $j = 1, \ldots, n$ are dummy variables, so this implies that

$$(A - \lambda I)G(\lambda)u = u + \rho^{2\alpha}(x)\sum_{i=1}^{n}\sum_{r=1}^{m}\gamma_{i_r}(x)\frac{\partial}{\partial x_i}(A_r - \lambda I)^{-1}\varphi_r$$

$$- \sum_{i,\, j=1}^{n}\sum_{r=1}^{m}\{\rho^{2\alpha}(x)a_{ij}(x)q_r(x)(\varphi_r)'_{x_i}\}'_{x_i}((A_r - \lambda I)^{-1}\varphi_r u).$$

Now, since

$$\{\rho^{2\alpha}(x)a_{ij}(x)q_r(x)(\varphi_r)'_{x_i}\}'_{x_j}$$

$$= \frac{\partial}{\partial x_j}(\rho^{2\alpha}(x))a_{ij}(x)q_r(x)(\varphi_r)'_{x_i} + \rho^{2\alpha}(x)(a_{ij}(x)q_r(x)(\varphi_r)'_{x_i})'_{x_j},$$

$$\left|\frac{\partial}{\partial x_j}\rho^{2\alpha}(x)\right| \le M_S\rho^{2\alpha-1}(x); \quad \rho^{2\alpha}(x) = \rho^{2\alpha-1}(x)\rho(x).$$

Therefore, the above equality is equivalent to

$$\rho^{2\alpha-1}(x)\beta'_r + \rho^{2\alpha-1}(x)\rho(x)\beta''_r.$$

To the end, we will have

$$(A - \lambda I)G(\lambda) = I + \rho^{2\alpha-1}(x)\sum_{r=1}^{m}\beta_r(x)(A_r - \lambda I)^{-1}\varphi_r$$

$$+ \rho^{2\alpha}(x)\sum_{i=1}^{n}\sum_{r=1}^{m}\gamma_{i_r}(x)\frac{\partial}{\partial x_i}(A_r - \lambda I)^{-1}\varphi_r.$$

Now, the proof of (3.4) is completed. Let us take the right side of (3.4) equal to $I + T(\lambda)$. Thus, we will have

$$(A - \lambda I)G(\lambda) = I + T(\lambda). \tag{3.6}$$

Now, according to Section 2, if we put $A = A_r$ for $r = 1, \ldots, m$ in (2.2), then we will have

$$\|(A_r - \lambda I)^{-1}\| \le M1_S|\lambda|^{-1}, \quad \left\|\rho^{\alpha}\frac{\partial}{\partial x_i}(A_r - \lambda I)^{-1}\right\| \le M'_S\|\lambda\|^{-\frac{1}{2}}.$$

Owing to the definition of $T(\lambda)$ in the (3.5), easily it follows that

$$\|T(\lambda)\| \le M_S|\lambda|^{-\frac{1}{2}} \quad (\lambda \in \Phi, |\lambda| > 1). \tag{3.7}$$

Since $|\lambda|$ is sufficiently large number which easily implies that $\|T(\lambda)\| < \frac{1}{2} < 1,$ from this and using the well-known theorem in the operator theory, we conclude that $I + T(\lambda)$ and so $(A - \lambda I)G(\lambda)$ are invertible. Hence, $((A - \lambda I)G(\lambda))^{-1}$ exists and equals to

$$(G(\lambda))^{-1}(A - \lambda I)^{-1} = (I + T(\lambda))^{-1}. \tag{3.8}$$

By adding $+I$ and $-I$ to the right side of the (3.7), it follows that

$$(G(\lambda))^{-1}(A - \lambda I)^{-1} = (I + T(\lambda))^{-1} - I + I.$$

We now set

$$F(\lambda) = (I + T(\lambda))^{-1} - I.$$

Then

$$(G(\lambda))^{-1}(A - \lambda I)^{-1} = I + F(\lambda).$$

In view of $\| T(\lambda) \| < 1$ and (3.7), by estimating the following geometric series for the $F(\lambda),$ we will have

$$\| F(\lambda) \| \leq \sum_{i=2}^{+\infty} \| T^k(\lambda) \| \leq \| T(\lambda) \|^2 (1 + \| T(\lambda) \| + \| T(\lambda) \|^2 + \cdots)$$

$$\leq \| T(\lambda) \|^2 M_S (1 + 1/2 + 1/4 + \cdots) \leq 2M_S (M'_S |\lambda|^{-1/2})^2,$$

i.e., $\| F(\lambda) \| \leq 2M1_S |\lambda|^{-1}.$ By (3.3) and $\| (A_r - \lambda I)^{-1} \| \leq M1_S |\lambda|^{-1},$ we will have

$$\| G(\lambda) \| = \left\| \sum_{r=1}^{m} \varphi_r (A_r - \lambda I)^{-1} \varphi_r \right\| \leq M''_S \| (A_r - \lambda I)^{-1} \| \leq M''_S M1_S |\lambda|^{-1},$$

i.e., $\| G(\lambda) \| \leq M2_S |\lambda|^{-1}.$ Now from (3.8), we have

$$(A - \lambda I)^{-1} = G(\lambda)(I + T(\lambda))^{-1} = G(\lambda)(I + F(\lambda)).$$

Therefore

$$\| (A - \lambda I)^{-1} \| = \| G(\lambda) \| \| (I + F(\lambda)) \| \leq M2_S |\lambda|^{-1} \| (1 + 2M1_S |\lambda|^{-1}) \|,$$

i.e., here the assertion of Theorem 3.1 is proved. Therefore, to complete the proof of Theorem 3.1, we must prove the estimate in (3.2), to the end, we have according to latter inequality

$$\| (A - \lambda I)^{-1} \| \leq M2_S |\lambda|^{-1} + 2M2_S M1_S |\lambda|^{-1} |\lambda|^{-1},$$

and since $|\lambda|^{-1}|\lambda|^{-1} = |\lambda|^{-2} \leq |\lambda|^{-1}$, it follows that

$$\| (A - \lambda I)^{-1} \| \leq M_S |\lambda|^{-1}, \quad (|\lambda| \geq C, \ \ \lambda \in \Phi).$$

This completes the proof of Theorem 3.1.

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