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NUMERICAL SIMULATES FOR THE REGULARIZED LONG-WAVE EQUATION

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Abstract

In this paper, testing the numerical scheme for the regularized long wave (RLW) equation and study of motion, interaction and development of the solitary wave solution are presented. These schemes impose the amplitude of a one solitary wave solution and predict the progress of the wave solutions with errors and the demonstrating the shape, height and velocity of an undular bore consistency.

1. Introduction

We consider the RLW equation

$$U_t + U_x + \varepsilon U U_x - \mu U_{xxt} = 0, \tag{1}$$

where ε and μ are parameters and the subscripts x and t denote differentiation. The physical boundary conditions require $U \to 0$ as $x \to \pm\infty$. Boundary conditions will be selected from the homogeneous boundary conditions:

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$$U(a, t) = \beta_1, \quad U(b, t) = \beta_2,$$

$$U_x(a, t) = 0, \quad U_x(b, t) = 0, \quad t \in (0, T],$$

$$U_{xx}(a, t) = 0, \quad U_{xx}(b, t) = 0, \tag{2}$$

and its initial condition

$$U(x, t) = f(x), \quad x \in [a, b], \tag{3}$$

where $f(x)$ is a localized disturbance inside the interval $[a, b]$.

This equation is the favorite nonlinear wave equation which can be used to model a large number of problems arising in various areas of applied sciences [1, 2]. In 1966, Peregrine [6] derived the RLW equation to model development of an undular bore. In 1984, Morrison et al. [4] derived the one-dimensional nonlinear dispersive waves which is accurate and equally valid model for the same wave simulated by the Korteweg-de Vries (KdV) equation and RLW equation. In 1972, Benjamin et al. [1] discovered the BBM equation that is also known as the RLW equation. In this paper, we have used a collocation method with quintic *B*-spline to investigate the motion of one solitary wave solution, and an undular bore for the RLW equation in Eq. (1) to predict the progressive wave with small error norms.

2. Quintic *B*-spline Collocation Method

We divided the interval $[a, b]$ by nodes x_m such that $a < x_1 < \cdots < x_N = b$ and $h = \dfrac{b-a}{N} = x_{m+1} - x_m$, $m = 0, 1, 2, ..., N$. We use 10 knots of the interval $[a, b]$ as

$$x_{-5} < x_{-4} < x_{-3} < x_{-2} < x_{-1} \text{ and } x_{N+1} < x_{N+2} < x_{N+3} < x_{N+4} < x_{N+5},$$

and applied quintic *B*-spline analyzed solutions of the RLW equation.

The quintic *B*-spline $K_m(x)$ with basis form over the domain $[a, b]$, that is, $\{K_{-1}, K_{-2}, K_{-3}, K_{-4}, K_{-5}\}$ is given by

$$K_m(x)=\frac{1}{h^2}\begin{cases}(x-x_{m-3})^5, & [x_{m-3}, x_{m-2}],\\ (x-x_{m-3})^5-6(x-x_{m-2})^5, & [x_{m-2}, x_{m-1}],\\ (x-x_{m-3})^5-6(x-x_{m-2})^5+15(x-x_{m-1})^5, & [x_{m-1}, x_m],\\ (x-x_{m-3})^5-6(x-x_{m-2})^5+15(x-x_{m-1})^5 & \\ \quad -20(x-x_m)^5, & [x_m, x_{m+1}],\\ (x-x_{m-3})^5-6(x-x_{m-2})^5+15(x-x_{m-1})^5 & \\ \quad -20(x-x_m)^5+15(x-x_{m+1})^5, & [x_{m+1}, x_{m+2}],\\ (x-x_{m-3})^5-6(x-x_{m-2})^5+15(x-x_{m-1})^5 & \\ \quad -20(x-x_m)^5+15(x-x_{m+1})^5-6(x-x_{m+2})^5, & [x_{m+2}, x_{m+3}],\\ 0, & \text{otherwise}.\end{cases} \tag{4}$$

The set of quintic B-spline $K_m(x), m=-2, ..., N+2$, forms a basis over the interval $[a, b]$. A global interpolation $U_N(x, t)$ to the analytic solutions $U(x, t)$ is given by

$$U_N(x, t)=\sum_{m=-2}^{N+2}\delta_m(t)K_m(x), \tag{5}$$

where $\delta_m(t)$ are time-dependent parameters to be determined from the conditions in Eq. (2) and Eq. (3). The function of quintic B-spline and its derivatives are continuous. Similarly, the trial solutions with derivatives are continuous. The node of U and its derivatives at the knots x_m in terms of parameters δ_m used from the B-spline function in Eq. (4) and the trial solutions in Eq. (5), are as follows

$$\begin{aligned} U_m &= U(x_m) = \delta_{m-2}+26\delta_{m-1}+66\delta_m+\delta_{m+2},\\ U'_m &= U'(x_m) = \frac{5}{h}(\delta_{m+2}+10\delta_{m+1}-10\delta_{m-1}-\delta_{m-2}),\\ U''_m &= U''(x_m) = \frac{20}{h^2}(\delta_{m+2}+2\delta_{m+1}-6\delta_m+2\delta_{m-1}+\delta_{m-2}),\\ U'''_m &= U'''(x_m) = \frac{60}{h^3}(\delta_{m+2}-2\delta_{m+1}+2\delta_{m-1}-\delta_{m-2}),\\ U^{(4)}_m &= U^{(4)}(x_m) = \frac{120}{h^4}(\delta_{m+2}-4\delta_{m+1}+6\delta_m-4\delta_{m-1}+\delta_{m-2}). \end{aligned} \tag{6}$$

Applying the knots x_i, $i = 0, 1, \ldots, N$ and substituting the variables U_m, U''_m and U'''_m in Eq. (6) into Eq. (1), we get the nonlinear ordinary differential equation:

$$\dot{\delta}_{m-2} + 26\dot{\delta}_{m-1} + 66\dot{\delta}_m + 26\dot{\delta}_{m+1} + \dot{\delta}_{m+2} + \frac{5c}{h}(\delta_{m+2} + 10\delta_{m+1} - 10\delta_{m-1} - \delta_{m-2})$$

$$- \frac{20}{h^2}(\dot{\delta}_{m+2} + 2\dot{\delta}_{m+1} - 6\dot{\delta}_m + 2\dot{\delta}_{m-1} + \dot{\delta}_{m-2}) = 0, \tag{7}$$

where . denotes derivative with respect to time and $c = 1 + \varepsilon d_m$, $d_m = \delta_{m-2} + 26\delta_{m-1} + 66\delta_m + 26\delta_{m+1} + \delta_{m+2}$.

Interpolating time parameters δ_m carried out time step of Eq. (7) and using the Crank-Nicholson and forward difference scheme with its time derivatives $\dot{\delta}_m$ between time level n and $n+1$ as

$$\delta_m = \frac{\delta_m^{n+1} + \delta_m^n}{2}, \; \dot{\delta}_m = \frac{\delta_m^{n+1} - \delta_m^n}{\Delta t}, \tag{8}$$

we get a recurrence relationship between time level n and $n+1$ successive unknown parameters δ_i^{n+1} and δ_i^n, $i = m-2, \ldots, m+2$,

$$\alpha_{m1}\delta_{m-2}^{n+1} + \alpha_{m2}\delta_{m-1}^{n+1} + \alpha_{m3}\delta_m^{n+1} + \alpha_{m4}\delta_{m+1}^{n+1} + \alpha_{m5}\delta_{m+2}^{n+1}$$

$$= \alpha_{m5}\delta_{m-2}^n + \alpha_{m4}\delta_{m-1}^n + \alpha_{m3}\delta_m^n + \alpha_{m2}\delta_{m+1}^n + \alpha_{m5}\delta_{m+2}^n, \tag{9}$$

where

$$\alpha_{m1} = 2h^2 - 5ch\Delta t - 40\mu, \quad \alpha_{m2} = 52h^2 - 50ch\Delta t - 80\mu,$$

$$\alpha_{m3} = 132h^2 + 240\mu, \quad \alpha_{m4} = 2h^2 + 50ch\Delta t - 80\mu,$$

$$\alpha_{m5} = 2h^2 + 5ch\Delta t - 40\mu, \quad m = 0, 1, \ldots, N. \tag{10}$$

The nonlinear system before solving has $(N+5)\times(N+5)$ dimension. We applied three boundary conditions to the system in Eq. (9) and eliminated the parameters δ_{-2}^{n+1}, δ_{-1}^{n+1}, δ_{N+1}^{n+1}, δ_{N+2}^{n+1}.

We determine boundary conditions $U(a, t) = \beta_1$, $U_x(a, t) = 0$, $U(b, t) = \beta_2$ and $U_x(b, t) = 0$, the result of this system changes to $(N+1)\times(N+1)$ dimension. Therefore, we applied the Gauss elimination procedure at every time step to solve the matrix system. To increase the accuracy of this system iterate the procedure at least two or three times, before moving to next step that solves the unknown parameters,

$$(\delta^*)^{n+1} = \delta^n + \frac{1}{2}(\delta^{n+1} - \delta^n). \tag{11}$$

Applying the von Neumann stability analysis verifies the stability of the nonlinear system in Eq. (9). Let U in the term of nonlinear UU_x be a locally constant p for the RLW equation and assume terms d_m are also equal to a constant p. The Fourier $\delta_m^n = \hat{p}^n e^{im\varphi}$ substituted into the difference scheme in Eq. (9) obtains

$$\begin{aligned}\hat{p}^{n+1} = \hat{p}^n \{&[(\alpha_{m1} + \alpha_{m5})\cos 2\varphi + (\alpha_{m2} + \alpha_{m4})\cos\varphi + \alpha_{m3}] \\ &+ i[(\alpha_{m1} - \alpha_{m5})\sin 2\varphi + (\alpha_{m2} - \alpha_{m4})\sin\varphi]\}/\{[(\alpha_{m1} + \alpha_{m5})\cos 2\varphi \\ &+ (\alpha_{m2} + \alpha_{m4})\cos\varphi + \alpha_{m3}] + i[(\alpha_{m5} - \alpha_{m1})\sin 2\varphi + (\alpha_{m4} - \alpha_{m2})\sin\varphi]\}\end{aligned}$$

the difference equation is given by

$$\hat{p}^{n+1} = \hat{p}^n q,$$

where q is defined by

$$q = \frac{x + iy}{x - iy},$$

where

$$\begin{aligned}x &= (\alpha_{m1} + \alpha_{m5})\cos 2\varphi + (\alpha_{m2} + \alpha_{m4})\cos\varphi + \alpha_{m3}, \\ y &= (\alpha_{m1} - \alpha_{m5})\sin 2\varphi + (\alpha_{m2} - \alpha_{m4})\sin\varphi,\end{aligned}$$

where $c = 1 + \varepsilon p$ and α_{mi}, $i = 1, 2, 3, 4, 5$ are given in Eq. (10). Difference scheme in Eq. (9) satisfies the von Neumann's condition $|q| \le 1$ that is unconditionally stable.

3. The Conversation Laws and the Error Norms

Partial differential equations posses an infinite number of conversation laws. An important state in the development of the general method of the solution for the RLW equation is that solutions obey a number of independent conversation laws [6, Definition, pp. 21-22].

For the RLW equation there are only three conversation laws [5],

(i) $C_1 = \int_{-\infty}^{\infty} U dx,$

(ii) $C_2 = \int_{-\infty}^{\infty} [U^2 + \mu(U_x)^2] dx,$

(iii) $C_3 = \int_{-\infty}^{\infty} [U^3 + 3U^2] dx.$

Numerical method of nonlinear equation can be imposed by the properties, time assessed migration of the solitary wave solutions. We measured the accuracy of the numerical algorithm by L_2 and L_∞ norms as

$$L_2 = \|U^{exact} - U^2\|_2 = \left[\Delta x \sum_1^N |U_j^{exact} - U_j^n|^2\right]^{\frac{1}{2}},$$

and

$$L_\infty = \|U^{exact} - U^2\|_\infty = \max_j |U_j^{exact} - U_j^n|.$$

4. Numerical Solutions of Equation

4.1. One solitary wave solution

The analytical solution of the RLW equation is

$$U(x, t) = 3c\operatorname{sech}^2(k[x - x_0 - 1(1 + \varepsilon c)t]),$$

which represents one solitary wave solution with amplitude $3c$, $\upsilon = 1 + \varepsilon c$ is the wave velocity and $k = \frac{1}{2}(\varepsilon c/\mu(1 + \varepsilon c))^{\frac{1}{2}}$.

The initial condition

$$U(x, 0) = 3c\operatorname{sech}^2(k(x - x_0)),$$

and we choose the boundaries $\beta_1 = 0, \beta_2 = 0, -40 \le x \le 60$ and time $0 \le x \le 20$. The parameters $h = 0.125$, $\Delta t = 0.1$, $c = 0.3;\ 0.09$ and $\varepsilon = \mu = 1$ are used the same with the previous method [3, 7]. The program recorded the values of quantities C_1, C_2, C_3 at the time steps, values L_2 and L_∞ norms.

This is algorithm of one solitary wave solution of amplitude 0.3 at time $t = 20$, to an L_∞ error norm with the value 0.082×10^3, while the quantities of conversation laws (C_1, C_2, C_3) change by less than 0.002. In the simulation of one solitary wave solution with amplitude 0.3 the collocation method with quartic-QBCM at time $t = 20$, to an L_∞ error norm with value 0.083×10^3, when the quantities of conversation laws (C_1, C_2, C_3) change by less than 0.003. In the procedure of quadratic *B*-spline with the error norm at time $t = 20$ is only 0.086×10^3 and the quantities of conversation laws (C_1, C_2, C_3) change by less than 7×10^{-6}. Cubic spline with time $t = 20$ has value 67.35×10^3 and also found the changing of the quantities of conversation laws less than 0.05, the error in this simulation is so poor.

We see that for one solitary wave solution with amplitude 0.3 using quintic *B*-spline collocation that its solution of quintic *B*-spline is more accurate than the solution of quartic-QBCM 2 and Galerkin-quadratic, but this solution is nearly the same quartic-QBCM 1. As the finite difference scheme is the least accurate of all methods. In Figure 1, comparing the initial wave profile with time $t = 20$, we observed that the wave amplitude and any non-physical oscillation have a small change. In Figure 2 shown the error of the wave maximum and oscillates smoothly between -2×10^{-4} and 3×10^{-5}.

In Table 2, we change the amplitude to 0.09 and get the same results of each method but they have the better accuracy of the error norms. The quintic collocation, quartic and Galerkin quadratic have nearly the results of the error but finite difference cubic is poor in compare with other methods but better than that in Table 1.

Table 1. Invariants and error norm for one solitary wave solution with amplitude 0.3, $\Delta x = 0.125$, $\Delta t = 0.1$ and $-40 \leq x \leq 60$

Method	Time	C_1	C_2	C_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
Quintic collocation	20	3.97993	0.810445	2.57900	0.214	0.082
Quartic-QBCM 1	20	3.97995	0.81046	2.57901	0.215	0.083
Quartic-QBCM 2	20	3.97995	0.81046	2.57901	0.357	0.129
Galerkin-quadratic	20	3.97989	0.808650	2.57902	0.220	0.086
f.d cubic	20	4.41219	0.897342	0.85361	196.1	67.35

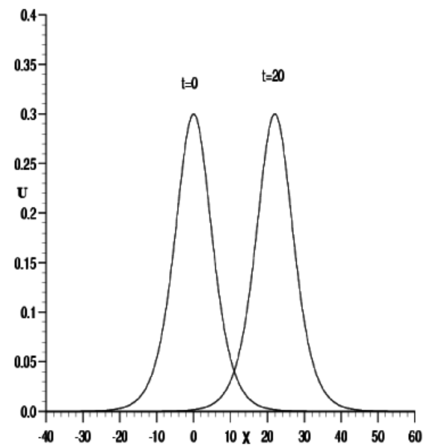

Figure 1. Profiles of the solitary wave at $t = 0$ and $t = 20$.

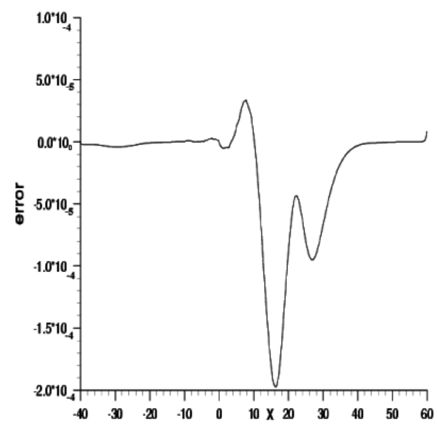

Figure 2. The error = exact − numerical numerical solution at $t = 20$ in Figure 1.

Table 2. Invariants and error norm for one solitary wave solution with amplitude 0.09, $\Delta x = 0.125$, $\Delta t = 0.1$ and $-40 \le x \le 60$

Method	Time	C_1	C_2	C_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
Quintic collocation	20	2.10830	0.127303	0.388809	0.355	0.298
Quartic-QBCM 1	20	2.10832	0.12909	0.38881	0.359	0.302
Quartic-QBCM 2	20	2.10831	0.12913	0.38881	0.356	0.295
Galerkin-quadratic	20	2.10460	0.127302	0.388803	0.563	0.432
f.d cubic	20	2.333	0.140815	0.430052	14.45	3.996

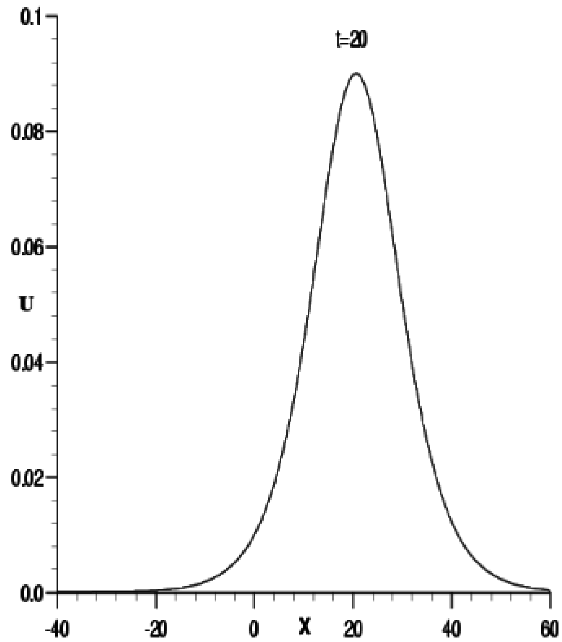

Figure 3. Solitary wave solution, amplitude 0.09 at $t = 20,\ \Delta x = 0.125,\ \Delta t = 0.1,\ -40 \le x \le 60.$

4.2. Undular bore and modeling

We study the development of an undular bore, follow Peregrine [6] and use as initial condition

$$U(x, 0) = 0.5U_0\left[1 - \tanh\left(\frac{x - x_0}{d}\right)\right],$$

and boundary conditions

$$U(a, t) = U_0, \quad U(b, t) = 0,$$

where $U(x, 0)$ represents the elevation of the water above the equilibrium surface at time $t = 0$. The constant U_0 is the change in water level that is centered on $x = x_c$ and d denotes the slope between the still water and deeper water. For the algorithm we choose the value of parameters as follows, $\varepsilon = 1$, $\mu = 0.16666667$, $U_0 = 0.1$ and $d = 5$.

Table 3. The amplitudes of the undular bore at time $t = 400$

	$d = 5$		$d = 2$	
	Position	Amplitude	Position	Amplitude
Present method				
Leading undulation	263.896	0.177	264.788	0.181
Second undulation	254.632	0.150	256.356	0.164
Third undulation	242.383	0.122	248.125	0.146
Cubic *B*-spline				
Leading undulation	264.962	0.178	265.922	0.182
Second undulation	253.923	0.153	254.163	0.162
Third undulation	244.823	0.132	244.082	0.145

The physical boundary conditions are $U \to 0$ as $x \to \infty$ and $U \to U_0$ as $x \to -\infty$.

In Table 3 are shown the maximum position and amplitude of the undular bore at time $t = 400$. The amplitude of the present method closes to the amplitude of cubic *B*-spline for each slope. The difference between the amplitude of leading undulation for each slope is $0.181 - 0.177 = 0.004$; it is the same result of cubic *B*-spline. We see that the undulations were nearly the same velocity for each steep slope. The position of leading undulation is 264.788 when $d = 2$, while with $d = 5$ the position of leading undulation is 263.896. In Figures 4 and 5 are shown the undulation and a space/time of the gentle slope $d = 5$ at time $t = 400$.

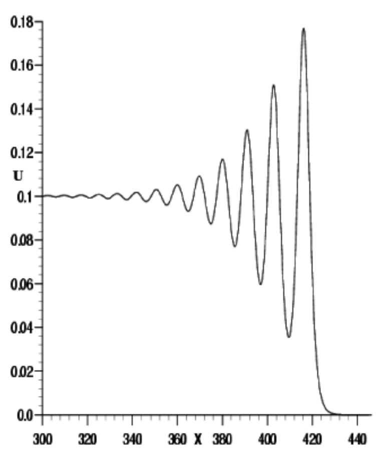

Figure 4. A gentle slope $d = 5$ of the undulation at $t = 400$.

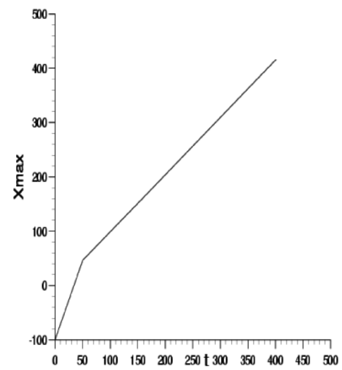

Figure 5. The undulation $d = 5$ for a space/time.

In Figures 6 and 7 are shown the undulation and the growth of the amplitude with slope $d = 2$ at time $t = 400$.

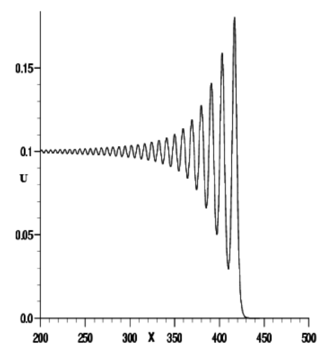

Figure 6. The undulation at $t = 400$ with slope $d = 2$.

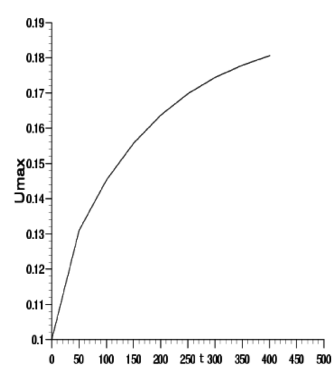

Figure 7. The undulation $d = 2$ with growth of the amplitude.

5. Conclusions

The numerical scheme for the RLW equation has shown the amplitude of one solitary wave solution in each time step and predicts wave progress of an undular bore with the small error that is examined by the error norms L_2 and L_∞.

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