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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF HIGHER ORDER NONLINEAR NEUTRAL DIFFERENCE EQUATIONS

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Abstract

In this paper, we study the asymptotic behavior of solutions of higher order neutral difference equations of the form

$$\Delta^m(x_n + px_{n-k}) + f(n, x_n) = h_n,$$

where $m \geq 1$ is an integer. We establish conditions under which all nonoscillatory solutions are asymptotic to $an^{m-1} + b$ with $a, b \in \mathbb{R}$. The obtained results extend those that are known for non-neutral higher order difference equations.

1. Introduction

In this paper, we study the asymptotic behavior of non-oscillatory solutions of the neutral difference equations of the form

$$\Delta^m(x_n + px_{n-k}) + f(n, x_n) = h_n, \tag{1}$$

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where $n \in N(n_0) = \{n_0, n_0 + 1, ...\}$, n_0 is a positive integer, m, $k \geq 1$ are integers, $\{h_n\}$ is a sequence of real numbers, p is a non-negative real number, and $f : \mathbb{N}(n_0) \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The forward difference operator Δ is defined as usual, that is, $\Delta x_n = x_{n+1} - x_n$. The higher order difference for a positive integer m is defined as $\Delta^m x_n = \Delta(\Delta^{m-1} x_n)$, $\Delta^0 x_n = x_n$.

By a solution of equation (1), we mean a real sequence $\{x_n\}$ which is defined for all $n \geq n_0 - k$ and satisfies equation (1) for all $n \in \mathbb{N}(n_0)$. As is customary, a nontrivial solution $\{x_n\}$ of equation (1) is said to be *nonoscillatory* if the terms x_n of the sequence $\{x_n\}$ are either eventually positive or eventually negative and oscillatory otherwise.

The neutral delay difference equations arise in a number of important applications including problems in population dynamics when maturation and gestation are included, in “cobweb” models in economics where demand depends on current price but supply depends on the price at an earlier time, and in electrical transmission in lossless transmission lines between circuits in high speed computers.

Oscillation theory of higher-order neutral difference equations has developed very rapidly in recent years. It has concerned itself largely with the oscillatory and asymptotic behavior of solutions (see, e.g., [1-3, 6, 8-14] and the references containing therein). The asymptotic behavior of nonoscillatory solutions of equation (1) has been studied using fixed point theorems and summation averaging techniques with the condition $uf(n, u) > 0$ for $u \neq 0$ and for all $n \in \mathbb{N}(n_0)$ (see, for example [1, 2, 3, 9, 10, 12, 14] and the references cited therein). Motivated by this observation, in this paper, we investigate the asymptotic behavior of nonoscillatory solutions of equation (1) without using the above said conditions on the nonlinear function. For a general background on difference equations and inequalities we can refer to [1] and [7].

2. Asymptotic Behavior of Nonoscillatory Solutions

In this section, we investigate the asymptotic behavior of nonoscillatory solutions of equation (1). We begin with the following lemma which can used to obtain useful information about the properties of nonoscillatory solutions of equation (1).

Lemma 1. *Let* $\{x_n\}$ *be an eventually positive or eventually negative sequence and*

$$w_n = x_n + p\left(\frac{n-k}{n}\right)^{m-1} x_{n-k}, \quad n \in \mathbb{N}(n_0), \tag{2}$$

where $0 \leq p < \infty$, k *is a positive integer. If* $\lim_{n\to\infty} w_n = c$, *then* $\lim_{n\to\infty} y_n = \frac{c}{1+p}$.

Proof. Assume that $\{x_n\}$ is eventually positive since the proof for the case $\{x_n\}$ is eventually negative is similar. First note that $\{x_n\}$ is bounded since $x_n \leq w_n$, for all $n \in \mathbb{N}(n_0)$. Next, let $\{n_i\}$ and $\{n_j\}$ be divergent subsequences of positive integers such that $\lim_{i\to\infty} x_{n_i} = \liminf_{n\to\infty} x_n = \alpha$ and $\lim_{j\to\infty} x_{n_j} = \limsup_{n\to\infty} x_n = \beta$. In view of (2), we see that

$$c = \lim_{i\to\infty} w_{n_i} = \lim_{i\to\infty}\left(x_{n_i} + p\left(\frac{n_i-k}{n_i}\right)^{m-1} x_{n_i-k}\right)$$

$$\geq (\liminf_{n\to\infty} x_n)\left(1 + p\lim_{i\to\infty}\left(\frac{n_i-k}{n_i}\right)^{m-1}\right) = \alpha(1+p).$$

Hence $\alpha \leq \frac{c}{1+p}$. Similarly, one can show that $\frac{c}{1+p} \leq \beta$. The proof is now complete. □

Next, we state the discrete type Bihari inequality established by Hull and Luxemburg [5].

Lemma 2. *Let* $\{y_n\}$ *and* $\{f_n\}$ *be non-negative sequences defined on* $\mathbb{N}(n_0)$ *and let* $W : \mathbb{R}_+ \to (0, \infty)$ *be continuous and nondecreasing. If*

$$y_n \leq d + \sum_{s=n_0}^{n-1} f_s W(y_s),$$

for all $n \in \mathbb{N}(n_0)$, *where* d *is a nonnegative constant, then for* $n_0 \leq n \leq n_1$, $n_1 \in \mathbb{N}(n_0)$, *we have*

$$y_n \leq G^{-1}\left(G(d) + \sum_{s=n_0}^{n-1} f_s\right), \tag{3}$$

where $G(u) = \int_{u_0}^{u} \frac{ds}{W(s)}$, $u > 0$, $u_0 > 0$ *arbitrary, and* G^{-1} *is the inverse of G such that* $G(d) + \sum_{s=n_0}^{n-1} f_s \in Dom(G^{-1})$, *for* $n_0 \le n \le n_1$.

Remark 3. If $\int_{u_0}^{u} \frac{ds}{W(s)} \to \infty$ as $u \to \infty$, then (3) is valid for all $n \in \mathbb{N}(n_0)$.

Theorem 4. *Assume that* $0 \le p < \infty$ *and k is a positive integer. If there exists a positive real sequence* $\{q_n\}$ *and a continuous nondecreasing function* $g : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$| f(n, u) | \le q_n g\left(\frac{|u|}{n^{m-1}}\right), \tag{4}$$

$$\sum_{n=n_0}^{\infty} q_n < \infty \ \ and \ \ \sum_{n=n_0}^{\infty} | h_n | < \infty, \tag{5}$$

and

$$G(u) = \int_{u_0}^{u} \frac{ds}{g(s)} \to \infty, \quad as \ u \to \infty,$$

then every nonoscillatory solution $\{x_n\}$ *of equation* (1) *is asymptotic to* $an^{m-1} + b$, *where a and b are real constants.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1). Define

$$z_n = x_n + px_{n-k}. \tag{6}$$

Then $| z_n | > | x_n |$ and from equation (1), we have

$$\Delta^m z_n = -f(n, x_n) + h_n. \tag{7}$$

If we denote $\Delta^i z_{n_0} = \alpha_i$, $0 \le i \le m-1$, then summing (7), m times from n_0 to $n-1$, we obtain

$$\Delta^{m-1} z_n = d_{m-1} - \sum_{s=n_0}^{n-1} f(s, x_s) + \sum_{s=n_0}^{n-1} h_s \tag{8}$$

and

$$z_n = \sum_{i=0}^{m-1} \frac{(n-n_0)^{(i)}}{i!} d_i - \sum_{s=n_0}^{n-1} \frac{(n-s-1)^{m-1}}{(m-1)!} f(s, x_s) + \sum_{s=n_0}^{n-1} \frac{(n-s-1)^{m-1}}{(m-1)!} h_s. \quad (9)$$

It follows from (9) that

$$z_n \le \left(\sum_{i=0}^{m-1} | d_i | n^{m-1} \right) + n^{m-1} \sum_{s=n_0}^{n-1} | f(s, x_s) | + n^{m-1} \sum_{s=n_0}^{n-1} | h_s | \quad (10)$$

and in view of (4), it is clear that

$$| f(n, x_n) | \le q_n g\left(\frac{| x_n |}{n^{m-1}} \right) \le q_n g\left(\frac{| z_n |}{n^{m-1}} \right).$$

Then from (10), we have

$$\frac{| z_n |}{n^{m-1}} \le \sum_{i=0}^{m-1} | d_i | + \sum_{s=n_0}^{n-1} q_s g\left(\frac{| z_s |}{S^{m-1}} \right) + \sum_{s=n_0}^{n-1} | h_s |. \quad (11)$$

From condition (5), there exists a positive constant d_m such that $\sum_{s=n_0}^{n-1} | h_s | \le d_m,$ for all $n \in \mathbb{N}(n_0),$ and from (11), we have

$$\frac{| z_n |}{n^{m-1}} \le \delta + \sum_{s=n_0}^{n-1} q_s g\left(\frac{| z_s |}{S^{m-1}} \right), \quad (12)$$

where $\delta = d_m + \sum_{i=0}^{m-1} | d_i |.$ Applying Lemma 2 on (12), we obtain

$$\frac{| z_n |}{n^{m-1}} \le G^{-1}\left(G(\delta) + \sum_{s=n_0}^{n-1} q_s \right),$$

where G^{-1} is the inverse function of G. Let

$$\delta_1 = G(\delta) + \sum_{s=n_0}^{m-1} q_s < \infty.$$

Since G^{-1} is increasing, we conclude that

$$\frac{| z_n |}{n^{m-1}} \le \delta_2 = G^{-1}(\delta_1) < \infty.$$

On the other hand, by (4), we have

$$\sum_{s=n_0}^{n-1} |f(s, x_s)| \le \sum_{s=n_0}^{n-1} q_s g\left(\frac{|x_s|}{S^{m-1}}\right) \le \sum_{s=n_0}^{n-1} q_s g\left(\frac{|z_s|}{S^{m-1}}\right)$$

$$\le g(\delta_2)\sum_{s=n_0}^{n-1} q_s < \delta_3.$$

Therefore, $\sum_{n=n_0}^{\infty} |f(n, x_n)|$ exists and from (8), we see that there exists a constant $a_1 \in \mathbb{R}$ such that $\lim_{n\to\infty} \Delta^{m-1} z_n = a_1$. Then by Stolz's theorem [4], we have

$$\lim_{n\to\infty} \frac{z_n}{n^{m-1}} = \lim_{n\to\infty} \Delta^{m-1} z_n = a_1.$$

Now, we put $w_n = \frac{z_n}{n^{m-1}}$, then (6) implies $w_n = y_n + p\left(\frac{n-k}{n}\right)^{m-1} y_{n-k}$, where $y_n = \frac{x_n}{n^{m-1}}$, Lemma 1 implies that

$$\lim_{n\to\infty} \frac{x_n}{n^{m-1}} = a = \frac{a_1}{1+p}.$$

This completes the proof. □

Remark 5. If in the proof of Theorem 4, we choose d_{m-1} sufficiently large so that $\lim_{n\to\infty} \Delta^{m-1} z_n = d_{m-1} \neq 0$, then the corresponding solution $\{x_n\}$ of equation (1) is asymptotic to $an^{m-1} + b$, where $a \neq 0$.

Corollary 6. *Consider the equation*

$$\Delta^m (x_n + p x_{n-k}) + e_n x_n = h_n, \quad n \in \mathbb{N}(n_0), \tag{13}$$

where p is a non-negative real number and k is a positive integer, and

$$\sum_{n=n_0}^{\infty} n^{m-1} |e_n| < \infty \quad \textit{and} \quad \sum_{n=n_0}^{\infty} n^{m-1} |h_n| < \infty.$$

Then every nonoscillatory solution $\{x_n\}$ *of equation* (13) *is asymptotic to* $an^{m+1} + b$ *as* $n \to \infty$.

Proof. The conclusion follows from Theorem 4 with $q_n = n^{m-1}|e_n|$ and $g(u) = u.$ □

Remark 7. For $p = 0$ and $h_n = 0$ Corollary 6 corresponds to the well-known result [7] for the equation $\Delta^m y_n + e_n y_n = 0.$

Corollary 8. *Consider the equation*

$$\Delta^m(x_n + px_{n-k}) + e_n x_n^{\alpha} = 0, \quad n \in \mathbb{N}(n_0), \tag{14}$$

where $0 < \alpha < 1,\ 0 \le p < \infty,$ *k is a positive integer and*

$$\sum_{n=n_0}^{\infty} n^{\alpha(m-1)}|q_n| < \infty.$$

Then every nonoscillatory solution $\{y_n\}$ *of equation* (14) *is asymptotic to* $an^{m-1} + b$ *as* $n \to \infty.$

Proof. Apply Theorem 4 with

$$q_n = n^{\alpha(m-1)}|e_n|, \quad h_n \equiv 0 \text{ and } g(u) = u^{\alpha}.$$

As a final result of this section, we extend the conclusion of Theorem 4 to more general equation

$$\Delta^m(x_n + px_{n-k}) + f(n, y_n, y_{n+1}) = h_n, \tag{15}$$

where $m \ge 2.$

Theorem 9. *Assume that* $0 \le p < \infty,$ *k is a positive integer and* $f(n, u, v)$ *is continuous in* $D = \{(n, u, v), n \in \mathbb{N}(n_0), u, v \in \mathbb{R}\}.$ *If there exists a positive sequence* $\{q_n\}$ *and a continuous nondecreasing function* $g : \mathbb{R}_+ \to (0, \infty)$ *such that*

$$|f(n, u, v)| \le q_n g\left(\frac{|u|}{n^{m-1}}\right)|v|, \text{ on } D,$$

$$G(u) = \int_{u_0}^{u} \frac{ds}{sg(s)} \to \infty, \text{ as } u \to \infty$$

and

$$\sum_{n=n_0}^{\infty} n^{m-1} q_n < \infty \quad \text{and} \quad \sum_{n=n_0}^{\infty} | h_s | < \infty,$$

then every oscillatory solution $\{x_n\}$ *of equation* (15) *is asymptotic to* $an^{m-1} + b$ *as* $n \to \infty$, *where a and b are real constants.*

Proof. Proceeding as in the proof of Theorem 4, we have

$$\Delta^{m-1} z_n = d_{m-1} - \sum_{s=n_0}^{n-1} f(s, x_s, x_{s+1}) + \sum_{s=n_0}^{n-1} h_s$$

and

$$z_n = \sum_{i=0}^{m-1} \frac{(n-n_0)^{(i)}}{i!} d_i - \sum_{s=n_0}^{n-1} \frac{(n-s-1)^{m-1}}{(m-1)!} f(s, x_s, x_{s+1}) + \sum_{s=n_0}^{n-1} \frac{(n-s-1)^{m-1}}{(m-1)!} h_s.$$

If follows that

$$\frac{| z_n |}{n^{m-1}} \le \sum_{i=0}^{m-1} | d_i | + \sum_{s=n_0}^{n-1} q_s g\left(\frac{| x_s |}{S^{m-1}}\right) + \sum_{s=n_0}^{n-1} | h_s | \tag{16}$$

and

$$\frac{| \Delta z_n |}{n^{m-2}} \le \sum_{i=1}^{m-1} | d_i | + \sum_{s=n_0}^{n-1} q_s g\left(\frac{| x_s |}{S^{m-1}}\right) | x_{s+1} | + \sum_{s=n_0}^{n-1} | h_s |. \tag{17}$$

Since $| z_n | \ge | x_n |$ and g is nondecreasing, we have from (16) and (17)

$$\frac{| z_n |}{n^{m-1}} \le \sum_{i=0}^{m-1} | d_i | + \sum_{s=n_0}^{n-1} q_s g\left(\frac{| z_s |}{S^{m-1}}\right) | z_{s+1} | + \sum_{s=n_0}^{n-1} | h_s | \tag{18}$$

and

$$\frac{| z_n |}{n^{m-2}} \le \sum_{i=1}^{m-1} | d_i | + \sum_{s=n_0}^{n-1} q_s g\left(\frac{| z_s |}{S^{m-1}}\right) | z_{s+1} | + \sum_{s=n_0}^{n-1} | h_s |. \tag{19}$$

Let R_n be the right side of (18). Then we have $\frac{|z_n|}{n^{m-1}} \le R_n$ and $\frac{|\Delta z_n|}{n^{m-2}} \le R_n$. Hence $|z_{n+1}| \le n^{m-2}R_n + |z_n|$ or $|z_{n+1}| \le (1+n)n^{m-2}R_n$. Thus from (18), we obtain

$$R_n \le \sum_{i=0}^{m-1}|d_i| + \sum_{s=n_0}^{n-1}(1+s)s^{m-2}R_s g(R_s) + \sum_{s=n_0}^{n-1}|h_s|.$$

The rest of the proof is similar to that of Theorem 4 and hence the details are omitted. □

3. Examples

In this section, we give some examples to illustrate the results obtained in Section 2.

Example 10. Consider the nonlinear, difference equation

$$\Delta^2\left(x_n + \frac{1}{2}x_{n-1}\right) + \frac{1}{(2n^2+1)^2}\left(\frac{x_n^2}{1+x_n^2}\right)$$

$$= \frac{1}{4n^4+5n^2+1} + \frac{3}{(n-1)(n+1)(n+2)}, \quad n \ge 2. \tag{20}$$

Set $m = 2$, $q_n = \frac{1}{n^4}$, $h_n = \frac{1}{4n^4+5n^2+1} + \frac{3}{(n-1)(n+1)(n+2)}$ and $g(u) = \frac{u^2}{1+u^2}$. Then applying Theorem 4, we see that for any nonoscillatory solution $\{x_n\}$ of (20) there exist reals a and b such that $x_n \to an+b$ as $n \to \infty$. Observe that $\{x_n\} = \left\{2n + \frac{1}{n}\right\}$ is a solution of (20) which is asymptotic to $2n$ as $n \to \infty$.

Example 11. Consider the nonlinear difference equation

$$\Delta^3(x_n + 2x_{n-1}) + \frac{(18n+30)(n^6+2n^3+2)}{n(n^2-1)(n^2+5n+6)(n^3+1)^2}\left(\frac{x_n^2}{1+x_n^2}\right) = 0, \tag{21}$$

where $n \ge 2$. Set $m = 3$, $q_n = \frac{1}{n^4}$, $h_n = 0$ and $g(u) = \frac{u^2}{1+u^2}$. Then applying

Theorem 4, we see that for any nonoscillatory solution $\{x_n\}$ of (21), there exist reals a and b such that $an^2 + b$ as $n \to \infty$. Observe that $\{x_n\} = \left\{n^2 + \frac{1}{n}\right\}$ is a solution of (21) which is asymptotic to n^2 as $n \to \infty$.

Remark 12. The known results in the literature [1, 2, 3, 10, 12, 14] do not apply to the equation (20) and (21) since the condition $uf(n, u) > 0$ for $u \neq 0$ is not satisfied.

Example 13. Consider the following higher order Emden-Fowler type nonlinear difference equation

$$\Delta^m(x_n + px_{n-k}) + n^{\beta}x_n^{\alpha} = 0, \tag{22}$$

where $0 < \alpha < 1$ is a ratio of odd integers, $0 \leq p < 1$, k is positive integer. Then by Corollary 8 every nonoscillatory solution $\{x_n\}$ of equation (22) is asymptotic to $an^{m-1} + b$ as $n \to \infty$ provided $\alpha(m-1) + \beta + 1 < 0$.

4. Conclusion

If $m = 2$ and $h_n \equiv 0$, then the results obtained in this paper reduces to that of in [8]. Further, the results obtained in this paper can be extended to more general equation of the form

$$\Delta(a_n\Delta^{m-1}(x_n + px_{n-k})) + f(n, x_n) = h_n,$$

where $\{a_n\}$ is a positive real sequence, without much difficulty and hence the details are left to the reader.

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