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TWO REMARKS ON TORUS LIGHTS OUT PUZZLE

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Abstract

We present a method to solve $2^k \times 2^k$ Torus Lights Out puzzle, and a criterion for the solvability of $5^k \times 5^k$ Torus Lights Out puzzle. Both the method and the criterion are easy to memorize.

1. Introduction

Lights Out puzzle consists of a 5×5 square array of lighted buttons, each light may be on or off. Pushing a button changes the on/off state of the button itself and of the neighboring ones on the same row or column. Given an initial configuration of lights, the purpose is to turn all the lights out. See [1] for how to solve this puzzle with linear algebra mod 2. There are several variants of this puzzle, see [2, Chapter 6].

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In this note, we are interested in *Torus Lights Out* puzzle, namely Lights Out puzzle on a torus. See Section 2 for the precise formulation. Our first aim is to show that, given an initial configuration of $2^k \times 2^k$ Torus Lights Out, we can solve it by repeating the procedure:

push all buttons which are "on".

Our second aim is to give a criterion for the solvability of $5^k \times 5^k$ Torus Lights Out. We divide the set of $5^k \times 5^k$ buttons into five disjoint subsets in a certain way. Given an initial configuration, we count lighted buttons in each subsets and obtain five numbers. By doing the same thing for the transpose of the configuration, we obtain another five numbers. We shall show that, under some assumption, we can determine whether the configuration is solvable or not by looking at these ten numbers.

2. Torus Lights Out Puzzle

Lights Out puzzle can be formulated on an arbitrary locally finite graph, see for example [4]. In this note, however, we exclusively consider the case where the graph is an $n \times n$ torus as follows. Suppose $n \geq 3$ and let T_n be the finite undirected graph consisting of

- vertices: (x, y) with $x, y \in \{1, 2, \dots, n\}$,
- edges: two vertices (x, y), (x', y') are connected by an edge if and only if either $(x = x'$ and $y - y' \equiv \pm 1 \pmod n)$ or $(y = y'$ and $x - x' \equiv \pm 1 \pmod n)$.

Note that $T_n \cong C_n \times C_n$, the graph product (Cartesian product) of two copies of C_n, where C_n denotes the cycle graph with *n* vertices. Let V_n be the vertex set of T_n.

Let $\mathbb{F}_2$ be the finite field with 2 elements. A *configuration* of T_n is a map $f : V_n \to \mathbb{F}_2$, which we often identify with the $n \times n$ matrix

$$\begin{pmatrix} f((1, 1)) & f((1, 2)) & \cdots & f((1, n)) \\ f((2, 1)) & f((2, 2)) & \cdots & f((2, n)) \\ \vdots & \vdots & \ddots & \vdots \\ f((n, 1)) & f((n, 2)) & \cdots & f((n, n)) \end{pmatrix}.$$

We also regard f as a column vector in $\mathbb{F}_2^{n^2}$ by introducing the lexicographic order in V_n.

For a vertex $v = (x, y) \in V_n$, let b_v be the configuration, as an $n \times n$ matrix, whose (x', y')-entry is 1 if $(x', y') = (x, y)$ or (x', y'), (x, y) are adjacent (i.e., there exists an edge connecting the two vertices), 0 otherwise. For example,

$$b_{(1,1)} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

for the vertex $(1, 1) \in V_4$.

The rule of $n \times n$ *Torus Lights Out* puzzle is as follows: pushing a sequence of buttons located at vertices $v_1, v_2, \ldots, v_s$ changes a configuration f into $f + b_{v_1} + b_{v_2} + \cdots + b_{v_s}$, where the addition is done in $\mathbb{F}_2^{n^2}$. A *solution* to f is a sequence of vertices $v_1, v_2, \ldots, v_s$ such that

$$f + b_{v_1} + b_{v_2} + \cdots + b_{v_s} = O, \tag{1}$$

where O is the zero configuration. We say f is *solvable* if there is a solution to f. The purpose of this puzzle is to determine whether a given configuration is solvable or not, and to find a solution if it is solvable.

We note that 4×4 Torus Lights Out puzzle is known as *Mini Lights Out* or *Keychain Lights Out*, whose solution is given in [2], [3].

Example 1. The configuration

$$f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

has a solution $v_1 = (1, 1)$, $v_2 = (1, 3)$, $v_3 = (2, 3)$, $v_4 = (3, 2)$;

$$f = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

A solution $v_1, v_2, \ldots, v_s$ is said to be *minimal* if $v_i \neq v_j$ $(i \neq j)$. Each solution can be reduced to a minimal one, since $b_v + b_v = O$. Since the order of the vertices is irrelevant in (1), we can and will identify a minimal solution with a subset of V_n. Thus $S \subset V_n$ is a solution to f if and only if

$$f = \sum_{v \in S} b_v$$

holds. Let δ_S be the characteristic function of $S \subset V_n$:

$$\delta_S(v) = \begin{cases} 1 & v \in S, \\ 0 & v \notin S, \end{cases}$$

and let A be the adjacency matrix of T_n:

$$A = (a_{uv})_{u, v \in V_n}, \qquad a_{uv} = \begin{cases} 1 & \text{if } u, v \text{ are adjacent}, \\ 0 & \text{otherwise}. \end{cases}$$

Regarding configurations as column vectors, we have

$$\sum_{v \in S} b_v = (I + A)\delta_S,$$

where I is the identity matrix of size $n^2 \times n^2$. Thus we have the following:

Proposition 2. *A configuration f of T_n is solvable if and only if the equation*

$$(I + A)w = f$$

has a solution $w \in \mathbb{F}_2^{n^2}$. If this is the case, then the subset $S \subset V$ such that $w = \delta_S$ gives a minimal solution to f.

Example 3. The configuration

$$f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is not solvable. Indeed, since

$$I + A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

and

$$(I + A \mid f) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

have rank 5 and 6 over $\mathbb{F}_2$, respectively, the equation $(I + A)w = f$ has no solution.

By Proposition 2, the set of solvable configurations of T_n is the image of the linear transformation of $\mathbb{F}_2^{n^2}$ determined by $I + A$. Let

$$d(n) = n^2 - \mathrm{rank}_{\mathbb{F}_2}(I + A)$$

be the dimension of the kernel of this linear transformation. It follows that the ratio of solvable configurations to all configurations of V_n is $1/2^{d(n)}$, and also that there are $2^{d(n)}$ minimal solutions for each solvable configuration.

Table 1

n			3	4	5	6	7	8	9	10
$d(n)$			4	0	8	8	0	0	4	16
n	11	12	13	14	15	16	17	18	19	20
$d(n)$	0	16	0	0	12	0	16	8	0	32
n	21	22	23	24	25	26	27	28	29	30
$d(n)$	4	0	0	32	8	0	4	0	0	24

Example 4. There are sixteen minimal solutions for

$$f = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

They are:

- $\{(1, 1), (1, 2), (1, 3)\}$ and similar patterns (there are 6),
- $\{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$ and similar patterns (there are 9), and
- V_3.

Remark 5. We can observe interesting, mysterious, arithmetic properties of $d(n)$, see our paper in preparation or [5]. There have been known some relations to deep mathematics such as Chebyshev polynomials and elliptic curves. See, for example, [5] and references therein.

3. $2^k \times 2^k$ Torus Lights Out

For a configuration *f* of T_n, consider the procedure:

$$\text{push all buttons located at } v \text{ with } f(v) = 1. \tag{2}$$

We claim that, given an arbitrary configuration of T_{2^k} $(k \geq 2)$, we can change it into the zero configuration by repeating this procedure at most 2^{k-1} times.

Example 6. Let f be the configuration in Example 1:

$$f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Pushing buttons at v such that $f(v) = 1$, namely at

$$(1, 1),\ (2, 1),\ (2, 4),\ (3, 1),\ (3, 2),\ (4, 1),\ (4, 2),\ (4, 3),$$

changes f into

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Pushing buttons at

$$(1, 3),\ (2, 1),\ (2, 3),\ (2, 4),\ (3, 1),\ (4, 1),\ (4, 2),\ (4, 3)$$

then changes this into the zero configuration. The solution thus obtained is by no means minimal, but has the advantage that it is easy to memorize.

Since the procedure (2) changes f into $f + (I + A)f = Af$, our claim follows from the following theorem. In the statement and the proof, we shall work in the ring of integers, not in $\mathbb{F}_2$.

Theorem 7. $A^{2^{k-1}} \equiv O \pmod 2$ *holds for* T_{2^k} $(k \geq 2)$.

Proof. Put $n = 2^k$ and introduce the circulant matrix

$$B = \begin{pmatrix} 0 & 1 & & & 1 \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ 1 & & & 1 & 0 \end{pmatrix}$$

of size $n \times n$. Then we have

$$A = \begin{pmatrix} O & I & & & I \\ I & O & I & & \\ & \ddots & \ddots & \ddots & \\ & & I & O & I \\ I & & & I & O \end{pmatrix} + \begin{pmatrix} B & & & & \\ & B & & & \\ & & \ddots & & \\ & & & B & \\ & & & & B \end{pmatrix} = B \otimes I + I \otimes B,$$

where I is the identity matrix of size $n \times n$ and $X \otimes Y$ is the Kronecker product of X and Y. Since $n/2$ is a power of 2, we have

$$A^{n/2} \equiv B^{n/2} \otimes I + I \otimes B^{n/2} \pmod 2.$$

It is enough to show that $B^{n/2} \equiv O \pmod 2$. This follows, for example, from the observation that B is the adjacency matrix of the cycle graph C_n. Let the vertices of C_n be $v_1, v_2, \ldots, v_n$ connected in this order. The (i, j)-entry of $B^{n/2}$ is equal to the number of paths of length $n/2$ from v_i to v_j. Again since $n/2$ is a positive power of 2, it is easy to see that this number is

$$\begin{cases} \dbinom{n/2}{\left| n/2 - |i-j| \right|/2} & i \equiv j \pmod 2, \ |i-j| \neq n/2, \\ 2 & |i-j| = n/2, \\ 0 & i \not\equiv j \pmod 2, \end{cases}$$

and is even for all i, j. □

Since any configuration of T_{2^k} is solvable, we have the following (cf. [5, 2.12(f)]).

Corollary 8. $d(2^k) = 0$ $(k \geq 2)$.

Remark 9. Our method gives the solution

$$f + Af + \cdots + A^{2^{k-1}-1} f$$

to a configuration f of T_{2^k}. Its reduced minimal solution is the unique minimal solution to f. In the case $k = 2$, this minimal solution is identical to that given in [3].

Remark 10. Our method is also valid for $2^k \times 2^{k'}$ Torus Lights Out. Let f be a configuration of $C_{2^k} \times C_{2^{k'}}$ $(k > k' \geq 2)$. By considering the "tiled" configuration

$$\underbrace{(f\ f \cdots f)}_{2^{k-k'}},$$

we see that we can change f into the zero configuration by repeating the procedure (2) at most 2^{k-1} times.

4. $5^k \times 5^k$ Torus Lights Out

We shall give a criterion for the solvability of $5^k \times 5^k$ Torus Lights Out. For a configuration f of T_{5^k} and for $i = 0, 1, 2, 3, 4,$ define

$$N_i(f) = \#\{v = (x, y) \in V_{5^k} \mid x + 2y \equiv i \pmod 5),\ f(v) = 1\}.$$

Table 2. $(x + 2y)$ mod 5 in the case $k = 1$

$y \backslash x$	1	2	3	4	5
1	3	4	0	1	2
2	0	1	2	3	4
3	2	3	4	0	1
4	4	0	1	2	3
5	1	2	3	4	0

Lemma 11. *If a configuration f of* T_{5^k} *is solvable, then the congruences*

$$N_0(f) \equiv N_1(f) \equiv N_2(f) \equiv N_3(f) \equiv N_4(f) \pmod 2$$

and

$$N_0({}^t f) \equiv N_1({}^t f) \equiv N_2({}^t f) \equiv N_3({}^t f) \equiv N_4({}^t f) \pmod 2$$

hold. Here ${}^t f$ *is the transpose of f regarded as a square matrix.*

Proof. It is enough to show the first half, since if f is solvable, then so is ${}^t f$.

Let

$$S_i = \{v = (x, y) \in V_{5^k} \mid x + 2y \equiv i \pmod 5\},$$

and let $\delta_i = \delta_{S_i}$ be its characteristic function. We have

$$N_i(f) \equiv \langle f, \delta_i \rangle \pmod 2,$$

where $\langle\ ,\ \rangle$ is the standard inner product. In particular, $N_i(f)$ mod 2 is linear in *f*. If *f* is solvable, then $f = \sum_{v \in S} b_v$ for some $S \subset V_{5^k}$, and hence

$$N_i(f) \equiv \sum_{v \in S} N_i(b_v) \equiv |S| \pmod 2,$$

since $N_i(b_v) = 1$ for any *i* and *v*. This proves the first half. □

Example 12. The configuration

$$f = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in V_5$$

satisfies

$$N_0(f) = 2, \quad N_1(f) = 1, \quad N_2(f) = 0, \quad N_3(f) = 2, \quad N_4(f) = 0,$$

$$N_0({}^t f) = N_1({}^t f) = N_2({}^t f) = N_3({}^t f) = N_4({}^t f) = 1.$$

Therefore, *f* is not solvable.

We claim that the converse to Lemma 11 holds, under the assumption that $d(5^k) = d(5)$ holds. We have checked, assuming another conjecture (see our paper in preparation), that

$$d(p^k) = d(p)$$

holds for prime powers p^k up to 20000. We expect this to hold in general.

Theorem 13. *Suppose that* $d(5^k) = d(5)$ *holds. A configuration* f *of* T_{5^k} *is* *solvable if and only if the congruences*

$$N_0(f) \equiv N_1(f) \equiv N_2(f) \equiv N_3(f) \equiv N_4(f) \pmod 2$$

and

$$N_0({}^t f) \equiv N_1({}^t f) \equiv N_2({}^t f) \equiv N_3({}^t f) \equiv N_4({}^t f) \pmod 2$$

hold.

Proof. Let $n = 5^k$, $W = \mathbb{F}_2^{n^2}$, $\alpha : W \to W$ be the linear transformation determined by $I + A$, and $\beta : W \to \mathbb{F}_2^8$ be the linear map defined by

$$\beta(f) = \begin{pmatrix} N_1(f) - N_0(f) \\ N_2(f) - N_0(f) \\ N_3(f) - N_0(f) \\ N_4(f) - N_0(f) \\ N_1({}^t f) - N_0({}^t f) \\ N_2({}^t f) - N_0({}^t f) \\ N_3({}^t f) - N_0({}^t f) \\ N_4({}^t f) - N_0({}^t f) \end{pmatrix} \bmod 2.$$

We must show that $\operatorname{Im}\alpha = \operatorname{Ker}\beta$ holds. Since we have shown $\operatorname{Im}\alpha \subset \operatorname{Ker}\beta$ in Lemma 11, we have

$$d(n) = n^2 - \dim \operatorname{Im}\alpha \geq n^2 - \dim \operatorname{Ker}\beta = \dim \operatorname{Im}\beta.$$

By the assumption and the fact that $d(5) = 8$, we have $d(n) = 8$. Therefore, what we have to show is the surjectivity of β. Let f be as in Example 12. The image of

$$\begin{pmatrix} f & O & \cdots & O \\ O & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & O \end{pmatrix}$$

(regarded as a column vector in W) under β is ${}^t(1\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$. Similarly, we see ${}^t(0\ 1\ 0\ 0\ 0\ 0\ 0\ 0) \in \operatorname{Im}\beta$ and so forth. This completes the proof. □

Remark 14. Lemma 11 is also true for T_n, $n \equiv 0 \pmod 5$. But the converse does not hold in general, since $d(n)$ may exceed 8 (see Table 1).

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