Far East Journal of Applied Mathematics
Volume 37, Number 3, 2009, Pages 261-281
Published Online: December 18, 2009
This paper is available online at http://www.pphmj.com

EXISTENCE AND UNIQUENESS FOR THE 'WEAK' SOLUTION TO THE NON-STATIONARY, NONLINEAR PERMEABLE BOUNDARY NAVIER-STOKES FLOWS, USING TRACE-LIKE OPERATORS

JOE HLOMUKA

Department of Mathematics and Applied Mathematics
University of Venda, P. Bag X5050
Thohoyandou, 0950, South Africa
e-mail: joe.hlomuka@univen.ac.za

Abstract

In this paper, we analyse a special case of the "Sixth Problem of the Millenium: Navier-Stokes equations, existence and smoothness", whose method of solution was suggested by Ladyzhenskaya in [7]. Whereas, the latter proposed analysis of the problem for a homogeneous boundary condition, we analyse the problem, at hand, with a non-homogeneous boundary condition. Our situation of a non-homogeneous boundary condition arises out of the application of a boundary permeation model proposed by Sauer [9] for the second grade fluids. The model imposes a zero initial velocity that will be explained later. We have already applied the model and the trace-related canonical operators to confirm existence and uniqueness of the 'weak' solution to the stationary version of the current problem [3]. Our solutions are referred to as a 'weak' as they possess weak derivatives in the sense of distributions or test functions. Also, in [2] we set the necessary and sufficient conditions for the existence of weak solutions to the problem at hand. In this paper, we wish to confirm existence and uniqueness using trace-related canonical operators; thus under different conditions to those proposed by Ladyzhenskaya in [7].

2000 Mathematics Subject Classification: 76D03, 76D05, 76D27.

Keywords and phrases: nonlinear, non-stationary, permeable boundary, flows.

Received July 7, 2009

1. Symbols Used

The following symbols will be used:

$\mathbf{x} = (x_1;\ x_2;\ x_3)$:	position in a 3-dimensional space
$\mathbf{v}(\mathbf{x})$:	velocity field in the fluid; $\mathbf{x} \in \Omega$
ρ :	fluid volume density
μ :	coefficient of viscosity; assumed constant
$\mathbf{n}(\mathbf{x})$:	the unit exterior normal to Γ, where $\mathbf{x} \in \Gamma$
ds :	Lebesgue measure on the surface area on Γ
$da(\mathbf{x})$:	the effective area measure on Γ
$\varsigma(\mathbf{x})$:	density function in terms of the area measure da, i.e., $da = \varsigma ds;\quad 0 < \varsigma(\mathbf{x}) < 1$
$\delta(\mathbf{x})$:	surface thickness at any point $\mathbf{x} \in \Gamma$
$\sigma(\mathbf{x})$:	surface density of fluid at any $\mathbf{x} \in \Gamma$, i.e., $\sigma(\mathbf{x}) = \delta(\mathrm{x})\varsigma(\mathbf{x})\rho$
$\gamma_0\mathbf{v}(\mathbf{x})$:	velocity at $\mathbf{x} \in \Gamma$
p :	fluid pressure
$\eta_v(\mathbf{x})$:	the normal velocity component at $\mathbf{x} \in \Gamma$ (we shall assume that $\gamma_0\mathbf{v}(\mathrm{y}, t) = -\eta_v(\mathbf{y},\ t)\mathbf{n}(\mathbf{x})$)
$\mathbf{D}(\mathbf{v})$:	the rate of deformation tensor
κ :	mean curvature of the inner container
$H^2(\Omega)$:	Sobolev space 2, on the bounded domain $\Omega \subset R^3$
$L^2(\Omega)/L^2(\Gamma)$:	Spaces of Lebesque integrable functions defined on Ω and its boundary Γ, respectively

2. Introduction

The proposed problem analysis in [7] is in the spaces $W_m^s(\Omega)$, whereas our version is analysed in the $H_m^s(\Omega)$ spaces. The $W_m^s(\Omega)$ spaces are measurable functions $\mathbf{v} : \Omega \to R^l$ in $L^m(\Omega)$ with generalized derivatives $\partial_x^k \mathbf{v}$ with respect to x_k up to order s. The norm in these spaces is given by, $\| \mathbf{v} \|_{W_m^s(\Omega)} := \| \mathbf{v} \|_{L^m(\Omega)} + \sum_{0 \le k \le s} \|\partial_x^k \mathbf{v}\|_{L^m(\Omega \times [0, T])}$. The analysis of the problem confirms the existence and uniqueness of a *generalized solution*.

We define the $H_m^s(\Omega)$ spaces as follows:

$$H_m^s(\Omega) := \left\{ \mathbf{v} \in L^m(\Omega) : \frac{\partial^{|\kappa|}\mathbf{v}}{\partial x_{\alpha_1} \partial x_{\alpha_2} \cdots \partial x_{\alpha_n}} \in L^m(\Omega);\ \kappa \le s \right\};$$

where $|\kappa| := \kappa_1 + \kappa_2 + \cdots + \kappa_n$; and has the norm defined by $\| \mathbf{v} \|_{H_m^s(\Omega)} := \left[\sum_{0 \le |\alpha| \le s} \left\| \frac{\partial^{|\kappa|}\mathbf{v}}{\partial x_{\alpha_1} \partial x_{\alpha_2} \cdots \partial x_{\alpha_n}} \right\|_{L^m(\Omega)} \right]^{1/m}$. The derivatives $\frac{\partial^{|\kappa|}\mathbf{v}}{\partial x_{\alpha_1} \partial x_{\alpha_2} \cdots \partial x_{\alpha_n}}$ are weak derivatives, in the sense of distributions, with a compact support which is a subset of Ω.

In this case we will confirm the existence and uniqueness of a *weak solution* to the problem in hand. However, for the sake of the regularity of the time-derivative, we will seek our weak solution in the space $L^m([0, T], H_m^s(\Omega))$; $T < \infty$; $1 \le m < \infty$.

Our boundary condition is the time-dependent boundary permeation model (1); hence we have a *non-homogeneous* and *time-dependent* boundary condition. This, obviously, contrasts sharply with the boundary condition in [7]. Our boundary condition, applied in the statement for the conservation of linear momentum, leads to the *dynamic boundary equation*, $\sigma\partial_t[\eta_\nu] + \rho\zeta\eta_\nu^2 + 2\mu\kappa\eta_\nu(y, t) + \gamma_0 p(y) = P_0(t)$; $y \in \Gamma$ (see (3) on p. 4 of [2], for the statement of the conservation of linear momentum).

The preceding evolution boundary equation strengthens our boundary condition further and is subsequently used in the analysis of the problem. It could also be useful to compare the energy statements for the two problems (see (7) in this paper, and (4.2_1) on p. 272 of [7]).

3. The Setting for the Problem

The setting for the problem is a Navier-Stokes fluid, in a container which, in turn, is immersed in a larger container, with the same fluid. The boundary of the inner container is permeable and denoted by Γ; whilst the boundary of the outer container is solid and denoted by Γ_0.

The body of fluid between Γ and Γ_0 is denoted by an open bounded domain $\Omega \subset R^3$, with the *cone property* whilst that inside the inner container is denoted by Ω_0.

In this setting, Γ is the boundary for Ω and is assumed smooth and infinitely differentiable.

4. The Sauer-Maritz Boundary Permeation Model

We denote the unit exterior normal to Γ by $\mathbf{n}(\mathbf{x})$ and the trace operator γ_0 will be used to denote restriction to Γ.

The original model assumes that *Fluid particles are accelerated from rest in the domain* Ω_0 *across the boundary* Γ *into* Ω, *or they are decelerated from* Ω *across* Γ *and come to rest in* Ω_0.

It is assumed the velocity field $\mathbf{v}(x, t)$ *always satisfies the homogeneous Dirichlet condition*: $\mathbf{v}(., t) = 0$, *on* Γ_0 *for* $\mathrm{t} > 0$.

At the permeable boundary Γ, *we shall assume that*

$$\gamma_0 \mathbf{v}(\mathbf{x}, t) = -\eta_v(\mathbf{x}, t)\mathbf{n}(\mathbf{x}). \tag{1}$$

The scalar valued function η_v, defined on Γ, is unknown, and is determined by a dynamic boundary condition, which is an evolution equation. Also, the incompressibility of the fluid leads to the condition $\nabla . v(\mathbf{x}, t) = 0$, which, in turn leads to

$$\int_\Gamma \eta_v ds = 0.$$

5. Initial Problem Formulation

We look for $\mathbf{v}(.,t) \in H^2(\Omega)$ such that

$$\begin{cases} \text{(a) } \rho\,\partial_t \mathbf{v}(x,t) + \rho(\nabla.\mathbf{v}(x,t))\mathbf{v}(x,t) - \mu\Delta\mathbf{v}(x,t) \\ = -\nabla p(x) + \mathbf{f}(x,t);\ x \in \Omega \subset R^3;\ \mathbf{f} \in L^2(\Omega);\ \text{subject to} \\ \nabla.\mathbf{v}(x,t) = 0; \\ \gamma_0\mathbf{v}(y,t) = -\eta_v(y,t)\mathbf{n}(y);\ y \in \Gamma(\text{boundary permeation model}). \\ \text{(b) } \sigma\partial_t[\eta_v] + \rho\zeta\eta_v^2 + 2\mu\kappa\eta_v(y,t) + \gamma_0 p(y) = P_0(t). \end{cases} \tag{2}$$

For this formulation, we choose the following test functions:

$$\Theta := \{\mathbf{v}(x,t) \in L^2([0,T], H^2(\Omega)),\ \gamma_0\mathbf{v}(y,t)$$
$$= -\eta_v(y,t)\mathbf{n}(y);\ \text{on } \Gamma;\ \mathbf{v}(x,t) = 0 \text{ on } \Gamma_0;\ \nabla.\mathbf{v}(x,t) = 0;\ T < \infty\}.$$

For further analysis we transform problem (4): we multiply 3(a) by $\rho^{-\frac{1}{2}}$ and 3(b) by $\sigma^{-\frac{1}{2}}$ to obtain the following equations:

$$\begin{cases} \rho^{\frac{1}{2}}\partial_t\mathbf{v}(x,t) + \rho^{\frac{1}{2}}(\nabla.\mathbf{v}(x,t))\mathbf{v}(x,t) - \rho^{-\frac{1}{2}}\mu\Delta\mathbf{v}(x,t) + \rho^{-\frac{1}{2}}\nabla p(x) = \rho^{-\frac{1}{2}}\mathbf{f}(x,t), \\ \sigma^{\frac{1}{2}}\partial_t[\eta_v] + \rho\sigma^{-\frac{1}{2}}\zeta\eta_v^2 + 2\mu\kappa\sigma^{-\frac{1}{2}}\eta_v(y,t) + \sigma^{-\frac{1}{2}}\gamma_0 p(y) = \sigma^{-\frac{1}{2}}P_0(t). \end{cases} \tag{3}$$

Remarks 5.1. (a) Assuming that $(\tau_1, \tau_2, \mathbf{n})$ are the unit vectors in tangential and normal directions for an arbitrary point on the surface Γ of the container, then the velocity field on the surface would be given by, $\gamma_0\mathbf{v}(y,t) = \gamma_0\mathbf{v}_1(y,t)\tau_1 + \gamma_0\mathbf{v}_2(y,t)\tau_2 - \eta_v(y,t)\mathbf{n};\ y \in \Gamma$; where $\gamma_0\mathbf{v}_1$ and $\gamma_0\mathbf{v}_2$ are tangential to the surface Γ.

(b) We, however, observe that the permeation model (1) and the definition of the set Θ of our test functions assume $\gamma_0\mathbf{v}_1$ and $\gamma_0\mathbf{v}_2$ are not effective. Thus, for the purpose of the permeation model and the choice of our weak solutions, $\gamma_0\mathbf{v}_1 = \gamma_0\mathbf{v}_2 = \mathbf{0}$. Later, this will be crucial in the characterisation of some operators.

(c) With the trace-operator restriction as defined by $\mathbf{v} \mapsto \gamma_0 \mathbf{v}$, being injective, then $\gamma_0 \mathbf{v} = \mathbf{0}$, will imply that $\mathbf{v} = \mathbf{0}$. On the other hand, $\eta_v = 0$ will imply that $\mathbf{v} = \mathbf{0}$. Hence, $\eta_v = 0$ implies that $\gamma_0 \mathbf{v} = \mathbf{0}$; which in turn implies that $\mathbf{v} = \mathbf{0}$, in terms of the definition of Θ.

6. Problem Reformulation as an Implicit Evolution Equation; the Energy Identity

First, we rewrite the problem now represented by (5) as follows:

$$\partial_t \begin{pmatrix} \rho^{\frac{1}{2}} \mathbf{v} \\ \sigma^{\frac{1}{2}} \eta_v \end{pmatrix} + \begin{pmatrix} \rho^{\frac{1}{2}} (\mathbf{v}.\nabla)\mathbf{v} \\ \rho\sigma^{-\frac{1}{2}} \varsigma \eta_{\mathbf{v}}^2 \end{pmatrix} + \mu \begin{pmatrix} -\rho^{-\frac{1}{2}} \Delta \mathbf{v} \\ 2\kappa \eta_v \sigma^{-\frac{1}{2}} \end{pmatrix} + \begin{pmatrix} \rho^{-\frac{1}{2}} \nabla p \\ \sigma^{-\frac{1}{2}} \gamma_0 p \end{pmatrix} = \begin{pmatrix} \rho^{-\frac{1}{2}} \mathbf{f} \\ \sigma^{-\frac{1}{2}} P_0(t) \end{pmatrix},$$

where the coupling is provided by the restriction of the trace operator defined by, $\mathbf{v} \mapsto \gamma_0 \mathbf{v}$, with $\gamma_0 \mathbf{v} = -\eta_v \mathbf{n}$.

We then conclude that

$$\partial_t B\mathbf{v} + L\mathbf{v} + N(\mathbf{v}) + \ell \mathbf{p} = \mathbf{F}; \tag{4}$$

where

$$B\mathbf{v} := \left\langle \rho^{-\frac{1}{2}} \mathbf{v},\ \sigma^{\frac{1}{2}} \eta_v \right\rangle \in H^2(\Omega) \times L^2(\Gamma) = Y;$$

$$L\mathbf{v} := \left\langle -\mu\rho^{-\frac{1}{2}} \Delta \mathbf{v},\ 2\mu\kappa\eta_v \sigma^{-\frac{1}{2}} \right\rangle \in Y;$$

$$N(\mathbf{v}) := \left\langle \rho^{\frac{1}{2}} (\mathbf{v}.\nabla)\mathbf{v},\ \rho\sigma^{-\frac{1}{2}} \varsigma \eta_{\mathbf{v}}^2 \right\rangle \in Y;$$

$$\ell p := \left\langle -\rho^{-\frac{1}{2}} \nabla p,\ \sigma^{-\frac{1}{2}} \gamma_0 p \right\rangle \text{ and } \mathbf{F} := \left\langle \rho^{-\frac{1}{2}} \mathbf{f},\ \sigma^{-\frac{1}{2}} P_0(t) \right\rangle \in Y.$$

Remarks 6.1. (a) The said coupling relation provided for by the restriction of the trace operator also "relates" all the preceding canonical operators to the trace-operator.

(b) It should not be hard to show that $H^2(\Omega) \times L^2(\Gamma) = Y$ is a real Hilbert space.

(c) The trace-related canonical operators will be used in the analysis of the problem at hand. As in general these operators are not closed, a suitable subspace of $H^2(\Omega)$ will be defined for the analysis of the problem.

Thus, we construct: $\Lambda := \overline{dom(B) \cap dom(L) \cap dom(N)} \supset \Theta$.

(d) In the definition of the space Λ, we left out $\ell : p \to \left\langle -\rho^{-\frac{1}{2}} \nabla p, \sigma^{-\frac{1}{2}} \gamma_0 p \right\rangle$ since the relation $(\ell \mathrm{p}, B\mathbf{w})_Y = 0$ (see p. 22 of [3]), renders the pressure ineffective in the analysis of the problem. Thus, we reformulate the problem as follows:

We seek $\mathbf{w}(., t) \in L^2([0, T], Y)$; $T < \infty$, $\mathbf{v} \in \Theta$, such that the following conditions are satisfied:

$$\begin{cases} \partial_t \mathbf{w} + L\mathbf{v} + N(\mathrm{v}) + \ell \mathbf{p} = \mathbf{F}, \\ \text{subject to :} \\ \mathbf{w} := \langle 0, 0 \rangle; \text{ for a fluid particle starting from the position of rest.} \end{cases} \tag{5}$$

Remarks 6.2. (a) The initial conditions denote the status of a particle starting from rest, either from outside or inside of the container.

(b) $\mathbf{w} := B\mathbf{v}$. The solution to (5) will lead to the solution to (2), through the bijection, $\chi : L^2([0, T], H^2(\Omega)) \to L^2([0, T], Y)$; $T < \infty$, as referred to on p. 2 of [2]. This bijection owes its existence to the Lions/Magenes trace theorem, on p. 39, Theorem 8.3 of [8]. Also see Remarks 5.1(a) and 5.1(c) in this paper.

We now derive the energy identity for the problem: Taking the scalar product of (4) with $B\mathbf{v}$ we obtain the following interim expression:

$$(\partial_t \mathbf{w}, \mathbf{w})_Y + (L\mathbf{v}, \mathbf{w})_Y + (N(\mathbf{v}), \mathbf{w})_Y + (\ell \mathbf{p}, \mathbf{w})_Y = (\mathbf{F}, \mathbf{w})_Y. \tag{6}$$

To simplify (6), we use the following identities:

$$(\partial_t \mathbf{w}, \mathbf{w})_Y = \frac{1}{2} \frac{d}{dt} \left[\rho \| \mathbf{v} \|^2_{H^2(\Omega)} + \sigma \| \eta_v \|^2_{L^2(\Gamma)} \right];$$

$(L\mathbf{v}, \mathbf{w})_Y = 2\mu \| D(\mathbf{v}) \|^2_{L^2(\Omega)}$; by (1) of Proposition 3.3.1 in [4];

$$(N(\mathbf{v}),\ \mathbf{w})_Y = \rho\int_\Gamma\left[\varsigma - \frac{1}{2}\right]\eta_v^3 ds;\ \text{by (2) of Proposition 3.3.1 in [4]};$$

$$(\ell\mathbf{p},\ \mathbf{w})_Y = 0;\ \text{by 7(b); p. 22 of [3]};$$

$$(\mathbf{F},\ \mathbf{w})_Y = (\mathbf{f},\ \mathbf{v})_{L^2(\Omega)}.$$

We then simplify (6) to obtain the energy identity:

$$\frac{1}{2}E^{/}(t) + 2\mu\| D(\mathrm{v})\|^2_{L^2(\Omega)} + \rho\int_\Gamma\left[\varsigma - \frac{1}{2}\right]\eta_v^3 ds = (\mathbf{f},\ \mathbf{v})_{L^2(\Omega)}; \tag{7}$$

where $E(t) := \rho\|\mathbf{v}\|^2_{H^2(\Omega)} + \sigma\|\eta_v\|^2_{L^2(\Gamma)}$.

As our external body forces are mainly gravitational, and hence conservative, we have, $\mathbf{f}(\mathbf{x}) := -\nabla E_p(\mathbf{x})$, where $E_p(\mathbf{x})$ is the gravitational potential energy of the fluid particle at $\mathbf{x}$. Hence $\mathbf{f}$ may be part of the expression $\frac{1}{2}E^{/}(t)$, where, $\frac{1}{2}[E(t) - \nabla E_p(\mathbf{x})]^{/} = \frac{1}{2}E^{/}(t)$. In that case we may then set $\mathbf{f} := \mathbf{0}$; leading to the vanishing of the right hand expression of (7).

We thus end up with the energy identity:

$$\frac{1}{2}E^{/}(t) + 2\mu\| D(\mathbf{v})\|^2_{L^2(\Omega)} + \rho\int_\Gamma\left[\varsigma - \frac{1}{2}\right]\eta_v^3 ds = 0. \tag{8}$$

We rewrite (8) and subsequently obtain, $\frac{1}{2}E^{/}(t) + 2\mu\| D(\mathbf{v})\|^2_{L^2(\Omega)} = \rho\int_\Gamma\left[\frac{1}{2} - \varsigma\right]\eta_v^3 ds.$ By Theorem 7.1 of [2], the integral $\int_\Gamma\left[\frac{1}{2} - \varsigma\right]\eta_v^3 ds$ is bounded. Hence

$$\frac{1}{2}E^{/}(t) + 2\mu\| D(\mathbf{v})\|^2_{L^2(\Omega)} = (\partial_t\mathbf{w},\ \mathbf{w})_Y + (L\mathbf{v},\ \mathbf{w})_Y\ \text{ is bounded.}$$

7. Compact Imbeddings: The Rellich-Kondrachov Theorem

Remarks 7.1. (a) Let $\bigcup_{\ell=1}^{M}\Omega_\ell$ be the finite union of the k-dimensional planes in R^3 that intersect the bounded subdomain Ω_0 of Ω in accordance with the statement of the Rellich-Kondrachov theorem (see Theorem 6.2; p. 144 of [1]).

(b) Since Ω is bounded, then $\Omega = \Omega_0$ (see Remarks 6.3(4); p. 145 of [1]).

(c) Hence $\bigcup_{i=1}^{\ell} \Omega_i \cap \Omega = \Omega_0^k$. In our case of R^3, $k = 2$, and $\Gamma \subset \Omega_0^k$.

(d) Assuming the *cone property* for Ω; and since, $mp > n$ for $m = 2$; $p = 2$ and $n = 3$; by Part II of the Rellich_Kondrachov theorem, and in view of the boundedness of the trace $\gamma : u \to \gamma_0 u$, the imbedding $H^2(\Omega) \to L^2(\Gamma)$ is compact; a bounded sequence (u_n) in $H^2(\Omega)$ will have a convergent subsequence.

8. The Riesz Representation for the Problem

Consider the implicit evolution equation:

$$\partial_t \mathbf{w} + L\mathbf{v} + N(\mathbf{v}) + \ell\mathbf{p} - \mathbf{F} = \mathbf{0}. \tag{9}$$

The following proposition shows that the preceding equation may be rewritten in terms of an appropriate linear and bounded operator as the result of the application of Riesz's representation theorem:

Proposition 8.1 (Riesz's representation)**.** *For the implicit evolution equation*, *there exists a bounded linear operator*

$A : H^2(\Omega) \to Y;$ *such that*,

$\partial_t \mathbf{w} + L\mathbf{v} = A\mathbf{v};$ *and the implicit equation is reduced in the form*,

$A\mathbf{v} = -N(\mathbf{v}).$

Proof. We take the scalar product of (9) with $B\mathbf{v}$ to obtain,

$$(\partial_t \mathbf{w}, \mathbf{w})_Y + (L\mathbf{v}, \mathbf{w})_Y + (N(\mathbf{v}), \mathbf{w})_Y + (\ell\mathbf{p}, \mathbf{w})_Y - (\mathbf{F}, \mathbf{w})_Y = 0.$$

Since we have shown that $(\ell\mathbf{p}, \mathbf{w})_Y = 0$ and $(\mathbf{F}, \mathbf{w})_Y = 0$,

$$(\partial_t \mathbf{w}, \mathbf{w})_Y + (L\mathbf{v}, \mathbf{w})_Y = -(N(\mathbf{v}), \mathbf{w})_Y.$$

Finally, by (8),

$$(\partial_t \mathbf{w}, \mathbf{w})_Y + (L\mathbf{v}, \mathbf{w})_Y = \rho \int_\Gamma \left[\frac{1}{2} - \varsigma\right] \eta_v^3 ds.$$

We now define $J : H^2(\Omega) \times Y \to R$ as follows:

$$J(\mathbf{v},\ \mathbf{w}) := (\partial_t \mathbf{w},\ \mathbf{w})_Y + (L\mathbf{v},\ \mathbf{w})_Y.$$

We then observe the following:

(a) J is a sesquelinear form by the definition 3.8-3 on p. 191 of [5];

(b) J is bounded since $\int_\Gamma \left[\frac{1}{2} - \varsigma\right] \eta_v^3 ds$ is bounded by Theorem 7.1 of [2].

Then, by Theorem 3.8-4, on p. 192 of [5], there exists a bounded linear operator $A : H^2(\Omega) \to Y$ such that $J(\mathbf{v}, \mathbf{w}) = (A\mathbf{v}, \mathbf{w})_Y$ and hence, $\partial_t \mathbf{w} + L\mathbf{v} = A\mathbf{v}$; which implies that

$$A\mathbf{v} = -N(\mathbf{v});\ \text{and the result follows.} \tag{10}$$

9. Characterization of the Linear Operator $\partial_t B + L;\ t \in [0,\ T]T < \infty$

We note that in the following analysis, the abstract Cauchy problem (5) is not directly involved. However, due to the established relation, $H^2(\Omega) \cong H^2(\Omega) \times H^{3/2}(\Gamma)$, (see Proposition 10.4; (15) and (16) on p. 27 of [3]); the final result of the analysis will infer on (5) as well.

Proposition 9.1. (a) The operator $\partial_t B + L$ is self-adjoint;

(b) There exists $\xi > 0$ (independent of $\mathbf{v}$) such that

$$\| (\partial_t B + L)\mathbf{v} \|_Y^2 \geq \alpha \| \mathbf{v} \|_{H^2(\Omega)}^2.$$

Proof. (a) For $\theta,\ \varphi \in \Theta$,

$$\begin{aligned}((\partial_t B + L)\theta, B\varphi)_Y &= (\partial_t B\theta,\ B\varphi)_Y + (L\theta,\ B\varphi)_Y \\ &= \frac{1}{2}\frac{d}{dt}(B\theta,\ B\varphi)_Y + (L\theta,\ B\varphi)_Y \\ &= \frac{1}{2}\frac{d}{dt}[\rho(\theta,\ \varphi)_{H^2(\Omega)} + \sigma(\eta_\theta,\ \eta_\varphi)_{L^2(\Gamma)}] + 2\mu(D(\theta),\ D(\varphi))_{L^2(\Omega)},\end{aligned}$$

$$\begin{aligned}(B\theta, (\partial_t B + L)\varphi)_Y &= (B\theta, \partial_t B\varphi)_Y + (B\theta, L\varphi)_Y \\ &= \frac{1}{2}\frac{d}{dt}(B\theta, B\varphi)_Y + (B\theta, L\varphi)_Y \\ &= \frac{1}{2}\frac{d}{dt}[\rho(\theta, \varphi)_{H^2(\Omega)} + \sigma(\eta_\theta, \eta_\varphi)_{L^2(\Gamma)}] + 2\mu(D(\theta), D(\varphi))_{L^2(\Omega)}.\end{aligned}$$

Hence, if

$$((\partial_t B + L)\theta, B\varphi)_Y = (B\theta, (\partial_t B + L)\varphi)_Y.$$

Since $H^2(\Omega) \cong H^2(\Omega) \times H^{3/2}(\Gamma)$, then $\mathbf{v}$ is identifiable with $B\mathbf{v}$ and the result follows. □

(b) We now put $\theta = \varphi = \mathbf{v}$.

Then

$$\begin{aligned}((\partial_t B + L)\mathbf{v}, B\mathbf{v})_Y &= \frac{1}{2}\frac{d}{dt}\| B\mathbf{v} \|_Y^2 + (L\mathbf{v}, B\mathbf{v}) \\ &= \frac{1}{2}\frac{d}{dt}\left[\rho\| \mathbf{v} \|_{H^2(\Omega)}^2 + \sigma\| \eta_v \|_{L^2(\Gamma)}^2\right] + 2\mu\| D(\mathbf{v}) \|_{L^2(\Omega)}^2.\end{aligned}$$

Therefore,

$$((\partial_t B + L)\mathbf{v}, B\mathbf{v})_Y = \rho\int_\Gamma \left[\frac{1}{2} - \varsigma\right]\eta_v^3 ds.$$

For the permeable boundary flows $(\eta_{\mathbf{v}} \neq 0)$, there are two possibilities:

Case I. $((\partial_t B + L)\mathbf{v}, B\mathbf{v})_Y \geq 0$,

$$\text{for } \begin{cases} \varsigma \leq \dfrac{1}{2} \text{ and } \eta_v > 0 \text{ (outward flows)}, \\ \qquad\qquad \text{or} \\ \varsigma \geq \dfrac{1}{2} \text{ and } \eta_v < 0 \text{ (inward flows)}. \end{cases}$$

By definition (2); on p. 470 of [5], $\partial_t B + L > 0$. Hence, there exists $C > 0$, such that,

$$\begin{aligned}\| (\partial_t B + L)\mathbf{v} \|_Y^2 &\geq C\| B\mathbf{v} \|_Y^2 \\ &= C[\rho\| \mathbf{v} \|_{H^2(\Omega)}^2 + \sigma\|\eta_v\|_{L^2(\Gamma)}^2] \\ &\geq C\rho\| \mathrm{v} \|_{H^2(\Omega)}^2.\end{aligned}$$

With $\xi = C\rho$, the result follows.

Case II. $((\partial_t B + L)\mathbf{v},\ B\mathbf{v})_Y \le 0,$

$$\text{for}\begin{cases} \varsigma \ge \dfrac{1}{2} \text{ and } \eta_\nu > 0\,(\text{outward flows}), \\ \qquad\quad \text{or} \\ \varsigma \le \dfrac{1}{2} \text{ and } \eta_\nu < 0\,(\text{inward flows}). \end{cases}$$

We now consider the spectrum of the operator $\partial_t B + L$ as defined by the following eigenvalue problem:

$$(\partial_t B + L)\mathbf{v} = \lambda B\mathbf{v};\ \lambda \in \sigma(\partial_t B + L).$$

Therefore,

$$\begin{aligned}((\partial_t B + L)\mathbf{v},\ B\mathbf{v})_Y &= \lambda(B\mathbf{v},\ B\mathbf{v})_Y \\ &= \lambda\| B\mathbf{v} \|_Y^2.\end{aligned}$$

Hence, for this case, $\lambda < 0$. Note that $\lambda = 0$ is excluded for flows.

Proceeding, we have,

$$\begin{aligned}\| (\partial_t B + L)\mathbf{v} \|_Y^2 &\ge \lambda^2 \| B\mathbf{v} \|_Y^2 \\ &= \lambda^2[\rho\| \mathbf{v} \|_{H^2(\Omega)}^2 + \sigma\|\eta_\nu\|_{L^2(\Gamma)}^2] \\ &\ge \lambda^2\rho\| \mathbf{v} \|_{H^2(\Omega)}^2 \ge \frac{\rho}{(r_\sigma[\partial_t B + L])^2} \| \mathbf{v} \|_{H^2(\Omega)}^2;\end{aligned}$$

where $r_\sigma(\partial_t B + L)$ is the spectral radius of the operator $\partial_t B + L$, defined by, $r_\sigma(\partial_t B + L) := \lim\limits_{n\to\infty} \sqrt[n]{\|(\partial_t B + L)^n\|}$ (see (11) on p. 378 of [5]). Since $\partial_t B + L$ is bounded, the limit does exist. With $\xi := \dfrac{\rho}{(r_\sigma[\partial_t B + L])^2}$, the result follows. □

Proposition 9.2. *The operator* $\partial_t B + L$,

(a) *is invertible, and* $(\partial_t B + L)^{-1}$ *is a bounded linear operator for* $t \in [0, T]$, $T < \infty$;

(b) *is compact*.

Proof. (a) We put $(\partial_t B + L)\mathbf{v} = \langle 0, 0\rangle$. Then $\langle -\rho^{1/2}(\mathbf{v}.\nabla)\mathbf{v}, -\rho\sigma^{-1/2}\varsigma\eta_v^2\rangle$ $= \langle 0, 0\rangle$. However, this implies that, $\rho\sigma^{-1/2}\varsigma\eta_v^2 = 0$; which in turn, implies that $\eta_v = 0$.

Using general surface coordinates, any point on Γ may be represented through the basis $(\tau_1, \tau_2, \mathbf{n})$, where τ_1 and τ_2 are tangential and $\mathbf{n}$ normal to Γ.

Hence, in our situation, the velocity field at any point on the surface Γ of the container may be represented in the form $\gamma_0\mathbf{v} = \gamma_0 v_1\tau_1 + \gamma_0 v_2\tau_2 - \eta_v\mathbf{n}$.

Our permeation model, $\gamma_0\mathbf{v} = -\eta_v\mathbf{n}$, implies that, for our analysis, $\gamma_0 v_1 = \gamma_0 v_2$ $= 0$. Hence, $\eta_v = 0$ implies that $\gamma_0\mathbf{v} = 0$; which in turn, implies that $\mathbf{v} = 0$, by the permeation model and the injection defined by, $\mathbf{v} \mapsto \gamma_0\mathbf{v}$, for any fluid particle racing to the surface Γ.

Therefore, $(\partial_t B + L)\mathbf{v} = \langle 0, 0\rangle$ implies that $\operatorname{Ker}(\partial_t B + L) = \{0\}$. By Theorem 2.6-10; p. 88 of [5], $(\partial_t B + L)^{-1}$ exists; is linear and bounded on $\Theta \subset \Lambda$. □

(b) By the Riesz's representation (see Proposition 7.1), there exists a bounded linear operator A such that, $\partial_t B + L = A$; which implies that $\partial_t B + L$ is bounded, for $t \in [0, T]$; $T < \infty$.

Further, the operator, L defined by $L\mathbf{v} := \left\langle -\mu\rho^{-1/2}\Delta\mathbf{v}, 2\mu\sigma^{-1/2}\kappa\eta_v\right\rangle$ is compact in $\Theta \subset \Lambda$; by Proposition 10.1; p. 25 of [3].

Since $\partial_t B + L$ is bounded, $\partial_t B$ is bounded for $t \in [0, T]$; $T < \infty$.

Let $(\mathbf{v}_n)$ be a bounded sequence in Θ.

Then $(\partial_t B\mathbf{v}_n)$ is a bounded sequence in Y. Then there exists a subsequence $(\partial_t B\mathbf{v}_q^{/})$ of $(\partial_t B\mathbf{v}_n)$ in Y; where, $B\mathbf{v}_n := \left\langle \rho^{1/2}\mathbf{v}_n, \sigma^{1/2}\eta_{\mathbf{v}_n}\right\rangle$. By the compact imbedding of the Rellich-Kondrachov theorem (Remarks 7.1(d)), the subsequence $(\partial_t B\mathbf{v}_q^{/}) = \partial_t\left\langle \rho^{1/2}\mathbf{v}_q^{/}, \sigma^{1/2}\eta_{\mathbf{v}_q^{/}}\right\rangle$, due to the compact imbedding $H^2(\Omega) \to L^2(\Gamma)$, converges in Y. Therefore, $\partial_t B$ is compact for $t \in [0, T]$; $T < \infty$. This implies that $\partial_t B + L$ is compact for $t \in [0, T]$; $T < \infty$, and the result follows. □

Remarks 9.3. (a) In general, for our permeable boundary Γ, with the arbitrary surface coordinate system $(\tau_1, \tau_2, \mathbf{n})$, the velocity field is given by

$$\gamma_0\mathbf{v} := \gamma_0 v_1\tau_1 + \gamma_0 v_2\tau_2 + \eta_v\mathbf{n} \in L^2(\Gamma).$$

(b) We observe that whilst $\gamma_0 v_1\tau_1 + \gamma_0 v_2\tau_2 \in L^2(\Gamma)$, $\eta_v\mathbf{n} \in L^2(\Gamma)^{\perp}$. This is not surprising as $L^2(\Gamma)$ is a Hilbert space.

(c) But by the trace theorem and the permeation model, $\gamma_0\mathbf{v} := -\eta_v\mathbf{n} \in H^{3/2}(\Gamma)$; which renders the component $\gamma_0 v_1\tau_1 + \gamma_0 v_2\tau_2$ "redundant" for our current problem. Hence, in this particular special case, we assert that, $\dim[H^{3/2}(\Gamma)] = 1$; and hence the relation, $H^2(\Omega) \cong H^2(\Omega) \times H^{3/2}(\Gamma)$ is once more confirmed.

(d) In general, it may not be necessarily true that $H^{3/2}(\Gamma) = L^2(\Gamma)^{\perp}$ (see the definition of the Nikol'skii spaces in [8]).

10. Existence and the Uniqueness of the Weak Solution to the Problem

Proposition 10.1. *The operator* $(\partial_t B + L)^{-1}(-N)$ *is compact in* Θ.

Proof. We have, $\partial_t B + L = -N$, which implies that $-N$ is compact, by Proposition 8.2(b). By Proposition 8.2(a) and Theorem 2.6 -10; on p. 88 of [5], $(\partial_t B + L)^{-1}$ exists; is linear and bounded in $\Theta \subset \Lambda$.

We also have that

$$(\partial_t B + L)^{-1} : H^2(\Omega) \times H^{3/2}(\Gamma) \to H^2(\Omega);$$

$$-N : H^2(\Omega) \to H^2(\Omega) \times H^{3/2}(\Gamma).$$

Since $H^2(\Omega) \cong H^2(\Omega) \times H^{3/2}(\Gamma)$ (see (15) and (16) of Theorem 10.4 on p. 27 of [3]), $(\partial_t B + L)^{-1}$ and $(-N)$ satisfy the functions space requirement of Lemma 8.3-2; on p. 422 of [5]. Hence, the operator $(\partial_t B + L)^{-1}(-N)$ is compact on Θ. $\square$

Lemma 10.2. *The solution to the equation*, $\mathbf{v} = -\alpha(\partial_t B + L)^{-1} N(\mathbf{v})$; $\alpha \in (0, 1)$, *is uniformly bounded in* Θ.

Proof. The solution of, $\mathbf{v} = -\alpha(\partial_t B + L)^{-1} N(\mathbf{v})$; $\alpha \in (0, 1)$ is the same as the solution of $(\partial_t B + L)\mathbf{v} = -\alpha N(\mathbf{v})$. Now, $\| (\partial_t B + L)\mathbf{v} \|_Y^2 = \alpha^2 \| N(\mathbf{v}) \|_Y^2$. However, By Proposition 9.1(b), there exists $\gamma > 0$ such that, $\| (\partial_t B + L)\mathbf{v} \|_Y^2 \geq \xi \| \mathbf{v} \|_{H^2(\Omega)}^2$ Therefore, $\xi \| \mathbf{v} \|_{H^2(\Omega)}^2 \leq \alpha^2 \| N(\mathbf{v}) \|_Y^2$.

This implies that $\sqrt{\xi} \| \mathbf{v} \|_{H^2(\Omega)} \leq \alpha \| N(\mathbf{v}) \|_Y = \alpha \| A(\mathbf{v}) \|$; by (10). Since operator A is bounded, it is therefore continuous. Hence, for any arbitrarily given $\varepsilon > 0$, there exists $\delta > 0$ such that, $0 < \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^2(\Omega)} < \delta$, implies that, $\| A(\mathbf{v}_1) - A(\mathbf{v}_2) \|_{H^2(\Omega)} < \varepsilon$.

Also, choosing $\delta < \dfrac{\varepsilon}{\| A \|}$, we have, $0 < \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^2(\Omega)} < \dfrac{\varepsilon}{\| A \|}$ implies that, $\| A \| \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^2(\Omega)} < \varepsilon$. This, in turn, implies that $\| A(\mathbf{v}_1) - A(\mathbf{v}_2) \|_{H^2(\Omega)} < \varepsilon$, since A is a bounded linear operator.

On the other hand, by (10), $\| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^2(\Omega)} < \dfrac{\varepsilon}{\| A \|}$, implies that, $\sqrt{\xi} \| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^2(\Omega)} < \alpha\varepsilon$; which in turn implies that, $\| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^2(\Omega)} < \dfrac{\alpha\varepsilon}{\sqrt{\xi}}$. Therefore, the continuity of N or A demands that, $\| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^2(\Omega)} < \varepsilon \min\left(\dfrac{\alpha}{\sqrt{\xi}}, \dfrac{1}{\| A \|} \right)$; and the result follows for $\mathbf{v} := \mathbf{v}_1 - \mathbf{v}_2$. □

Main result. Although our analysis up to now has involved $\mathbf{v}(\mathbf{x}, \mathbf{t})$, the solution of our original formulation, as opposed to $\mathbf{w}(t)$, the solution of our second formulation of the problem as an implicit abstract problem, in our following main result, we show that the solution to one problem implies the solution to the other problem. Strictly speaking, we should be referring to the solution pair $\mathbf{v}(\mathbf{x}, t)$; $p(\mathbf{x})$ or $\mathbf{w}(t)$; $p(\mathbf{x})$. However, since pressure is eliminated through $(\ell p, B\mathbf{v})_Y = 0$, and does not take part in our analysis, the main result is confirmed in terms of the velocity field only. The unique velocity field may then be used to calculate the corresponding pressure.

Theorem 10.3 (Existence and uniqueness)**.** *For* $\mathbf{v} \in \Theta$, *the following statements are equivalent*:

(I) *the solution to the equation*, $\mathbf{v} = -\alpha(\partial_t B + L)^{-1} N(\mathbf{v})$; $\alpha \in (0, 1)$, *exists and is unique*.

(II) *the solution to the equation*, $\partial_t \mathbf{w} + L\mathbf{v} + N(\mathbf{v}) + \ell\mathbf{p} = 0$, *exists and is unique*.

Proof. (I) By Lemma 10.2, the solution to the equation is uniformly bounded. By Proposition 10.1, the operator, $(\partial_t B + L)^{-1}(-N)$ is compact. Therefore, by the Leray-Schauder Theorem, p. 245 in [10], the solution to the given equations exists.

To prove uniqueness to the solution: Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the solutions to the given equation. Then

$$\begin{aligned}\| \mathbf{v}_1 - \mathbf{v}_2 \|_{H^2(\Omega)} &= \alpha\|(\partial_t B + L)^{-1} N(\mathbf{v}_1 - \mathbf{v}_2)\|_Y \\ &= \alpha\|(\partial_t B + L)^{-1} A(\mathbf{v}_1 - \mathbf{v}_2)\|_Y, \text{ by (10)} \\ &\leq \alpha\|(\partial_t B + L)^{-1} A\| \|\mathbf{v}_1 - \mathbf{v}_2\|_{H^2(\Omega)}, \\ &\text{since } A \text{ is linear and bounded.}\end{aligned}$$

Since, $\|(\partial_t B + L)^{-1} A\| = 1$, by Proposition 8.1, $\|\alpha(\partial_t B + L)^{-1} A\| < 1$, and the uniqueness of the solution follows. On the other hand, the linear operator γ_0 defined by $\mathbf{v} \mapsto \gamma_0\mathbf{v}$ is the bijective restriction of the trace; $\gamma : H^2(\Omega) \to H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$, since Γ is assumed to be infinitely differentiable.

From the bijection γ_0 it can be easily shown that if $\mathbf{v}$ is unique, then $\gamma_0\mathbf{v}$ is unique. Hence, $\mathbf{w}(t) := B\mathbf{v} := \langle \rho\mathbf{v}, \sigma\gamma_0\mathbf{v}\rangle$ is unique as the solution of the abstract implicit Cauchy problem $\partial_t \mathbf{w} + L\mathbf{v} + N(\mathbf{v}) + \ell\mathbf{p} = 0$, provided $t \in (0, T)$. □

(II) The solution to the abstract Cauchy problem (5) is given by,

$$\begin{aligned}\mathbf{w}(t) &:= \mathbf{w}(0) - \int_0^t [L\mathbf{v}(\phi) + N(\mathbf{v}(\phi)) + \ell\mathbf{p}(\mathbf{x})] d\phi; \; t \in (0, T), T < \infty \\ &= \mathbf{w}(0) - \int_0^t [L(B^{-1}\mathbf{w}(\phi)) + N(B^{-1}\mathbf{w}(\phi)) + \ell\mathbf{p}(\mathbf{x})] d\phi;\end{aligned}$$

since the operator B is bijective. The solution is unique if and only if, $L(B^{-1}\mathbf{w}(\phi)) + N(B^{-1}\mathbf{w}(\phi)) + \ell\mathbf{p}(\mathbf{x})$ is Lipschitz on $\mathbf{w}$.

We now put

$$G(\mathbf{w}(t),\ p(\mathbf{x})) := L(B^{-1}\mathbf{w}(t)) + N(B^{-1}\mathbf{w}(t)) + \ell\mathbf{p}(\mathbf{x}),$$

$$\| G(\mathbf{w}_2(t),\ p(\mathbf{x})) - G(\mathbf{w}_1(t),\ p(\mathbf{x})) \|_Y$$

$$\leq \| LB^{-1}\mathbf{w}_2(t) - LB^{-1}\mathbf{w}_1(t)\|_Y + \| NB^{-1}\mathbf{w}_2(t) - NB^{-1}\mathbf{w}_1(t)\|_Y$$

$$= \| LB^{-1}\mathbf{w}_2(t) - LB^{-1}\mathbf{w}_1(t)\|_Y + \| AB^{-1}\mathbf{w}_1(t) - AB^{-1}\mathbf{w}_2(t)\|_Y;\ \text{by (10)}$$

$$\leq \| \mathbf{w}_2(t) - \mathbf{w}_1(t) \|_Y \| LB^{-1} + AB^{-1} \|;\ \text{due to the linearity of } LB^{-1} \text{ and } AB^{-1}$$

$$= \mathrm{Lip} \| \mathbf{w}_2(t) - \mathbf{w}_1(t) \|_Y,$$

where

$$\mathrm{Lip} := \| LB^{-1} + AB^{-1} \|,$$

since the operators L, A and B^{-1} are bounded in Θ, then G is Lipschitz on $\mathbf{w}(t)$ and the solution of the abstract implicit Cauchy problem (5) is unique If $\mathbf{w}(t) := \langle \rho\mathbf{v}(t),\ \gamma_0\mathbf{v}(t)\rangle$ is unique, then $\mathbf{v}(t)$ and $\gamma_0\mathbf{v}(t)$ are unique. Then the solution to the equation $\mathbf{v} = -\alpha(\partial_t B + L)^{-1} N(\mathbf{v});\ \alpha \in (0, 1)$, exists and is unique. □

In Section 11, we will show that the existence and uniqueness is global on time.

11. Exponential Stability for the Flows

By (12), on p. 9 of [3], for the problem in hand, we have the following energy inequality:

$$E^{/}(t) \leq -2\mu \| D(\mathbf{v}) \|^2_{L^2(\Omega)} \beta(t);\quad 0 < \beta(t) \leq \frac{3}{2}. \tag{11}$$

We now rewrite (5) as follows: $\partial_t \mathbf{w} = -L\mathbf{v} - \ell p - N(\mathbf{v})$ and consider the following eigenvalue problem: $-L\mathbf{v} - \ell p = \lambda B\mathbf{v}$; $\mathbf{v} \in \Theta$ and $\lambda \neq 0$ (it can be shown that $\lambda = 0$ implies no flows). Then

$$(-L\mathbf{v} - \ell p,\, B\mathbf{v})_Y = \lambda \| B\mathbf{v} \|_Y^2.$$

This implies that

$$(-L\mathbf{v},\, B\mathbf{v})_Y - (\ell p,\, B\mathbf{v})_Y = \lambda \| B\mathbf{v} \|_Y^2;$$

which implies that

$$-2\mu \| D(\mathbf{v}) \|_{L^2(\Omega)}^2 = \lambda \| B\mathbf{v} \|_Y^2, \tag{12}$$

since

$$(\ell p,\, B\mathbf{v})_Y = 0$$

and

$$(L\mathbf{v},\, B\mathbf{v})_Y = 2\mu \| D(\mathbf{v}) \|_{L^2(\Omega)}^2.$$

From (12), we deduce that $\lambda < 0$ for the flows.

We rewrite (12), using (11) to obtain

$$\frac{E^{/}(t)}{\beta(t)} \leq \lambda \| B\mathbf{v} \|_Y^2 = \lambda\rho^2 \| \mathbf{v} \|_{H^2(\Omega)}^2 + \lambda\sigma^2 \| \eta_v \|_{L^2(\Gamma)}^2 = \lambda E(t);$$

from which we have the inequality

$$E^{/}(t) - \lambda\beta(t)E(t) \leq 0. \tag{13}$$

The solution to the inequality (13) is

$$E(t) \leq E_0 \exp(\lambda\beta(t)t); \tag{14}$$

where

$$E(0) := E_0.$$

Since $\lambda < 0$ and $\beta(t) > 0$, the flows experience exponential energy decay for $t > 0$. In terms of the observation in the middle of p. 14 in [3], where $E(t) < \frac{1}{2}\left[\frac{\sqrt{3}-1}{C_2}\right]^2$ for all t, and $C_2 > 0$, then we may re-write (14) as the following exponential stability statement:

$$E(t) \leq \frac{1}{2}\left[\frac{\sqrt{3}-1}{C_2}\right]^2 \exp(\lambda\beta(t)t). \tag{15}$$

From (14), we have that $\ln E(t) \leq \ln E(0) + \lambda\beta(t)t \leq \ln E(0) + \frac{3}{2}\lambda t$; with max $[\beta(t)] = \frac{3}{2}$ (see p. 9 of [2]). Then $t \leq \frac{2}{3}\left[\frac{\ln E(t) - \ln E(0)}{\lambda}\right]$, since $\lambda < 0$, from (12). From the preceding inequality, since ln $E(t) \to \infty$ as $E(t) \to 0$, then

$$t < \infty; \tag{16}$$

thus showing there is no restriction on time since the time interval for the existence and the uniqueness of the 'weak' solution would be $[0, \infty)$.

Thus, existence and uniqueness is global on time.

12. Conclusion

(a) The most critical requirement for our analysis is that both the open bounded domains Ω and Ω_0 be endowed with the or that $\partial\Omega := \Gamma$ is both smooth and infinitely differentiable. Without these requirements we cannot define the trace operator; hence our operators *B*, *N* and *L* would not make sense at all.

(b) The mean curvature κ of the permeable boundary Γ (as seen in the Main result of [2]) is critical in the confirmation of the existence of the weak solution. If the inner container were a rectangular prism, $\kappa = 0$ for the four walls; however, $\kappa = \infty$ for the corners. There would be some uncertainty as to the existence of the solution for the flows through the corners.

(c) On the outer boundary Γ_0, we have the same boundary condition as in [7]; namely, that $\mathbf{v}(\mathbf{x}, t) = 0$. However, this condition does not play the same critical role, in our analysis, as it does in [7]. Reading through [4], however, the condition gives rise to the Poincare's inequality which partly defines the boundedness of the rate of deformation tensor $D(\mathbf{v})$. The latter boundedness is very crucial in this paper.

(d) The critical property of compactness for the operator $\partial_t B + L; t \in (0, T)$, $T < \infty$, is deduced on the set of our weak solutions, $\Theta \subset \Lambda$. The closed domain set Λ may coincide with $H^2(\Omega)$.

(e) The bijection $\chi : L^2([0, T], H^2(\Omega)) \to L^2([0, T], Y); T < \infty$, is the reason for the dual formulation of the problem in hand (see (2) and (5)). This has been confirmed by Theorem 10.3

(f) By (16) it is now certain that our existence and uniqueness result is global on time. Note that when $t = 0$, $E(0)$ represents the gravitational potential energy for the fluid particles in the container.

References

[1] R. A. Adams, Sobolev spaces, Pure and Applied Mathematics, Vol. 65, Academic Press, New York, San Francisco, London, 1975.

[2] Joe Hlomuka, Solvability conditions for the nonlinear, non-stationary problem of the permeable boundary Navier-Stokes flows, Int. J. Nonlinear Oper. Theory Appl. 1(1) (2006), 1-15.

[3] Joe Hlomuka, The existence and uniqueness of the solution to the stationary permeable boundary Navier-Stokes flows, using trace-related canonical operators, Int. J. Nonlinear Oper. Theory Appl. 1(1) (2006), 17-33.

[4] Joe V Hlomuka and N. Sauer, Stability of Navier-Stokes flows through permeable boundaries, Navier-Stokes Equations: Theory and Numerical Methods, R. Salvi, ed., Marcel-Dekker, Inc., New York, Basel, 2001, pp. 33-43.

[5] E. Kreyszig, Introductory Functional Analysis with Applications, John Wiley & Sons, Inc., New York, Chichester, Brisbane, Toronto, Singapore, 1978.

[6] O. A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flow, Gordon and Breach, New York, 1963.

[7] O. A. Ladyzhenskaya, The sixth millennium problem: Navier-Stokes equations, existence and smoothness, Russian Math. Surveys 58(2) (2003), 251-286.

[8] J. L. Lions and E. Magenes, Non-homogeneous Boundary Value Problems and Applications, Vol. II, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[9] R. Maritz and N. Sauer, Stability of uniqueness of second grade fluids in regions with permeable boundaries, Navier-Stokes Equations and Related Nonlinear Problems, (Palanga, 1997), 153-164, VSP, Utrecht, 1998.

[10] E. Zeidler, Non-linear Functional Analysis and its Applications (Fixed-point Theorems); Vol. I, Springer-Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, Hong Kong, Barcelona, Budapest, 1992.