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ARITHMETIC VERSION OF BOOLEAN ALGEBRA

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Abstract

In this paper, we discuss that the logical results in Boolean algebra can equally be derived with ordinary algebraic operations. We establish arithmetic versions of the common logical propositions inclusive of Sheffer stroke (Nand connective) and Peirce's arrow (Nor connective) which are very important to design circuit diagrams. We present the comparison of some basic logical Boolean expressions and their arithmetic versions through the truth tables. Finally, we establish the fundamental logical equivalent proposition via arithmetic versions.

Introduction

Logic developed by Aristotle (322-384 BCE) has been used to develop many areas of theology, philosophy and mathematics [1-3, 5]. It is a pivotal tool in mathematics and computer science [4]. Just like in any language, simple sentences are combined using common connectives like, "and, or, not, if, only if, then and if and only if" to form compound sentences. Likewise, logicians have given logical meaning to these connectives to form compound statements from simple statements. Logicians prefer to deal specifically with Boolean algebra [6], mostly overlooking the fact that ordinary algebra can very well do the same. In this paper, we will interplay the same to have a flavor of arithmetic version of Boolean expressions.

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Results and Discussion

The following are the arithmetic versions of the common logical propositions.

- Arithmetic version of a true or false statement will be denoted as 1 or 0, respectively.
- Arithmetic version of negation is defined as $\sim p = 1 - p$.
- Conjunction statement is defined as "if p and q are two statements, then p and q equivalently $p \wedge q$ is true if both p and q are true otherwise false". We define conjunction arithmetically by ordinary multiplication as $p \wedge q = pq$. Note that if p is true or false, then $pp = p$.
- Disjunction statement is defined as "if p and q are two statements, then p or q equivalently $p \vee q$ is false if both p and q are false otherwise true". Using De Morgan's Law, we derive arithmetic version of disjunction as:

$$p \vee q = \sim(\tilde{p} \wedge \tilde{q}) = \sim(\tilde{p}\tilde{q}) = 1 - [(1-p)(1-q)] = p + q - pq.$$

- Considering equivalent version of contrapositive statement, we derive its arithmetic version as:

If p, then $q \cong p \to q \cong (\sim p) \vee q = (1-p) \vee q = (1-p) + q - (1-p)q = 1 - p + q - q + pq = 1 - p + pq$.

- Considering equivalent version of bi-conditional statement, we derive its arithmetic version as:

p if and only if $q \cong p \Leftrightarrow q \cong (p \to q) \wedge (q \to p) = (1 - p + pq)(1 - q + qp)$ $= 1 - p - q + 2pq = 1 - (p-q)^2$.

- Arithmetic version of exclusive disjunction is defined as:

$p \veebar q \cong (p \wedge \sim q) \vee (\sim p \wedge q) = p(1-q) \vee (1-p)q = (p - pq) \vee (q - pq)$ $= p + q - 2pq = (p-q)^2$.

- Now, we will establish the arithmetic versions of Sheffer stroke and Peirce's arrow which are Nand and Nor connectors, respectively to design circuit diagrams.

Sheffer stroke: $p \mid q \cong \sim(p \wedge q) = 1 - pq$.

Peirce's arrow: $p \downarrow q \cong \sim(p \vee q) = 1 - p - q + pq$.

In the following few lines, we present the comparison of some basic logical Boolean expressions and their arithmetic versions through the truth tables.

1. Negation

Statement	Boolean	Statement	Arithmetic
p	$\sim p$		$1 - p$
T	F	1	0
F	T	0	1

2. Conjunction

Statement	Statement	Boolean			Arithmetic
p	q	p and q $(p \wedge q)$	p	q	pq
T	T	T	1	1	1
T	F	F	1	0	0
F	T	F	0	1	0
F	F	F	0	0	0

3. Disjunction

Statement	Statement	Boolean			Arithmetic
p	q	p or q $(p \vee q)$	p	q	$p + q - pq$
T	T	T	1	1	1
T	F	T	1	0	1
F	T	T	0	1	1
F	F	F	0	0	0

Now, we will establish the fundamental logical equivalent proposition via arithmetic versions.

1. Idempotent laws

(i) $p \wedge p = p \cdot p = p,$

(ii) $p \vee p = p + p - pp = p + p - p = p.$

2. Double negation

$$\sim(\sim p) = \sim(1 - p) = 1 - (1 - p) = p.$$

3. De Morgan's laws

(i) $(\sim p) \vee (\sim q) = (1 - p) \vee (1 - q) = (1 - p) + (1 - q) - (1 - p)(1 - q)$

$$= 1 - pq = \sim(p \wedge q),$$

(ii) $(\sim p) \wedge (\sim q) = (1 - p)(1 - q) = 1 - (p + q - pq) = \sim (p \vee q)$.

4. Commutative properties

(i) $p \wedge q = pq = qp = q \wedge p$,

(ii) $p \vee q = p + q - pq = q + p - qp = q \vee p$.

5. Associative laws

(i) $(p \wedge q) \wedge r = (pq)r = p(qr) = p \wedge (q \wedge r)$,

(ii) $(p \vee q) \vee r = (p + q - pq) \vee r = (p + q - pq) + r - (p + q - pq)r$

$p + (q + r - qr) - p(q + r - qr) = p \vee (q \vee r)$.

6. Distributive laws

(i) $(p \wedge q) \vee r = (pq) \vee r = pq + r - (pq)r = (p + r - pr)(q + r - qr)$

$= (p \vee r) \wedge (q \vee r)$,

(ii) $(p \vee q) \wedge r = (p + q - pq) \wedge r = (p + q - pq)r = pr + qr - pqr$

$= (p \wedge r) \vee (q \wedge r)$.

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