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OPTIMAL FAMILIES OF TWO AND THREE-DIMENSIONAL LATTICE PACKINGS FROM POLYNOMIALS WITH INTEGER COEFFICIENTS

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Abstract

Starting from single-variable quadratic and cubic polynomials with integer coefficients, we construct families of optimal lattice packings of rank two and three in Euclidean space. Each family has infinitely many members.

1. Introduction and Notation

An n-dimensional lattice is a discrete subgroup of $\mathbb{R}^n$ of rank n. More explicitly, it can be described as a set $\Lambda = \left\{\sum_{i=1}^{n} a_i v_i \mid a_i \in \mathbb{Z}\right\}$, where each v_i, $i = 1, \ldots, n$, is a $1 \times n$ vector with entries in $\mathbb{R}$, and the set $\{v_i\}_{i=1}^{n}$ is linearly independent over $\mathbb{R}$. The parameter that describes the packing properties of Λ is its center density $\delta(\Lambda) = \rho^n / |\det M|$, where ρ, the packing radius of Λ, is equal to half the minimal distance between lattice points, and M, the generator matrix, is an $n \times n$ matrix whose rows are $v_1, \ldots, v_n$. The absolute value of the determinant of M gives the volume of a fundamental region of Λ. Finding the densest lattice packing in each dimension is a classic problem in geometry of numbers, and it is part of Hilbert's 18th problem. For a complete account on the subject, the reader is referred to [2].

The purpose of this paper is to illustrate the fact that families of polynomials with integer coefficients can produce lattices with maximum achievable center density. Essentially, the method consists of constructing a generator matrix from the set of roots of a polynomial. With the technique, we construct two infinite families of dense lattices in $\mathbb{R}^2$, and an infinite family of dense lattices in $\mathbb{R}^3$. This is illustrated in Sections 2, 3, and 4. Although the examples are restricted to dimensions 2 and 3, the objective is to suggest that the technique be extended to higher dimensions.

The method resembles the one from geometry of numbers in which one starts with a number field K of degree n and its canonical embedding $\sigma : K \to \mathbb{R}^n$. Then a lattice is obtained as $\sigma(I)$, where I is an integral ideal in the ring of algebraic integers of K, see [1, Chapter 2, Section 3], [2, Chapter 8, Section 7], and [4, Chapter 8]. The lattice $\sigma(I)$ is the geometric representation of I. However, contrary to the method being presented, it is not known how to construct the densest three-dimensional lattice (that is, the face-centered cubic lattice) as the geometric representation of an ideal in a number field.

2. Quadratic Polynomials with Real Roots

Hereupon, given two points $u = (\alpha_1, ..., \alpha_n)$ and $v = (\beta_1, ..., \beta_n)$ in $\mathbb{R}^n$, their inner product will be denoted by $u \cdot v = \alpha_1\beta_1 + \cdots + \alpha_n\beta_n$, as is customary. The squared Euclidean distance from v to the origin, namely, $v \cdot v$, is denoted by $|v|^2$.

Let $f(X) = X^2 + aX + b \in \mathbb{Z}[X]$, where $a \neq 0$ and $a^2 > 4b$. Denote the roots of $f(X)$ by α_1 and α_2. Observe that the vectors $v_1 = (\alpha_1, \alpha_2)$ and $v_2 = (\alpha_2, \alpha_1)$ form a basis of $\mathbb{R}^2$ since

$$\det M = \begin{vmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_1 \end{vmatrix} = -a\sqrt{a^2 - 4b} \neq 0.$$

Define Λ_f as the lattice generated by v_1 and v_2.

Lemma 1. *With the above notation, let* $v = xv_1 + yv_2$ *be a point in* Λ_f, *where* $x, y \in \mathbb{Z}$. *Then*

$$|v|^2 = (a^2 - 2b)(x^2 + y^2) + 4bxy.$$

Proof. We have

$$|v|^2 = x^2 v_1 \cdot v_1 + y^2 v_2 \cdot v_2 + xy(v_1 \cdot v_2 + v_2 \cdot v_1).$$

Since $v_1 \cdot v_1 = v_2 \cdot v_2 = \alpha_1^2 + \alpha_2^2$ and $v_1 \cdot v_2 = v_2 \cdot v_1 = 2\alpha_1\alpha_2$, the result follows. □

Theorem 1. *Let* $f(X) = X^2 + aX + b \in \mathbb{Z}[X]$, *where* $a \neq 0$ *and* $a^2 = 6b$. *Let* α_1 *and* α_2 *be the distinct real roots of* $f(X)$, *and define* $v_1 = (\alpha_1, \alpha_2)$ *and* $v_2 = (\alpha_2, \alpha_1)$. *Under these conditions,*

$$\Lambda_f = \{\alpha_1 v_1 + a_2 v_2 \mid a_i \in \mathbb{Z}\} \tag{1}$$

has center density equal to $\frac{1}{2\sqrt{3}}$, *the maximum achievable in dimension* 2.

Proof. If $v = xv_1 + yv_2 \in \Lambda_f$, then by Lemma 1,

$$|v|^2 = 4b(x^2 + y^2 + xy).$$

For $x, y \in \mathbb{Z}$, the minimum value of the latter expression is equal to $4b$, attained at $(x, y) = (1, 0)$, for example. Hence, the packing radius of Λ_f is $\rho = \sqrt{b}$, and the volume of its fundamental region is

$$v(\Lambda_f) = |\det M| = 2b\sqrt{3}.$$

Therefore, the center density of Λ_f is

$$\delta(\Lambda_f) = \frac{1}{2\sqrt{3}}. \qquad \square$$

Observe that Theorem 1 yields an infinite family of quadratic polynomials whose associated lattices, that is, those defined in (1), have center density equal to that of the hexagonal lattice.

3. Quadratic Polynomials with Complex Roots

Let $f(X) = X^2 + aX + b \in \mathbb{Z}[X]$, where $a \neq 0$ and $a^2 < 4b$. Denote the roots of $f(X)$ by α_1 and α_2. Let $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of $z \in \mathbb{C}$, and define $v_1 = (\Re(\alpha_1), \Im(\alpha_1))$ and $v_2 = (\Re(\alpha_2), \Im(\alpha_2))$, that is, $v_1 = \left(\frac{-a}{2}, \frac{\sqrt{4b - a^2}}{2}\right)$ and $v_2 = \left(\frac{-a}{2}, \frac{\sqrt{4b - a^2}}{2}\right)$. The volume of the fundamental region of the lattice Λ_f generated by v_1 and v_2 is $\frac{|a|\sqrt{4b - a^2}}{2}$, and the expression which gives the squared distance of $v = xv_1 + yv_2 \in \Lambda_f$ to the origin is:

$$|v|^2 = \frac{a^2}{4}(x^2 + y^2 + 2xy) + \frac{4b - a^2}{4}(x^2 + y^2 - 2xy).$$

Hence,

$$|v|^2 = b(x - y)^2 + a^2 xy.$$

Theorem 2. *Let* $f(X) = X^2 + aX + b \in \mathbb{Z}[X]$, *where* $a \neq 0$ *and* $a^2 = 3b$. *Let* α_1 *and* α_2 *be the distinct complex roots of* $f(X)$, *and define* v_1 *and* v_2 *as in the previous paragraph*. *Under these conditions*,

$$\Lambda_f = \{a_1 v_1 + a_2 v_2 \mid a_i \in \mathbb{Z}\} \tag{2}$$

has center density equal to $\frac{1}{2\sqrt{3}}$, *the maximum achievable in dimension* 2.

Proof. Observe that in this case $|v|^2 = b(x^2 + y^2 + xy)$. For $x, y \in \mathbb{Z}$, the minimum value of this expression is equal to b, attained at $(x, y) = (1, 0)$, for example. Hence, the packing radius is $\frac{\sqrt{b}}{2}$, and the volume of the fundamental region is equal to $\frac{b\sqrt{3}}{2}$. Therefore, the center density of Λ_f is $\frac{1}{2\sqrt{3}}$. □

Theorem 2 also yields an infinite family of quadratic polynomials whose associated lattices, that is, those defined in (2), have the maximum achievable center density in dimension 2.

4. Cubic Polynomials with Real Roots

In this section, we consider cubic polynomials whose roots are real. A family of such polynomials whose associated lattices have a record center density in dimension 3 will be presented.

Lemma 2. *Let* $f(X) = X^3 + aX^2 + bX + c \in \mathbb{Z}[X]$ *have real roots* α_1, α_2, *and* α_3. *If*

$$M = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_3 & \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_1 \end{bmatrix},$$

then $\det(M) = -a(a^2 - 3b)$.

Proof. We have $\det M = \alpha_1^3 + \alpha_2^3 + \alpha_3^3 - 3\alpha_1\alpha_2\alpha_3$. From the Newton-Girard relations, it follows that $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = a^2 - 2b$. Multiplying the left-hand side of the latter equation by $\alpha_1 + \alpha_2 + \alpha_3$ and the right-hand side by $-a$, we get

$$\alpha_1^3 + \alpha_2^3 + \alpha_3^3 + \alpha_1\alpha_2^2 + \alpha_2\alpha_1^2 + \alpha_1\alpha_3^2 + \alpha_3\alpha_1^2 + \alpha_2\alpha_3^2 + \alpha_3\alpha_2^2 = -a(a^2 - 2b).$$

Expanding the left-hand side, we obtain

$$\alpha_1^3 + \alpha_2^3 + \alpha_3^3 + \alpha_1\alpha_2(-a - \alpha_3) + \alpha_1\alpha_3(-a - \alpha_2) + \alpha_2\alpha_3(-a - \alpha_1) = -a(a^2 - 2b).$$

The latter equality is equivalent to

$$\alpha_1^3 + \alpha_2^3 + \alpha_3^3 - 3\alpha_1\alpha_2\alpha_3 - ab = -a(a^2 - 2b),$$

whence the result follows. □

Lemma 3. *Let* $f(X) = X^3 + aX^2 + bX + c \in \mathbb{Z}[X]$ *have real roots* α_1, α_2, *and* α_3. *Define* $v_1 = (\alpha_1, \alpha_2, \alpha_3)$, $v_2 = (\alpha_3, \alpha_1, \alpha_2)$, *and* $v_3 = (\alpha_2, \alpha_3, \alpha_1)$. *If x, y, and z are integers and* $v = xv_1 + yv_2 + zv_3$, *then*

$$|v|^2 = (a^2 - 2b)(x^2 + y^2 + z^2) + 2b(xy + xz + yz).$$

Proof. We have $v = (\alpha_1 x + \alpha_3 y + \alpha_2 z, \alpha_2 x + \alpha_1 y + \alpha_3 z, \alpha_3 x + \alpha_2 y + \alpha_1 z)$. Then

$$|v|^2 = (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)(x^2 + y^2 + z^2) + 2(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)(xy + xz + yz),$$

which in turn is equal to $(a^2 - 2b)(x^2 + y^2 + z^2) + 2b(xy + xz + yz)$. □

Let $f(X) = X^3 + aX^2 + bX + c \in \mathbb{Z}[X]$. A necessary and sufficient condition for its roots α_1, α_2, and α_3 to be real and distinct is that its discriminant is strictly greater than zero [3, p. 48]. This is equivalent to imposing

$$c(27c + 4a^3 - 18ab) < b^2(a^2 - 4b).$$

Theorem 3. *Let* $f(X) = X^3 + aX^2 + bX + 1 \in \mathbb{Z}[X]$, *where* $a > 0$, $a^2 = 4b$, *and* $b \geq 9$. *As before, let* α_1, α_2, *and* α_3 *be the real roots of* $f(X)$, *and define* $v_1 = (\alpha_1, \alpha_2, \alpha_3)$, $v_2 = (\alpha_3, \alpha_1, \alpha_2)$, *and* $v_3 = (\alpha_2, \alpha_3, \alpha_1)$. *Then*

$$\Lambda_f = \{a_1 v_1 + a_2 v_2 + a_3 v_3 \mid a_i \in \mathbb{Z}\} \tag{3}$$

has center density equal to $\frac{\sqrt{2}}{8}$, *the maximum achievable in dimension* 3.

Proof. If $v = (\alpha_1 x + \alpha_3 y + \alpha_2 z,\ \alpha_2 x + \alpha_1 y + \alpha_3 z,\ \alpha_3 x + \alpha_2 y + \alpha_1 z)$, then from Lemma 3 it follows that

$$|v|^2 = 2b(x^2 + y^2 + z^2 + xy + xz + yz).$$

The minimum value of the latter expression is $2b$, attained, for example, when $(x, y, z) = (1, 0, 0)$. It follows that the packing radius of Λ_f is $\rho = \sqrt{b/2}$, and the volume of its fundamental region is:

$$v(\Lambda_f) = |\det M| = 2b\sqrt{b}.$$

Therefore, the center density of Λ_f is

$$\delta(\Lambda_f) = \frac{\sqrt{2}}{8}. \qquad \square$$

Theorem 3 yields an infinite family of polynomials whose associated lattices, that is, those defined in (3), have the same density as the face-centered-cubic lattice [2, p. 15].

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