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EXACT SOLUTION OF FREDHOLM INTEGRAL EQUATIONS SYSTEM WITH SEPARABLE KERNELS

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Abstract

In this paper, we apply the direct computation method for solving Fredholm integral equations system with separable kernels. It is worthnoting that the direct computation method determines the exact solution in the closed form.

1. Introduction

The direct computation method is an efficient method for solving Fredholm integral equations of the second kind with separable kernels [6]. This method gives exact solution of the Fredholm integral equation with separable kernel. In recent years, several researchers discussed a numerical method to approximate the solution of second kind Fredholm integral equations system. For example, Babolian et al. [1] proposed the Adomian Decomposition Method (ADM) for solving systems of linear and nonlinear Fredholm integral equations of the second kind. Maleknejad et al. [5] used Block-Pulse Functions (BPF) as a simple base

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for solving a system of integral equations. Javidi and Golbabai [4] applied the Homotopy Perturbation Method (HPM) for solving system of Fredholm integral equations. Golbabai and Keramati [3] applied a simple method to approximate the solution of linear Fredholm integral equations system based on ADM. Yusufoğlu [7] extended the HPM to solve the system of Fredholm and Volterra type integral equations. All these methods give an approximate solution. De Bonis and Laurita [2] presented a projection method based on the Lagrange interpolation for the numerical solution of second kind Fredholm integral equations system on $[-1, 1]$.

In this paper, we use direct computation method to give the exact solution of the Fredholm integral equations system of the second kind with separable kernels.

Consider the second kind Fredholm integral equations system of the following form:

$$\mathbf{U}(t) = \mathbf{F}(t) + \int_0^1 \mathbf{K}(t, s)\mathbf{U}(s)ds, \tag{1}$$

where

$$\mathbf{U}(t) = [u_1(t), u_2(t), \cdots, u_n(t)],$$

$$\mathbf{F}(t) = [f_1(t), f_2(t), \cdots, f_n(t)],$$

$$\mathbf{K}(t, s) = [k_{ij}(t, s)], \quad i, j = 1, 2, \cdots, n.$$

In equation (1), the functions $\mathbf{K}$ and $\mathbf{F}$ are given, and $\mathbf{U}$ is the solution to be determined. We assume that (1) has a solution.

The paper is organized as follows: In Section 2, we describe the direct computation method for linear Fredholm integral equations system. In Section 3, we give some numerical examples.

2. The Direct Computation Method

Recall that our attention will be focused on separable or degenerate kernels $k_{i,j}(t, s)$ expressed in the following form:

$$k_{ij}(t, s) = \sum_{k=1}^{L_{ij}} g_{ij}^{(k)}(t)h_{ij}^{(k)}(s), \quad i, j = 1, 2, \cdots, n.$$

Without loss of generality and for simplicity reasons, we may assume that the kernel $k_{ij}(t, s)$ can be expressed as

$$k_{ij}(t, s) = g_{ij}(t)h_{ij}(s), \quad i, j = 1, 2, \cdots, n.$$

Accordingly, equations (1) become

$$u_i(t) = f_i(t) + \sum_{j=1}^{n} g_{ij}(t) \int_0^1 h_{ij}(s)u_j(s)ds, \quad i = 1, 2, \cdots, n. \tag{2}$$

It is clear that the definite integral at the right hand side of (2) reveals that the integrand depends on one variable, namely, variable s. This means that the definite integrals in the right hand side of (2) are equivalent to numerical values α_{ij}, where α_{ij} are constants. In other words, we may write

$$\alpha_{ij} = \int_0^1 h_{ij}(s)u_j(s)ds. \tag{3}$$

It follows that equations (2) become

$$u_i(t) = f_i(t) + \sum_{j=1}^{n} \alpha_{ij} g_{ij}(t), \quad i = 1, 2, \cdots, n. \tag{4}$$

It is thus obvious that the solution $u_i(t)$ is completely determined by (4) upon evaluating the constants α_{ij}. This can be easily done by substituting equation (4) in equation (3),

$$\alpha_{ij} = \int_0^1 h_{ij}(s)\left(f_j(s) + \sum_{k=1}^{n} \alpha_{jk} g_{jk}(s) \right) ds \tag{5}$$

$$= \int_0^1 h_{ij}(s) f_j(s) ds + \sum_{k=1}^{n} \alpha_{jk} \int_0^1 h_{ij}(s) g_{jk}(s) ds, \quad i, j = 1, 2, \cdots, n, \tag{6}$$

we set

$$b_{ij} = \int_0^1 h_{ij}(s) f_j(s) ds, \quad i, j = 1, 2, \cdots, n$$

and

$$c_{ijk} = \int_0^1 h_{ij}(s) g_{jk}(s) ds, \quad i, j, k = 1, 2, \cdots, n.$$

Accordingly, we may write (6) as

$$\alpha_{ij} = b_{ij} + \sum_{k=1}^{n} \alpha_{jk} c_{ijk}, \quad i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, n,$$

we obtain simultaneous equations for $\alpha_{ij} = [\alpha_{11}, \cdots, \alpha_{1n}, \alpha_{21}, \cdots, \alpha_{nn}]^T$,

$$(\mathbf{I} - \mathbf{C})\alpha = \mathbf{b}, \tag{7}$$

where $\mathbf{I}$ is an identity matrix of order n^2, $\mathbf{C}$ is an $n^2 \times n^2$ matrix and $\mathbf{b}$ is an $n^2 \times 1$ vector

$$\mathbf{C} = [\mathbf{C}_{ij}]_{n^2 \times n^2},$$

$$\mathbf{b} = [\mathbf{b}_{ij}]_{n^2 \times 1} = [b_{11}, \cdots, b_{1n}, b_{21}, \cdots, b_{nn}]^T$$

and $\mathbf{C}_{ij}$ is an $n \times n$ matrix in which all rows are zero except row j, where row j is $[c_{ij1} \quad c_{ij2} \quad \cdots \quad c_{ijn}]$, i.e.,

$$\mathbf{C}_{ij} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ c_{ij1} & c_{ij2} & \cdots & c_{ijn} \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{n \times n}.$$

So, after solving system (7), we can find $\alpha_{ij}, i, j = 1, 2, \cdots, n$.

Then by inserting α_{ij} in (4), we give $u_i(t), \ i = 1, 2, \cdots, n$, the exact solution of the integral equations system under discussion.

Also, this method can be used for solving some of nonlinear Fredholm integral equations systems with separable kernels. In this case, we obtain all exact solutions [6].

3. Numerical Examples

In this section, we will give some examples to use the method given in this paper. We calculate the exact solution of the Fredholm integral equations system with separable kernels.

The following examples will be helpful to illustrate the main results of this paper:

Example 1. Consider the following linear Fredholm integral equations system:

$$\begin{cases} u_1(t) = \dfrac{2}{3}e^t - \dfrac{1}{4} + \displaystyle\int_0^1 \left(\dfrac{1}{3}e^t s u_1(s) + s^2 u_2(s)\right) ds, \\ u_2(t) = \dfrac{3}{2} - t^2 + \displaystyle\int_0^1 (t^2 e^{-s} u_1(s) - t u_2(s)) ds. \end{cases}$$

For this problem, we have $n = 2$ and

$$\mathbf{U}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \quad \mathbf{F}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} = \begin{bmatrix} \dfrac{2}{3}e^t - \dfrac{1}{4} \\ \dfrac{3}{2} - t^2 \end{bmatrix},$$

$$g_{11}(t) = \frac{1}{3}e^t, \quad g_{12}(t) = 1, \quad g_{21}(t) = t^2, \quad g_{22}(t) = t,$$

$$h_{11}(t) = t, \quad h_{12}(t) = t^2, \quad h_{21}(t) = e^{-t}, \quad h_{22}(t) = 1.$$

Let $\alpha_{ij} = \int_0^1 h_{ij}(s) u_j(s) ds.$ Then, we have

$$\begin{cases} u_1(t) = \dfrac{2}{3}e^t - \dfrac{1}{4} + \dfrac{1}{3}e^t \alpha_{11} + \alpha_{12}, \\ u_2(t) = \dfrac{3}{2} - t^2 + t^2 \alpha_{21} - t\alpha_{22}. \end{cases} \tag{8}$$

$$\mathbf{b} = \left[\frac{13}{24}, \frac{3}{10}, \frac{5}{12} + \frac{1}{4e}, \frac{7}{6}\right]^T,$$

$$\mathbf{c} = [c_{ijk}] = \left[\frac{1}{3}, \frac{1}{2}, \frac{1}{5}, \frac{-1}{4}, \frac{1}{3}, \frac{-1+e}{e}, \frac{1}{3}, \frac{-1}{2}\right]^T,$$

$$\mathbf{C} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{-1}{4} \\ \frac{1}{3} & \frac{-1+e}{e} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{-1}{2} \end{bmatrix}.$$

By solving system $(\mathbf{I}-\mathbf{C})\alpha = \mathbf{b}$, we have

$$\alpha = [\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}]^T = \left[1, \frac{1}{4}, 1, 1\right]^T,$$

by substituting α_{ij} in (8), we give

$$\begin{cases} u_1(t) = \frac{2}{3}e^t - \frac{1}{4} + \frac{1}{3}e^t + \frac{1}{4} = e^t, \\ u_2(t) = \frac{3}{2} - t^2 + t^2 - t = \frac{3}{2} - t. \end{cases}$$

Example 2. Consider the following linear Fredholm integral equations system:

$$\begin{cases} u_1(t) + \int_0^1 (t+s)u_1(s)ds + \int_0^1 (t+2s^2)u_2(s)ds = \frac{11}{6}t + \frac{11}{15}, \\ u_2(t) + \int_0^1 ts^2 u_1(s)ds + \int_0^1 t^2 s u_2(s)ds = \frac{5}{4}t^2 + \frac{1}{4}t, \end{cases}$$

we may rewrite this system in the following form:

$$\begin{cases} u_1(t) = \frac{11}{6}t + \frac{11}{15} - t\int_0^1 u_1(s)ds - \int_0^1 su_1(s)ds - t\int_0^1 u_2(s)ds - 2\int_0^1 s^2 u_2(s)ds \\ \quad = \frac{11}{6}t + \frac{11}{15} - \alpha_{11}t - \alpha_{12} - t\alpha_{13} - 2\alpha_{14}, \\ u_2(t) = \frac{5}{4}t^2 + \frac{1}{4}t - t\int_0^1 s^2 u_1(s)ds - t^2\int_0^1 su_2(s)ds \\ \quad = \frac{5}{4}t^2 + \frac{1}{4}t - \alpha_{21}t - \alpha_{22}t^2. \end{cases} \tag{9}$$

We give

$$\mathbf{I}-\mathbf{C} = \begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{2} & 2 & 0 & 0 \\ \frac{1}{3} & \frac{3}{2} & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{5} \\ 0 & 0 & 0 & 1 & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{4} & \frac{2}{3} & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{5}{4} \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} \frac{32}{20} \\ \frac{44}{45} \\ \frac{13}{24} \\ \frac{5}{16} \\ \frac{253}{360} \\ \frac{19}{48} \end{bmatrix}$$

and then

$$\alpha = [\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{21}, \alpha_{22}]^T = \left[\frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}, \frac{1}{4}\right]^T.$$

By substituting α_{ij} in (9), we give

$$\begin{cases} u_1(t) = \frac{11}{6}t + \frac{11}{15} - \frac{1}{2}t - \frac{1}{3} - \frac{1}{3}t - \frac{2}{5} = t, \\ u_2(t) = \frac{5}{4}t^2 + \frac{1}{4}t - \frac{1}{4}t - \frac{1}{4}t^2 = t^2. \end{cases}$$

Example 3. Consider the following linear Fredholm integral equations system:

$$\begin{cases} u_1(t) = \frac{1}{18}t + \frac{17}{36} + \int_0^1 \frac{t+s}{3}(u_1(s) + u_2(s))ds, \\ u_2(t) = t^2 - \frac{19}{12}t + 1 + \int_0^1 ts(u_1(s) + u_2(s))ds. \end{cases}$$

Set

$$a = \int_0^1 (u_1(s) + u_2(s))ds, \qquad b = \int_0^1 s(u_1(s) + u_2(s))ds, \tag{10}$$

then

$$u_1(t) = \frac{1}{18}t + \frac{17}{36} + \frac{t}{3}a + \frac{b}{3}, \tag{11}$$

$$u_2(t) = t^2 - \frac{19}{12}t + 1 + bt. \tag{12}$$

By substituting (11) and (12) in (10), we give

$$\begin{cases} a - b = \frac{5}{4}, \\ -2a + 9b = \frac{79}{12}, \end{cases}$$

which gives $a = \frac{17}{6}$, $b = \frac{19}{12}$.

Substituting these values in (11) and (12) yields

$$u_1(t) = \frac{1}{18}t + \frac{17}{36} + \frac{t}{3}\frac{17}{6} + \frac{\frac{19}{12}}{3} = 1 + t,$$

$$u_2(t) = t^2 - \frac{19}{12}t + 1 + \frac{19}{12}t = 1 + t^2.$$

Example 4. Consider the following nonlinear Fredholm integral equations system:

$$\begin{cases} u_1(t) = t - \frac{5}{18} + \frac{1}{3}\int_0^1 (u_1(s) + u_2(s))ds, \\ u_2(t) = t^2 - \frac{2}{9} + \frac{1}{3}\int_0^1 ((u_1(s))^2 + u_2(s))ds. \end{cases} \tag{13}$$

Setting

$$a = \int_0^1 (u_1(s) + u_2(s))ds, \qquad b = \int_0^1 ((u_1(s))^2 + u_2(s))ds, \tag{14}$$

carries (13) into

$$\begin{cases} u_1(t) = t - \frac{5}{18} + \frac{a}{3}, \\ u_2(t) = t^2 - \frac{2}{9} + \frac{b}{3}. \end{cases} \tag{15}$$

Substituting (15) in equations (14) gives

$$a = \int_0^1 \left(t - \frac{5}{18} + \frac{a}{3} + t^2 - \frac{2}{9} + \frac{b}{3}\right)ds,$$

$$b = \int_0^1 \left(\left(t - \frac{5}{18} + \frac{a}{3}\right)^2 + t^2 - \frac{2}{9} + \frac{b}{3}\right)ds$$

which give

$$\begin{cases} 2a - b = 1, \\ a^2 - 6b = \frac{-667}{36} \end{cases}$$

or equivalently

$$3a^2 - 32a + \frac{295}{12} = 0,$$

$$b = 2a - 1,$$

so that $\begin{cases} a = \frac{5}{6} \\ b = \frac{2}{3} \end{cases}$ and $\begin{cases} a = \frac{59}{6}, \\ b = \frac{56}{3}. \end{cases}$

Accordingly, two real solutions are given by

$$\begin{cases} u_1(t) = t \\ u_2(t) = t^2 \end{cases} \text{ and } \begin{cases} u_1(t) = t - 3, \\ u_2(t) = t^2 + 6. \end{cases}$$

4. Conclusion

In this paper, we have pointed out that the direct computation method introduces the exact solution of Fredholm integral equations system in the closed form rather than a series form as in the case of decomposition method. However, if the separable kernels $k_{ij}(t, s)$ of the integral equations consist of a polynomial of one or two terms only, then the direct computation method provides the exact solution with the minimum value of calculations.

In addition, the numerical methods give only one approximate solution of nonlinear integral equations system but direct computation method gives all exact solutions.

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