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ON INVARIANT CURVES OF ANALYTIC REVERSIBLE MAPPINGS WITH DEGENERACY

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Abstract

In this paper, we consider small perturbation of analytic reversible mappings with degeneracy and prove the existence of invariant curve by KAM iteration. Moreover, the frequency of invariant curve persists without any drift.

1. Introduction and Main Results

In this paper, we are concerned with the existence of invariant curve of reversible mapping A:

$$\begin{cases} x_1 = x + \omega + \beta(y) + f(x, y), \\ y_1 = y + g(x, y), \end{cases} \tag{1.1}$$

where f and g are real analytic functions of periodic 2π for x, the variable y ranges

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in a neighborhood of the origin of the real line $\mathbb{R}$ and ω is a positive constant. We suppose that the mapping A is reversible with respect to the involution $R : (x, y) \to (-x, y)$, that is, $ARA = R$.

When $d\beta(y)/dy \neq 0$, ω satisfies the Diophantine condition (1.3), and f and g are sufficiently small, the existence of invariant curve with ω as its frequency has been proved in [3, 4, 5, 7, 14]. The natural question is when the condition $d\beta(y)/dy \neq 0$ is not satisfied, i.e., there is some r^* such that $d\beta(r^*)/dy = 0$, whether there exists invariant curve for mapping (1.1), whether its frequency can persist without any drift. For the above question, we can only prove the existence of invariant curve by using the similar methods as in [9]. However, there is no information on the persistence of the frequency of invariant curve.

In this paper, we investigate the persistence of the frequency of invariant curve, when $\beta(y) = y^{2n+1}$. More precisely, we consider the following reversible mapping:

$$\begin{cases} x_1 = x + \omega + y^{2n+1} + f(x, y), \\ y_1 = y + g(x, y), \end{cases} \tag{1.2}$$

where $(x, y) \in \mathbb{S}^1 \times I_{r_0}$, $I_{r_0} = [-r_0, r_0] \subset \mathbb{R}$, ω is a positive constant.

We prove that the mapping (1.2) still has an invariant curve with ω as its frequency, when ω satisfies the Diophantine condition (1.3), and f and g are sufficiently small.

Define

$$D(s, r_0) = \{(x, y) \in \mathbb{C}/2\pi\mathbb{Z} \times \mathbb{C} \mid \operatorname{Im} x \mid \leq s, \mid y \mid \leq r_0\}.$$

We expand

$$f(x, y) = \sum_{k \in \mathbb{Z}} f_k(y) e^{ikx},$$

then define

$$\| f(x, y) \|_{D(s, r_0)} = \sup_{D(s, r_0)} \left| \sum_{k \in \mathbb{Z}} f_k(y) e^{ikx} \right|.$$

The following theorem is the main result of this paper.

Theorem 1.1. *We consider the mapping* (1.2) *of an annulus, which is reversible with respect to the involution* $R:(x, y) \to (-x, y)$, *i.e.*, $ARA = R$. *Suppose the frequency* ω *satisfies the Diophantine condition*:

$$\left| \frac{k\omega}{2\pi} - l \right| \geq \frac{\alpha}{|k|^{\tau}}, \quad (k, l) \in \mathbb{Z} \times \mathbb{Z} \backslash \{0, 0\}, \tag{1.3}$$

where $\tau \geq 2$, α *is a positive constant. Suppose* $f(x, y)$ *and* $g(x, y)$ *are real analytic on* $D(s, r_0)$. *Then there is an* $\varepsilon > 0$ *such that if*

$$\| f \|_{D(s, r_0)} + \frac{1}{r_0} \| g \|_{D(s, r_0)} \leq \varepsilon,$$

the mapping (1.2) *has an invariant curve* γ *and the restriction of* (1.2) *onto* γ *is of the form* $x_1 = x + \omega$.

2. Proofs of the Main Results

At first, we introduce a parameter and change the mapping (1.2) to a parameterized mapping and this idea is used in [2, 6, 12] for Hamiltonian systems. Let $y = p + z$, where $|p| \leq \varrho$, $|z| \leq r \leq \varrho \leq r_0/2$. Then the above mapping (1.2) becomes

$$\begin{cases} x_1 = x + \omega + (z + p)^{2n+1} + f(x, z + p), \\ z_1 = z + g(x, z + p), \end{cases} \tag{2.4}$$

with

$$(z + p)^{2n+1} = p^{2n+1} + h(z; p)$$

and

$$h(z; p) = (2n+1)p^{2n} z + (2n+1)np^{2n-1} z^2 + \cdots + (2n+1)pz^{2n} + z^{2n+1}.$$

Let $f(x, z; p) = h(z; p) + f(x, z + p)$ and $g(x, z; p) = g(x, z + p)$. Then we write the above mapping (2.4) as follows:

$$\begin{cases} x_1 = x + \omega + p^{2n+1} + f(x, z; p), \\ z_1 = z + g(x, z; p), \end{cases}$$

where $(x, z) \in \mathbb{S}^1 \times I_r$ with $I_r = [-r, r]$; p is a parameter, $p \in I_\varrho = [-\varrho, \varrho]$.

Define

$$\Pi_\sigma = \{p \in \mathbb{C} \mid \operatorname{dist}\{p, I_\varrho\} \le \sigma\} \quad \text{with} \quad \sigma \le \varrho.$$

We choose $r = \varepsilon^{\frac{2}{2n+2}}$, $\varrho = \varepsilon^{\frac{1}{2n+2}}$ and have $\| h \|_{D(s, r)\times\Pi_\sigma} \le c\varepsilon$, where c is a constant depending on n. It follows that

$$\| f \|_{D(s, r)\times\Pi_\sigma} \le c\varepsilon.$$

Note that here and below we always use c to denote different constants in estimates. For simplicity, we write $\| \cdot \|_{s, r}$ for $\| \cdot \|_{D(s, r)\times\Pi_\sigma}$.

Now, we consider the following equivalent parameterized mapping:

$$\begin{cases} x_1 = x + \omega + H(p) + f(x, z; p), \\ z_1 = z + g(x, z; p), \end{cases} \tag{2.5}$$

where $H(p) = p^{2n+1}$. Moreover, f and g are small perturbations satisfying

$$\| f \|_{s, r} + \frac{1}{r} \| g \|_{s, r} \le c\varepsilon.$$

Theorem 2.1. *Let the parameterized reversible mapping* (2.5) *be real analytic on* $D(s, r) \times \Pi_\sigma$. *Let* ω *satisfy the Diophantine condition* (1.3). *Then there exists a sufficient small* $\varepsilon > 0$, *such that if* $(\| f \|_{s, r} + \| g \|_{s, r}/r) \le c\varepsilon$, *there exists* $p_* \in I_\varrho$ *such that the mapping* (2.5) *at* $p = p_*$ *has an invariant curve with* ω *as its frequency*.

In order to prove Theorem 2.1 effectively, we introduce an external parameter λ and consider the following mapping:

$$\begin{cases} x_1 = x + \omega + H(p) - \lambda + f(x, z; p), \\ z_1 = z + g(x, z; p), \end{cases} \tag{2.6}$$

where $\lambda \in I_1 = [-1, 1]$ is an outer parameter. The idea of introducing outer parameter was used in [9, 12, 14] for Hamiltonian systems and area preserving mapping. When $\lambda = 0$, the mapping (2.6) corresponds to the mapping (2.5).

Below we first consider the mapping (2.6) with parameters (p, λ). We will prove that there exists a smooth curve $\Gamma : \lambda = \lambda(p)$, $p \in I_\varrho$ such that for

$(p, \lambda) \in \Gamma$, the mapping (2.6) can be transformed to a norm form with $z = 0$ an equilibrium. Moreover, we can find $p_* \in I_\varrho$ such that $\lambda(p_*) = 0$ and then have the original mapping (2.5) with $p = p_*$.

Let $\Lambda = I_\varrho \times I_1$. Then define

$$M = \{(p, \lambda) \in \mathbb{C}^2 \,|\, \mathrm{dist}((p, \lambda), \Lambda) \leq \sigma\}.$$

Let $K > 0$ and $\tilde{\delta} = \dfrac{\pi\alpha}{K^{\tau+1}}$. Denote by $\mathcal{O}_{\tilde{\delta}}$, the complex $\tilde{\delta}$ neighborhood of ω. Then for all $\omega \in \mathcal{O}_{\tilde{\delta}}$, it follows that

$$\left| \frac{k\omega}{2\pi} - l \right| \geq \frac{\alpha}{2|k|^\tau}, \quad 0 < |k| \leq K.$$

Theorem 2.2. *Consider the parameterized reversible mapping* (2.6). *Suppose that the frequency* ω *satisfies the Diophantine condition* (1.3), *and* $f(x, z; p)$ *and* $g(x, z; p)$ *are real analytic on* $D(s, r) \times M$. *Then there exists a sufficiently small* $\varepsilon > 0$ *such that if* $(\| f \|_{s,r} + \| g \|_{s,r}/r) \leq c\varepsilon$, *then we have an analytic curve* Γ^* *defined by the equation*

$$p^{2n+1} - \lambda + h_*(p, \lambda) = 0,$$

where

$$|h_*(p, \lambda)| \leq 2\varepsilon, \quad |h_{*p}(p, \lambda)| + |h_{*\lambda}(p, \lambda)| \leq \frac{1}{2}, \tag{2.7}$$

such that for $(p, \lambda) \in \Gamma^*$, *there exists a transformation* V_*:

$$\begin{cases} x = \xi + p_*(\xi;\, p, \lambda), \\ z = \eta + q_*(\xi, \eta;\, p, \lambda), \end{cases}$$

which is affine in η, *the mapping* (2.6) *is transformed to*

$$\begin{cases} \xi_1 = \xi + \omega + f_*(\xi, \eta;\, p, \lambda), \\ \eta_1 = \eta + g_*(\xi, \eta;\, p, \lambda), \end{cases}$$

with $f_* = O(\eta)$, $g_* = O(\eta^2)$ *at* $\eta = 0$, *i.e.*, *the mapping* (2.6) *has an invariant curve* γ *such that the induced mapping on this curve is the translation* $\xi_1 = \xi + \omega$.

Remark. The derivatives in the estimates of (2.7) should be understood in the sense of Whitney [10]. In fact, we can extend $h_*(p, \lambda)$ to a neighborhood of Γ^* as a consequence in [1].

Now we use Theorem 2.2 to prove Theorem 2.1, and postpone the proof of Theorem 2.2 later. By the implicit theorem and the equation

$$p^{2n+1} - \lambda + h_*(p, \lambda) = 0,$$

we first have an analytic curve

$$\Gamma_* : \lambda = \lambda(p) = p^{2n+1} + \hat{h}_*(p), \quad p \in I_{\varrho}.$$

Moreover, if ε is sufficiently small, then we have $|\hat{h}_*(p)| \leq 2\varepsilon$. Due to $\varrho = \varepsilon^{\frac{1}{2n+2}}$, it follows that $\lambda(\pm\varrho) = (\pm\varrho)^{2n+1} + \hat{h}_*(\pm\varrho)$ must have different signs. Thus there exists $p_* \in I_{\varrho}$ such that $\lambda(p_*) = 0$. When $\lambda(p_*) = 0$, the mapping (2.6) corresponds to the mapping (2.5). Therefore, by Theorem 2.2, at p_*, the mapping (2.5) has an invariant curve with ω as its frequency.

Now it remains to prove Theorem 2.2. Our method is the stand KAM iteration. The difficulty is how to deal with parameters in KAM iteration.

KAM step. The KAM step can be summarized in the following lemma.

Lemma 2.1. *Consider the following real analytic mapping* A:

$$\begin{cases} x_1 = x + \omega + H(p, \lambda) + f(x, z; p, \lambda), \\ z_1 = z + g(x, z; p, \lambda) \end{cases} \tag{2.8}$$

on $D(s, r) \times M$, *where* $H(p, \lambda) = p^{2n+1} - \lambda + h(p, \lambda)$. *Suppose the mapping is reversible with respect to the involution* $R : (x, z) \to (-x, z)$, *i.e.*, $ARA = R$. *Let* $0 < E < 1$ *and* $0 < \rho < s/5$. *Suppose*

$$\| f \|_{s, r} + \frac{1}{r} \| g \|_{s, r} \leq \varepsilon = \alpha \rho^{\tau+2} E.$$

Suppose the function $h(p, \lambda)$ *satisfies that*

$$| h_p(p, \lambda) | + | h_{\lambda}(p, \lambda) | < \frac{1}{2}, \quad \forall (p, \lambda) \in M, \tag{2.9}$$

and then the equation

$$H(p, \lambda) = p^{2n+1} - \lambda + h(p, \lambda) = 0$$

defines implicitly an analytic mapping:

$$\lambda : p \in \Pi_\sigma \to \lambda(p)$$

such that $\Gamma = \{(p, \lambda(p)) | \, p \in \Pi_\sigma\} \subset M$. *Moreover, for* $K > 0$ *satisfying* $K^n e^{-K\rho} = E$, $\delta = \dfrac{\pi\alpha}{4K^{\tau+1}}$, *we have*

$$N(\Gamma, \delta) = \{(p, \lambda') \in \Pi_\sigma \times \mathbb{C}, (p, \lambda) \in \Gamma \,|\, |\lambda' - \lambda| \leq \delta\} \subset M.$$

Then there exist $M_+ \subset M$ *and* $D(s_+, r_+) \subset D(s, r)$ *such that for any* $(p, \lambda) \in M_+$ *there exists a mapping U*:

$$\begin{cases} x = \xi + u(\xi;\ p, \lambda), \\ z = \eta + v(\xi, \eta;\ p, \lambda), \end{cases}$$

which is affine in η, *such that the mapping* (2.8) *is transformed to* A_+:

$$\begin{cases} \xi_1 = \xi + \omega + H_+(p, \lambda) + f_+(\xi, \eta;\ p, \lambda), \\ \eta_1 = \eta + g_+(\xi, \eta;\ p, \lambda), \end{cases} \tag{2.10}$$

where $H_+(p, \lambda) = p^{2n+1} - \lambda + h_+(p, \lambda)$ *with* $h_+(p, \lambda) = h(p, \lambda) + \hat{h}(p, \lambda)$. *Moreover, the new perturbation satisfies*

$$\| f_+ \| + \frac{1}{r_+} \| g_+ \| \leq \varepsilon_+ = \alpha\rho_+^{\tau+2} E_+$$

on $D(s_+, r_+) \times M_+$, *where*

$$\rho_+ = \frac{1}{2}\rho, \quad \eta = \sqrt{E}, \quad s_+ = s - 5\rho, \quad r_+ = \eta r, \quad E_+ = c2^{\tau+2} E^{\frac{3}{2}},$$

and

$$M_+ = \left\{(p, \lambda') \in \mathbb{C} \times \mathbb{C} \,|\, p \in \Pi_{\sigma - \frac{1}{2}\delta}, (p, \lambda) \in \Gamma, |\lambda' - \lambda| \leq \frac{1}{2}\delta\right\}.$$

The term $\hat{h}(p, \lambda)$ *which may generate the drift of frequency after one KAM step satisfies that*

$$|\hat{h}(p, \lambda)| \le \varepsilon = \alpha\rho^{\tau+2}E, \quad \forall (p, \lambda) \in M$$

and

$$|\hat{h}_p(p, \lambda)| + |\hat{h}_\lambda(p, \lambda)| \le \frac{2\varepsilon}{\delta}, \quad \forall (p, \lambda) \in M_+.$$

Thus if

$$2\alpha\rho^{\tau+2}E \le \frac{1}{4}\delta, \tag{2.11}$$

then the equation

$$H_+(p, \lambda) = p^{2n+1} - \lambda + h_+(p, \lambda) = 0$$

determines implicitly an analytic curve in M_+ *given by*

$$\lambda_+ : p \in \Pi_{\sigma_+} \to \lambda_+(p)$$

with $\sigma_+ = \sigma - \frac{1}{2}\delta$, *and* λ_+ *satisfies*

$$|\lambda_+(p) - \lambda(p)| \le 2\varepsilon = 2\alpha\rho^{\tau+2}E \le \frac{1}{4}\delta, \tag{2.12}$$

and

$$\Gamma^+ = \{(p, \lambda_+(p)),\ p \in \Pi_{\sigma_+}\} \subset M_+. \tag{2.13}$$

Let $\delta_+ = \dfrac{\pi\alpha}{4K_+^{\tau+1}}$, *where* K_+ *satisfies* $K_+^n e^{-K_+\rho_+} = E_+$. *If*

$$\delta_+ \le \frac{1}{4}\delta, \tag{2.14}$$

then we have $N(\Gamma^+, \delta_+) \subset M_+$.

Proof of Lemma 2.1. The above lemma is actually one KAM step. We divide the KAM step into several parts.

A. *Truncation*. Let $Q_f = f(x, 0; p, \lambda)$ and $Q_g = g(x, 0; p, \lambda) + g_z(x, 0; p, \lambda)z$. It follows that

$$\| Q_f \| \le \varepsilon, \quad \| Q_g \| \le 2r\varepsilon$$

on $D(s, r) \times M$, and

$$\| f - Q_f \| \le c\eta\varepsilon, \quad \| g - Q_g \| \le c\eta^2 r\varepsilon \tag{2.15}$$

on $D(s, 2\eta r) \times M$.

Let

$$Q_f = \sum_{k \in \mathbb{Z}} Q_{fk}(p, \lambda) e^{ikx}$$

and

$$R_f = \sum_{|k| \le K} Q_{fk}(p, \lambda) e^{ikx}.$$

By the definition of norm, we have

$$\| Q_f - R_f \| \le cK^n e^{-K\rho}\varepsilon, \quad \| Q_g - R_g \| \le cK^n e^{-K\rho} r\varepsilon \tag{2.16}$$

on $D(s - \rho, r) \times M$.

B. *Construction of u, v*. From the theory of transformations, we know that after a canonical change of variables, the transformed mapping of a symplectic mapping is also symplectic. Analogously, for a reversible mapping, if the change of variables commutes with the involution *R*, then the transformed mapping is also reversible with respect to the same involution *R*. If the change of variable $U : (\xi, \eta) \to (x, z)$ is of the form

$$\begin{cases} x = \xi + u(\xi, \eta), \\ z = \eta + v(\xi, \eta), \end{cases}$$

then from the equality $RU = UR$, it follows that

$$\begin{cases} u(-\xi, \eta) = -u(\xi, \eta), \\ v(-\xi, \eta) = v(\xi, \eta). \end{cases} \tag{2.17}$$

In this case, the transformed mapping $U^{-1}AU$ of A is also reversible with respect to the involution $R : (\xi, \eta) \to (-\xi, \eta)$.

In the following, we will determine the unknown functions u and v to satisfy the condition (2.17) in order to guarantee that the transformed mapping $U^{-1}AU$ is also reversible.

Let $\omega(p, \lambda) = \omega + H(p, \lambda)$. As one does in Hamiltonian systems, one may solve u and v from the following equations:

$$\begin{cases} u(\xi + \omega(p, \lambda)) - u(\xi) = R_f(\xi) - [R_f(\xi)], \\ v(\xi + \omega(p, \lambda), \eta) - v(\xi, \eta) = R_g(\xi, \eta) - [R_g(\xi, \eta)]. \end{cases} \tag{2.18}$$

Indeed one can solve these functions from the above equations. But the problem is that such functions u and v do not, in general, satisfy the condition (2.17), i.e., the transformed mapping $U^{-1}AU$ is no longer a reversible mapping with respect to R. Therefore, we cannot use the above equations to determine the functions u and v.

Instead of solving the above equations (2.18), we may find these functions u and v from the following modified equations:

$$\begin{cases} u(\xi + \omega(p, \lambda)) - u(\xi) = \tilde{f}(\xi), \\ v(\xi + \omega(p, \lambda), \eta) - v(\xi, \eta) = \tilde{g}(\xi, \eta), \end{cases} \tag{2.19}$$

with

$$\begin{cases} \tilde{f}(\xi) = \frac{1}{2}(R_f(\xi) - [R_f(\xi)] + R_f(-\xi - \omega(p, \lambda)) - [R_f(-\xi - \omega(p, \lambda))]), \\ \tilde{g}(\xi, \eta) = \frac{1}{2}(R_g(\xi, \eta) - R_g(-\xi - \omega(p, \lambda), \eta)), \end{cases}$$

where $[\cdot]$ denotes the mean value of a function over the angular variable ξ.

It is easy to verify that $\tilde{f}(-\xi - \omega(p, \lambda)) = \tilde{f}(\xi)$ and $\tilde{g}(-\xi - \omega(p, \lambda), \eta) = -\tilde{g}(\xi, \eta)$. So, by Lemma 3.1, the functions u and v meet the condition (2.17). In this case, the transformed mapping $U^{-1}AU$ is also reversible with respect to the involution $R : (\xi, \eta) \to (-\xi, \eta)$.

Because the right hand sides of equations (2.19) have the mean value zero, we can solve u and v from equations (2.19). But the difference equations introduce the small divisors. By the definition of $N(\Gamma, \delta)$, it follows that

$$\omega : (p, \lambda) \in N(\Gamma, \delta) \to \omega(p, \lambda) \in \mathcal{O}_{\tilde{\delta}}.$$

Thus, we have

$$\left| \frac{k\omega(p, \lambda)}{2\pi} - l \right| \geq \frac{\alpha}{2|k|^{\tau}}, \quad \forall 0 < |k| \leq K$$

for any $(p, \lambda) \in N(\Gamma, \delta)$.

Let $\tilde{f}_k$ and $\tilde{g}_k$ be Fourier coefficients of $\tilde{f}$ and $\tilde{g}$. Then, we have

$$u_k = \frac{\tilde{f}_k}{e^{ik\omega(p,\lambda)} - 1}, \quad v_k = \frac{\tilde{g}_k}{e^{ik\omega(p,\lambda)} - 1}, \quad 0 < |k| \leq K,$$

and $u_k = 0$, $v_k = 0$ for $k = 0$ or $|k| > K$. Moreover, u and v are affine in η.

C. *Estimates of* u *and* v. By the definition of norm, we have

$$\| \tilde{f} \| \leq c\varepsilon, \quad \| \tilde{g} \| \leq cr\varepsilon$$

on $D(s - \rho, r) \times N(\Gamma, \delta)$.

By Lemma 3.1, it follows that

$$\| u \| \leq \frac{c\varepsilon}{\alpha\rho^{\tau+1}}, \quad \| v \| \leq \frac{cr\varepsilon}{\alpha\rho^{\tau+1}} \tag{2.20}$$

on $D(s - 2\rho, r) \times N(\Gamma, \delta)$.

Using Cauchy's estimate on the derivatives of u and v in the domain $D(s - 3\rho, r/4) \times N(\Gamma, \delta)$, we obtain

$$\| u_{\xi} \| \leq \frac{c\varepsilon}{\alpha\rho^{\tau+2}}, \tag{2.21}$$

$$\| v_{\xi} \| < \frac{cr\varepsilon}{\alpha\rho^{\tau+2}}, \quad \| v_{\eta} \| < \frac{c\varepsilon}{\alpha\rho^{\tau+1}}. \tag{2.22}$$

In the sane way as in [5], we can verify that $U^{-1}AU$ is well defined in $D(s - 5\rho, \eta r) \times N(\Gamma, \delta)$ with $0 < \eta \leq 1/8$.

D. *Estimates of* f_+ *and* g_+. Since $\hat{h}(p, \lambda) = [R_f(\xi)]$, the estimate for $\hat{h}$ holds. Let M_+ be defined as in Lemma 2.1. Then we have $\operatorname{dist}(M_+, \partial M) \geq \frac{1}{2}\delta$. By Cauchy's estimates, it follows the estimates for $\hat{h}_p$ and $\hat{h}_\lambda$.

Moreover, by (2.11) and the implicit function theorem, if

$$|h_{+\lambda}(p, \lambda)| \leq \frac{1}{2}, \quad \forall (p, \lambda) \in M_+,$$

then the equation

$$H_+(p, \lambda) = p^{2n+1} - \lambda + h_+(p, \lambda) = 0$$

determines an analytic mapping

$$\lambda_+ : p \in \Pi_{\sigma_+} \to \lambda_+(p).$$

It is easy to see that the conclusions (2.12) and (2.13) hold. By (2.14), we have $N(\Gamma^+, \delta_+) \subset M_+$.

We try to transform the mapping (2.8) into a new mapping (2.10) by U. Due to $U^{-1}AU = A_+$, we have

$$f_+(\xi, \eta) = u(\xi) - u(\xi_1) - \hat{h}(p, \lambda) + f(\xi + u, \eta + v).$$

By the first difference equation of (2.19), we have

$$f_+ = u(\xi + \omega(p, \lambda)) - u(\xi_1) + f(\xi + u, \eta + v) - \tilde{f}(\xi) - \hat{h}(p, \lambda).$$

From the reversibility of A, it follows that

$$\begin{cases} f(-x - \omega(p, \lambda) - f, z + g) - f(x, z) = 0, \\ g(-x - \omega(p, \lambda) - f, z + g) + g(x, z) = 0. \end{cases}$$

Hence, we have

$$\begin{aligned} & f(\xi, \eta) - \tilde{f}(\xi) - \hat{h}(p, \lambda) \\ = & \frac{1}{2}(f(\xi, \eta) - R_f(\xi) + f(\xi, \eta) - R_f(-\xi - \omega(p, \lambda))) \end{aligned}$$

$$= \frac{1}{2}(f(\xi, \eta) - R_f(\xi) + f(-\xi - \omega(p, \lambda), \eta) - R_f(-\xi - \omega(p, \lambda))$$

$$- f(-\xi - \omega(p, \lambda), \eta) + f(-\xi - \omega(p, \lambda) - f, \eta + g)),$$

which yields that

$$\| f(\xi, \eta) - \tilde{f}(\xi) - \hat{h}(p, \lambda) \| \leq c\eta\varepsilon + cK^n e^{-K\rho}\varepsilon + \frac{2\varepsilon^2}{\rho}.$$

By (2.15)-(2.16) and (2.20)-(2.22), the following estimate of f_+ holds:

$$\begin{aligned} \| f_+ \| &\leq \| u_\xi \| \cdot \| \hat{h}(p, \lambda) + f_+ \| + \| f_\xi \| \cdot \| u \| + \| f_\eta \| \cdot \| v \| \\ &\quad + c\eta\varepsilon + cK^n e^{-K\rho}\varepsilon + \frac{2\varepsilon^2}{\rho} \\ &\leq \frac{c\varepsilon}{\alpha\rho^{\tau+2}} \| f_+ \| + \frac{c\varepsilon^2}{\alpha\rho^{\tau+2}} + c\eta\varepsilon + cK^n e^{-K\rho}\varepsilon. \end{aligned} \tag{2.23}$$

Similarly, for g_+, we get

$$g_+ = v(\xi + \omega(p, \lambda), \eta) - v(\xi_1, \eta_1) + g(\xi + u, \eta + v) - \tilde{g}(\xi, \eta)$$

and

$$\frac{1}{r_+}\| g_+ \| \leq \frac{c\varepsilon}{\alpha\eta\rho^{\tau+2}} \| f_+ \| + \frac{c\varepsilon}{\alpha\rho^{\tau+1}} \frac{\| g_+ \|}{r_+} + \frac{c\varepsilon^2}{\alpha\eta\rho^{\tau+2}} + c\eta\varepsilon + \frac{cK^n e^{-K\rho}\varepsilon}{\eta}. \tag{2.24}$$

Now if one chooses ε sufficiently small such that

$$\frac{c\varepsilon}{\alpha\eta\rho^{\tau+2}} < \frac{1}{2}, \tag{2.25}$$

then combing with (2.23) and (2.24), we have

$$\| f_+ \| + \frac{1}{r_+}\| g_+ \| \leq \frac{c\varepsilon^2}{\alpha\eta\rho^{\tau+2}} + c\eta\varepsilon + \frac{cK^n e^{-K\rho}\varepsilon}{\eta}.$$

By the definition of ρ_+ and E_+, we have

$$\| f_+ \| + \frac{1}{r_+}\| g_+ \| \leq \frac{c\varepsilon^2}{\alpha\eta\rho^{\tau+2}} \leq c\alpha\rho_+^{\tau+2} 2^{\tau+2} E^{\frac{3}{2}} = \alpha\rho_+^{\tau+2} E_+$$

on $D(s_+, r_+) \times M_+$. Thus this ends the proof of Lemma 2.1.

KAM iteration. If the conditions (2.11), (2.14) and (2.25) hold for every mapping A_j, then the iteration lemma is valid for all $j \geq 0$. In the following, we choose some suitable parameters to ensure that the above iteration can go on infinitely.

At the initial step, set $s_0 = s$, $r_0 = r$, $\rho_0 = s/20$, $\varepsilon_0 = \alpha\rho_0^{\tau+2}E_0$, $\eta_0 = E_0^{\frac{1}{2}}$. Let K_0 satisfy $K_0^n e^{-K_0\rho_0} = E_0$.

Assume ρ_j, s_j, r_j and E_j are all well defined for jth step. Then, we define $\rho_{j+1} = \rho_j/2$, $s_{j+1} = s_j - 5\rho_j$, $\eta_j = E_j^{\frac{1}{2}}$, $r_{j+1} = \eta_j r_j$, $E_{j+1} = c2^{\tau+2}E_j^{\frac{3}{2}}$, ε_{j+1}, η_{j+1} and K_{j+1} are defined similarly.

Let $M_0 = M$ and $D_0 = D(s_0, r_0)$. By the iteration lemma, we have a sequence of closed sets $\{M_j\}$ with $M_{j+1} \subset M_j$, and a sequence of mappings U_j:

$$\begin{cases} x = \xi + u_j(\xi;\ p, \lambda), \\ z = \eta + v_j(\xi, \eta;\ p, \lambda), \end{cases}$$

such that for any $(p, \lambda) \in M_j$, $U_j : D_j \to D_{j-1}$, where $D_j = D(s_j, r_j)$. Moreover, U_j satisfies

$$\| u_j \| + \| v_j \| \leq \frac{c\varepsilon_j}{\alpha\rho_j^{\tau+1}}$$

and

$$\| u_{j\xi} \| + \| v_{j\xi} \| < \frac{c\varepsilon_j}{\alpha\rho_j^{\tau+2}}, \quad \| v_{j\eta} \| < \frac{c\varepsilon_j}{\alpha\rho_j^{\tau+1}}$$

on $D_j \times M_j$. So the transformation $V_j = U_0 \circ U_1 \circ \cdots \circ U_j$ is well defined in $D_j \times M_j$ and is seen to take A_0 into

$$A_j = V_j^{-1} A_0 V_j.$$

More precisely, if we write A_0 as

$$\begin{cases} x_1 = x + \omega + H_0(p, \lambda) + f(x,\ z;\ p), \\ z_1 = z + g(x,\ z;\ p) \end{cases}$$

with $H_0(p, \lambda) = p^{2n+1} - \lambda$, and express V_j in the form

$$\begin{cases} x = \xi + p_j(\xi;\ p, \lambda), \\ z = \eta + q_j(\xi, \eta;\ p, \lambda), \end{cases}$$

then A_0 is transformed into A_j:

$$\begin{cases} \xi_1 = \xi + \omega + H_j(p, \lambda) + f_j(\xi, \eta;\ p, \lambda), \\ \eta_1 = \eta + g_j(\xi, \eta;\ p, \lambda), \end{cases}$$

where $H_j(p, \lambda) = p^{2n+1} - \lambda + h_j(p, \lambda)$, $h_j = \sum_{i=0}^{j-1} \hat{h}_i$. Let $\delta_j = \dfrac{\pi\alpha}{4K_j^{\tau+1}}$ and $\sigma_j = \sigma_{j-1} - \frac{1}{2}\delta_{j-1}$ with $\sigma_0 = \sigma$. From the iteration lemma, we know that the equation

$$H_j(p, \lambda) = p^{2n+1} - \lambda + h_j(p, \lambda) = 0$$

on M_j defines implicitly an analytic mapping $\lambda = \lambda_j(p)$, $p \in \Pi_{\sigma_j}$, whose graph in M_j forms an analytic curve Γ^j.

Define

$$M_{j+1} = \left\{(p, \lambda') \in \mathbb{C} \times \mathbb{C} \,|\, p \in \Pi_{\sigma_{j+1}}, (p, \lambda) \in \Gamma^j, |\lambda' - \lambda| \le \frac{1}{2}\delta_j\right\}.$$

Obviously, it follows that $M_{j+1} \subset M_j$ and $\operatorname{dist}(M_{j+1}, \partial M_j) \ge \frac{1}{2}\delta_j$. Let

$$\hat{h}_j(p, \lambda) = H_{j+1}(p, \lambda) - H_j(p, \lambda).$$

Then, we have

$$|\hat{h}_j(p, \lambda)| \le \varepsilon_j, \quad \forall (p, \lambda) \in M_j$$

and

$$|\hat{h}_{jp}(p, \lambda)| + |\hat{h}_{j\lambda}(p, \lambda)| \le \frac{2\varepsilon_j}{\delta_j}, \quad \forall (p, \lambda) \in M_{j+1}.$$

Furthermore, we have

$$|\lambda_{j+1}(p) - \lambda_j(p)| \le 2\varepsilon_j, \quad \forall (p, \lambda) \in M_{j+1}$$

and

$$\| f_j \| + \frac{1}{r_j} \| g_j \| \le \varepsilon_j = \alpha \rho_j^{\tau+2} E_j$$

on $D_j \times M_j$.

In the following, we will check the assumptions in the iteration lemma to ensure KAM step is valid for all $j \ge 0$.

Let $F_j = \frac{2\varepsilon_j}{\delta_j}$. Then it follows that

$$\frac{F_{j+1}}{F_j} \le \frac{x_{j+1}^{\tau+2}}{x_j^{\tau+2}} \frac{E_{j+1}}{E_j} = 2^{n-1} \frac{x_{j+1}^{n+\tau+1} e^{-x_{j+1}}}{x_j^{n+\tau+1} e^{-x_j}},$$

where $x_j = K_j \rho_j$. By the iteration $E_{j+1} = c 2^{\tau+2} E_j^{\frac{3}{2}}$, if E_0 is sufficiently small, E_j are all sufficiently small and so $K_j \rho_j$ are sufficiently large. Since the function $x^{n+\tau+1} e^{-x}$ decreases as $x > n + \tau + 1$, we can choose a small E_0 such that $\frac{F_{j+1}}{F_j} \le \frac{1}{4}$ and $F_j \le \frac{1}{4}$, $\forall j \ge 0$. Moreover,

$$\frac{\delta_{j+1}}{\delta_j} = \left(\frac{1}{2}\right)^{\tau+1} \left(\frac{x_j}{x_{j+1}}\right)^{\tau+1} \le \frac{1}{4}. \tag{2.26}$$

It is obvious that $\frac{c\varepsilon_j}{\alpha \eta_j \rho_j^{\tau+2}} < c E_0^{\frac{1}{2}} < \frac{1}{2}$. Thus the assumptions (2.11), (2.14) and (2.25) hold.

Convergence of the iteration. Now we prove convergence of the KAM iteration. Let $V_j = U_0 \circ U_1 \circ \cdots \circ U_j : D_j \times M_j \to D_0 \times M_0$. Writing V_j in the form:

$$\begin{cases} x = \xi + p_j(\xi;\ p,\ \lambda), \\ z = \eta + q_j(\xi,\ \eta;\ p,\ \lambda), \end{cases}$$

we have to show that p_j, q_j and their first derivatives converge uniformly to some functions p_* and q_*. In fact, let $D_* = D(s/2,\ 0)$, $M_* = \bigcap_{j \geq 0} M_j$ and $V_* = \lim_{j \to \infty} V_j$. In the same way as in [5, 13] , we have on $D_* \times M_*$,

$$\| p_* \| + \| q_* \| \leq c\rho_0 E_0$$

and

$$\| DV_* - I \| \leq cE_0.$$

Since V_j is affine in η, we have the convergence of V_j to V_* on $D(s/2,\ r/2)$.

Now we prove the convergence of h_j. By iteration, we have $h_j = \sum_{i=0}^{j-1} \hat{h}_i$. Combining with the estimates for $\hat{h}_j$, we have for all $(p,\ \lambda) \in M_j$,

$$| h_j(p,\ \lambda) | \leq \sum_{i=0}^{j-1} \varepsilon_i \leq 2\varepsilon.$$

Similarly, it follows that for all $(p,\ \lambda) \in M_j$,

$$| h_{jp}(p,\ \lambda) | + | h_{j\lambda}(p,\ \lambda) | \leq \sum_{i=0}^{j-1} F_i \leq 2F_0 \leq \frac{8}{\pi} \frac{x_0^{n+\tau+1} e^{-x_0}}{\rho_0^{n-1}}.$$

So if E_0 is sufficiently small, then we have

$$| h_{jp}(p,\ \lambda) | + | h_{j\lambda}(p,\ \lambda) | \leq \frac{1}{2}, \quad \forall (p,\ \lambda) \in M_j,$$

and the assumption (2.9) holds.

Let $h_* = \lim_{j \to \infty} h_j$. Then, for all $(p,\ \lambda) \in M_*$, we have

$$| h_*(p,\ \lambda) | \leq 2\varepsilon \tag{2.27}$$

and

$$| h_{*p}(p,\ \lambda) | + | h_{*\lambda}(p,\ \lambda) | \leq \frac{8}{\pi} \frac{x_0^{n+\tau+1} e^{-x_0}}{\rho_0^{n-1}} \leq \frac{1}{2}.$$

Let $\sigma_* = \sigma - \frac{1}{2}\sum_{j=0}^{\infty}\delta_j$. By (2.26), it follows that $\sigma_* \geq \sigma - \frac{2}{3}\delta_0$. If E_0 is sufficiently small such that $\delta_0 \leq \sigma$, then we have $\sigma_* \geq \frac{1}{3}\sigma$. Thus $\Pi_{\sigma_*} \subset \bigcap_{j\geq 0}\Pi_{\sigma_j}$.

Similarly, we can prove the convergence of λ_j on Π_{σ_*}. In fact, we can choose E_0 sufficiently small such that $F_j \leq \frac{1}{4}$ for all $j \geq 0$. Then, for $i \geq j$, it follows that

$$|\lambda_i(p) - \lambda_j(p)| \leq \sum_{l=j}^{i-1} F_l\delta_l \leq 2F_j\delta_j \leq \frac{\delta_j}{2}.$$

Since $\Gamma^j = \{(p, \lambda_j(p)) | p \in \Pi_{\sigma_j}\} \subset M_j$ and λ_j are all analytic on Π_{σ_*}, so is the limit λ. Let $\lambda_i(p) \to \lambda(p)$, $p \in \Pi_{\sigma_*}$. Then

$$|\lambda(p) - \lambda_j(p)| \leq \frac{\delta_j}{2}.$$

This implies that $\Gamma^* = \{(p, \lambda(p)) | p \in \Pi_{\sigma_*}\} \subset M_j$ and so $\Gamma^* \subset M_* = \bigcap_{j\geq 0} M_j$. Obviously, for $(p, \lambda) \in \Gamma^*$, we have

$$p^{2n+1} - \lambda + h_*(p, \lambda) = 0$$

with $h_*(p, \lambda)$ satisfying (2.27).

In the same way as in [6], we can prove that f_* and g_* are C^{∞}-smooth with respect to (p, λ) on M_*. Thus the proof of Theorem 2.2 is complete.

3. Appendix

In this section, we formulate a lemma which has been used in the previous section. For detailed proofs, we refer to [3]. In the construction of the transformation in Lemma 2.1, we will meet the following difference equation:

$$l(x + \omega) - l(x) = g(x). \tag{3.28}$$

Lemma 3.1. *Suppose that* $l(x)$ *and* $g(x)$ *are real analytic on* $D(s)$, ω *satisfies the Diophantine condition* (1.3). *Then, for any* $0 < s' < s$, *the difference equation* (3.28) *has the unique solution* $l(x) \in D(s')$ *satisfying*

$$\| l(x) \|_{s'} \leq \frac{c}{\alpha(s - s')^{\tau+1}} \| g(x) \|_{s}.$$

Moreover, if $g(-x-\omega) = g(x)$, *then* $l(x)$ *is odd in x*; *if* $g(-x-\omega) = -g(x)$, $l(x)$ *is even in x*.

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