Far East Journal of Mathematical Sciences (FJMS)
Volume 35, Number 3, 2009, Pages 281-293
Published Online: November 19, 2009
This paper is available online at http://www.pphmj.com

CIRCULAR SYMMETRY OF SPECTRA OF COMPOSITION OPERATORS

SIVARAM K. NARAYAN and DAWN E. SADIR*

Department of Mathematics
Central Michigan University
Mount Pleasant MI 48859, U. S. A.
e-mail: sivaram.narayan@cmich.edu

Department of Mathematics
Pensacola Junior College
Pensacola, FL 32504, U. S. A.
e-mail: dsadir@pjc.edu

Abstract

For a certain class of composition operators C_ϕ on the Hardy space $H^2(\mathbf{D})$ with multivalent symbol ϕ, we verify the conjecture due to Cowen and MacCluer [3], which explains the presence of circular symmetry of the spectrum in this case.

1. Introduction

For ϕ analytic in the unit disk $\mathbf{D}$ such that $\phi(\mathbf{D}) \subset \mathbf{D}$, the composition operator C_ϕ on the Hardy space $H^2(\mathbf{D})$ of the unit disk is given by $C_\phi f = f \circ \phi$, for all f in $H^2(\mathbf{D})$. It is known that C_ϕ is a bounded operator, and other general properties of

2000 Mathematics Subject Classification: 47B38.

Keywords and phrases: composition operators, Hardy space, spectrum.

*This paper contains results from the doctoral dissertation of Dawn E. Sadir written at Central Michigan University.

Received August 13, 2009

C_ϕ have been established. Much of these developments can be found in [7] and [2]. Many properties of C_ϕ depend to a great extent on the behavior of the symbol map ϕ near its fixed points. A point b in $\overline{\mathbf{D}}$ is a fixed point of ϕ if $\lim_{r\to 1^-} \phi(rb) = b$. We will write $\phi'(b)$ for $\lim_{r\to 1^-} \phi'(rb)$. This limit obviously exists when $|b| < 1$. If $|b| = 1$, then the theorems of Julia, Carathéodory, and Wolff show that $\phi'(b)$ exists and $0 < \phi'(b) \le \infty$.

Theorem 1.1 (Wolff's Lemma) [2, p. 56]**.** *If ϕ is an analytic map of the unit disk into itself that has no fixed point in* $\mathbf{D}$, *then there is a unique fixed point b of ϕ on the unit circle with $0 < \phi'(b) \le 1$. If ϕ, not the identity, has a fixed point in* $\mathbf{D}$, *then $\phi'(b) > 1$ for all fixed points b of ϕ on the unit circle*.

We will call the distinguished fixed point b the *Denjoy-Wolff point* of ϕ, where if $|b| = 1$, then $0 < \phi'(b) \le 1$ and if $|b| < 1$, then $|\phi'(b)| < 1$. This Denjoy-Wolff point b of ϕ allows the study of composition operators based on its location $(|b| < 1 \text{ or } |b| = 1)$ and the value of $\phi'(b)$.

Much progress has been made in identifying the spectra of composition operators when the Denjoy-Wolff point is on the boundary of $\mathbf{D}$ [6, p. 133] as well as explaining the presence of circles in the spectrum [2, p. 287]. However, in the case of interior fixed point, results are incomplete. Let ϕ_m denote $\underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{m \text{ times}}$.

Kamowitz [6, p. 133] showed that if ϕ is analytic in a neighborhood of $\mathbf{D}$, has a fixed point b in $\mathbf{D}$, ϕ is not inner and C_ϕ is not power compact, then there is a positive integer m for which $S_m = \{\phi_m(w) : |\phi_m(w)| = 1\}$ is finite, nonempty, and consists only of fixed points of ϕ_m. Moreover, $c = \min\{\phi_m'(z) : z \in S_m\}$ is greater than 1 and $sp(C_\phi) = \left\{\lambda : |\lambda| \le c^{\frac{-1}{2m}}\right\} \cup \{(\phi'(b))^n : n \in \mathbf{N}\} \cup \{1\}$. Cowen and MacCluer [2, p. 289] proved the same result using different methods for univalent symbols ϕ without the additional assumption of analyticity in a neighborhood of $\mathbf{D}$.

Since $sp(C_\phi)$ in both theorems is a disc together with some isolated points, it has the property of circular symmetry. We say that the spectrum of an operator T has *circular symmetry* if whenever λ belongs to the spectrum and λ is not an isolated point, then $e^{i\theta}\lambda$ belongs to the spectrum for every real θ. Cowen and MacCluer conjectured that presence of circular symmetry can be explained by the presence of an invariant subspace for C_ϕ on which the restriction of C_ϕ is similar to $e^{i\theta}C_\phi$.

Conjecture 1.2 [3, p. 21]**.** If ϕ has a fixed point in $\mathbf{D}$ and the essential spectral radius of C_ϕ is positive, then there is an invariant subspace for C_ϕ on which the restriction of C_ϕ is similar to rotates of itself.

In this paper, we verify Cowen and MacCluer's conjecture for the class of symbols given by $\phi(z) = \dfrac{(1-2c)z^{k+1}}{1-2cz^k}$ with $0 < c < \dfrac{1}{2}$ and $k = 1, 2, 3, \ldots$. This is a class of multivalent functions having fixed points 0 and the kth roots of unity for $k \in \mathbf{N}$. In this case, $\phi'(0) = 0$ and from Kamowitz's theorem $sp(C_\phi)$ is a disc centered at the origin and 1.

In Section 5, we find a subspace M_k of $H^2(\mathbf{D})$ corresponding to each of the kth roots of unity on which C_ϕ^* behaves like a weighted shift. The subspace M_k is given using a basic sequence in $H^2(\mathbf{D})$ of differences of reproducing kernel functions determined by a sequence of backward iterates of the symbol map $\phi(z)$. This paper generalizes the result and techniques of Wahl [9]. We also show that the interior of the spectrum is a disc of eigenvalues of C_ϕ^*.

2. Analytic and Geometric Properties of $\phi(z)$

For the composition operator C_ϕ on $H^2(\mathbf{D})$ with symbol $\phi(z) = \dfrac{(1-2c)z^{k+1}}{1-2cz^k}$, $0 < c < \dfrac{1}{2}$, $k = 1, 2, 3, \ldots$ we first study the analytic and geometric properties of $\phi(z)$. It is easily verified that $\phi(\mathbf{D}) \subset \mathbf{D}$ and that ϕ is analytic in an open disc containing $\overline{\mathbf{D}}$. Also it is clear that $\phi(0) = 0$, $\phi'(0) = 0$ and ϕ is not univalent on $\mathbf{D}$. The following lemma records some easily proved properties of $\phi(z)$.

Lemma 2.1. *For* $\phi(z) = \dfrac{(1-2c)z^{k+1}}{1-2cz^k}$ *with* $0 < c < \dfrac{1}{2}$ *and* $k \in \mathbf{N}$ *the following hold*:

(1) *The fixed points of* ϕ *are precisely* 0 *and the kth roots of unity* $e^{\frac{2\pi ij}{k}}$ *for* $j = 1, 2, \ldots, k$.

(2) ϕ *maps* $[0, 1]$ *onto* $[0, 1]$.

(3) $\phi'\left(e^{\frac{2\pi ij}{k}}\right) = \dfrac{k+1-2c}{1-2c}$ *for* $j = 1, 2, \ldots, k$.

As a consequence of Lemma 2.1, we prove the following.

Lemma 2.2. *For* $\phi(z) = \dfrac{(1-2c)z^{k+1}}{1-2cz^k}$ *with* $0 < c < \dfrac{1}{2}$ *and* $k \in \mathbf{N}$, *for each* $j = 1, 2, \ldots, k$ *there exists an* $\varepsilon_j > 0$ *such that* ϕ *is univalent when restricted to* $D_j = \left\{z \in \mathbf{C} : \left|z - e^{\frac{2\pi ij}{k}}\right| < \varepsilon_j\right\}$ *and* $\left|\phi(z) - e^{\frac{2\pi ij}{k}}\right| > \left|z - e^{\frac{2\pi ij}{k}}\right|$ *for* $z \in D_j$.

Proof. Follows from $\phi'\left(e^{\frac{2\pi ij}{k}}\right) = \dfrac{k+1-2c}{1-2c} > 1$. □

Lemma 2.3. *For* $\phi(z) = \dfrac{(1-2c)z^{k+1}}{1-2cz^k}$ *with* $0 < c < \dfrac{1}{2}$ *and* $k \in \mathbf{N}$, $\phi(z)$ *is not inner*.

Proof. Let $z = e^{\frac{i\pi}{k}}$. Then $\left|\phi\left(e^{\frac{i\pi}{k}}\right)\right| = \dfrac{1-2c}{1+2c} < 1$. Since ϕ is analytic in a neighborhood of $\overline{\mathbf{D}}$, there is a small disc centered at $e^{\frac{i\pi}{k}}$ on which $|\phi(z)| < 1$. Thus, $\phi(z)$ is not inner. □

Lemma 2.4. *For* $\phi = \dfrac{(1-2c)z^{k+1}}{1-2cz^k}$ *with* $0 < c < \dfrac{1}{2}$ *and* $k \in \mathbf{N}$, *if* $z \in \mathbf{D}$ *and* $0 < \arg(z) \le \dfrac{\pi}{k+1}$, *then* $\arg(\phi(z)) > (k+1)\arg(z)$.

Proof. For $\phi = \dfrac{(1-2c)z^{k+1}}{1-2cz^k}$ write

$$\phi(z) = |\phi(z)|e^{i\arg(\phi(z))} = |\phi(z)|e^{i((k+1)\arg(z) - \arg(1-2cz^k))}.$$

Thus, $\arg(\phi(z)) = (k+1)\arg(z) - \arg(1-2cz^k) + 2m\pi$ for some integer m. Let $\arg(z)$ denote the principal argument of z, where $0 \le \arg(z) < 2\pi$. For $j = 0, \dots, k-1$, let $S_j = \left\{z \in \mathbf{D} : \frac{2\pi j}{k} < \arg(z) \le \frac{2\pi j}{k} + \frac{\pi}{k+1}\right\}$. If $z \in S_j$ for $j = 0, \dots, k-1$, then looking at the real and imaginary parts of $1 - 2cz^k$ we get $\operatorname{Re}(1-2cz^k) = 1 - 2c\operatorname{Re}(z^k) > 1 - 2c > 0$ and $\operatorname{Im}(1-2cz^k) = -2c\operatorname{Im}(z^k)$. Since $2\pi j < k\arg(z) \le 2\pi j + k\frac{\pi}{k+1}$ for $z \in S_j$, $j = 0, \dots, k-1$, we get $0 < \arg(z^k) < \pi$ and $\operatorname{Im}(1-2cz^k) < 0$. Hence, $\frac{3\pi}{2} < \arg(1-2cz^k) < 2\pi$ for $z \in S_j$, $j = 0, \dots, k-1$. This inequality together with $\frac{2\pi j}{k} < (k+1)\arg(z) - 2\pi j \le \frac{2\pi j}{k} + \pi$ results in

$$\frac{2\pi j}{k} < (k+1)\arg(z) + 2\pi(1-j) - \arg(1-2cz^k) \le \frac{2\pi j}{k} + \frac{3\pi}{2},$$

for $z \in S_j$, $j = 0, \dots, k-1$. For $j = 0$, we get

$$\arg(\phi(z)) = (k+1)\arg(z) + 2\pi - \arg(1-2cz^k) > (k+1)\arg(z). \qquad \square$$

Remark 2.5. In general taking the branch of argument with $\frac{2\pi j}{k} \le \arg(w) < \frac{2\pi j}{k} + 2\pi$, we have for $z \in S_j$, $\arg(\phi(z)) = (k+1)\arg(z) + 2\pi(1-j) - \arg(1-2cz^k)$. Thus, for $z \in S_j$, $j = 0, 1, \dots, k-1$, we get $\arg(\phi(z)) > \arg(z)$.

3. Spectra of C_ϕ

Since $\phi(z)$ has a finite angular derivative at $z = 1$, C_ϕ is not compact. Also, C_{ϕ_m} has a finite angular derivative at $z = 1$ and C_ϕ is not power compact. From Section 2, we know that ϕ is not inner and $\phi(0) = 0$. From Kamowitz's theorem stated in Section 1, we see that

$$sp(C_\phi) = \left\{\lambda : |\lambda| \le \sqrt{\frac{1-2c}{k+1-2c}}\right\} \cup \{1\}.$$

Clearly, $sp(C_\phi)$ has circular symmetry. The essential spectrum of C_ϕ is the disc without the isolated point. Hence, C_ϕ has positive essential spectral radius. Thus, the conditions of Cowen and MacCluer's conjecture hold for this class of composition operators with multivalent symbol. We show that there is an invariant subspace for C_ϕ on which C_ϕ is similar to rotates of itself.

4. Development of an Interpolating Sequence

In order to verify the conjecture of Cowen and MacCluer we find a subspace M of $H^2(\mathbf{D})$ on which C_ϕ^* behaves like a weighted shift and then apply well-known results on weighted shifts. A key ingredient in such a construction is the notion of an interpolating sequence.

Definition 4.1. A sequence $\{z_j\}$ in $\mathbf{D}$ is an interpolating sequence if for every bounded sequence of complex numbers $\{c_j\}$ there exists a bounded analytic function f on $\mathbf{D}$ with $f(z_j) = c_j$.

The following result gives a sufficient condition for a sequence to be an interpolating sequence.

Theorem 4.2 [5, p. 203]**.** *Suppose* $\{z_k\}$ *is a sequence of points in* $\mathbf{D}$ *and*

$$\frac{1 - |z_n|}{1 - |z_{n-1}|} < c < 1.$$

Then $\{z_k\}$ *is an interpolating sequence.*

We require in our construction that the interpolating sequence $\{z_j\}$ is a *backward iteration sequence* (i.e., the sequence $\{z_j\}$ satisfies $\phi(z_j) = z_{j-1}$). The following theorem is crucial to our construction of the interpolating sequence.

Theorem 4.3 [1, p. 88]**.** *Suppose for* $\phi : \mathbf{D} \to \mathbf{D}$ *there is a point* $b \in \partial\mathbf{D}$ *such that* $\lim_{r\to 1^-} \phi(rb) = b$ *and* $\phi'(b) > 1$. *If the non-constant sequence* $\{z_k\}_{k=0}^{\infty}$ *contained in* $\mathbf{D}$ *is such that* $\phi(z_k) = z_{k-1}$ *and* $\lim_{k\to\infty} z_k = b$, *then* $\{z_k\}_{k=0}^{\infty}$ *is an interpolating sequence.*

In our case for each boundary fixed point $b_j = e^{\frac{2\pi ij}{k}}$, $k \in \mathbf{N}$, $j = 0, ..., k-1$ an interpolating sequence $\{z_m^{(j)}\}_{m=0}^{\infty}$ can be constructed satisfying $\lim_{m\to\infty} z_m^{(j)} = b_j$. Since 1 is a fixed point of ϕ for all $k \in \mathbf{N}$, we will develop an interpolating sequence that approaches 1. A similar construction (with minor modifications) works for the other fixed points as well. For notational convenience, we give the details only for the boundary fixed point 1.

Theorem 4.4. *Let* $\phi = \dfrac{(1-2c)z^{k+1}}{1-2cz^k}$ *with* $0 < c < \dfrac{1}{2}$ *and* $k \in \mathbf{N}$. *Then there exists a sequence* $\{z_m\}_{m=0}^{\infty}$ *satisfying the following conditions*:

(1) $-1 < z_0 < 0$,

(2) $\mathrm{Im}(z_m) > 0$, *for* $m = 1, 2, ...$,

(3) $\phi(z_m) = z_{m-1}$, *for* $m = 1, 2, ...$,

(4) $\lim_{m\to\infty} z_m = 1$ *nontangentially*,

(5) $\{z_m\}_{m=0}^{\infty}$ *is an interpolating sequence*.

Proof. From Lemma 2.2, there exists $\varepsilon > 0$ such that ϕ is univalent on $D_1 = \{z \in \mathbf{C} : |z-1| < \varepsilon\}$ and $D_1 \subset \phi(D_1)$. Thus, we can define $\phi^{-1} : D_1 \to D_1$. By translating and dilating D_1 to $\mathbf{D}$, we can define a new function Φ that satisfies the hypothesis of Königs' Theorem [7, p. 91].

For $z \in \mathbf{D}$ define $\Phi(z) = \dfrac{\phi^{-1}(\varepsilon z+1)-1}{\varepsilon}$. It is easy to verify that $\Phi : \mathbf{D} \to \mathbf{D}$ is analytic, $\Phi(0) = 0$ and $\Phi'(0) = \dfrac{1-2c}{k+1-2c} \neq 0$. From Königs's Theorem, there exists $\Psi : \mathbf{D} \to \mathbf{C}$ such that $\Psi(0) = 0$, $\Psi'(0) = 1$, and $\Psi \circ \Phi = \Phi'(0)\Psi$. Since ϕ^{-1} is univalent on D_1, Φ is univalent on $\mathbf{D}$ and Königs's Theorem gives that Ψ is univalent on $\mathbf{D}$.

Define $\sigma : D_1 \to \mathbf{C}$ by $\sigma(z) = \Psi\left(\dfrac{z-1}{\varepsilon}\right)$. Then $\sigma(1) = 0$, $\sigma'(1) = \dfrac{1}{\varepsilon}$, and σ is univalent. For $z \in D_1$, we have

$$(\phi^{-1})'(1)\sigma(z) = \Phi'(0)\Psi\left(\frac{z-1}{\varepsilon}\right) = \Psi \circ \Phi\left(\frac{z-1}{\varepsilon}\right)$$

$$= \Psi\left(\frac{\phi^{-1}\left(\varepsilon\left(\frac{z-1}{\varepsilon}\right)+1\right)-1}{\varepsilon}\right) = \Psi\left(\frac{\phi^{-1}(z)-1}{\varepsilon}\right)$$

$$= \sigma(\phi^{-1}(z)) = \sigma \circ \phi^{-1}(z).$$

Henceforth we denote $a = \frac{k+1-2c}{1-2c}$. Let $w = \phi^{-1}(z)$ for $z \in D_1$. Then $w \in D_1$ and from above $(\phi^{-1})'(1)\sigma(\phi(w)) = \sigma \circ \phi^{-1}(\phi(w)) = \sigma(w)$. Thus, $\sigma(\phi(w)) = a\sigma(w)$. By iterating we obtain $\sigma(\phi_n(w)) = a^n\sigma(w)$.

Since $D_1 \cap \mathbf{D}$ is an open set with 1 on its boundary, $\sigma(D_1 \cap \mathbf{D})$ is an open set with $\sigma(1) = 0$ on its boundary.

Claim 4.5. $\sigma((1-\varepsilon, 1)) \subset (-\infty, 0)$.

Proof. Let $z \in (1-\varepsilon, 1)$. Then $z = 1-\delta$ for $0 < \delta < \varepsilon$. Since $\sigma(z) = \Psi\left(\frac{z-1}{\varepsilon}\right) = \Psi\left(\frac{-\delta}{\varepsilon}\right)$ and $-1 < \frac{-\delta}{\varepsilon} < 0$, it is enough to consider $\Psi((-1, 0))$. Let $\eta \in (-1, 0)$. Then $\varepsilon\eta + 1 \in (1-\varepsilon, 1)$. Since ϕ maps $(0, 1)$ onto $(0, 1)$, there is $\beta \in (0, 1)$ such that $\phi(\beta) = \varepsilon\eta + 1$. Thus

$$\Phi(\eta) = \frac{\phi^{-1}(\varepsilon\eta+1)-1}{\varepsilon} = \frac{\beta-1}{\varepsilon} \in (-1, 0).$$

Now iterating Φ, we get $\Phi_n\left(\frac{-\delta}{\varepsilon}\right) \in (-1, 0)$. From the proof of Königs' Theorem [7, pp. 91-92], we know that $\Psi(w) = \lim_{n\to\infty}(\Phi'(0))^{-n}\Phi_n(w)$. Since $\Phi'(0) > 0$, we conclude $\Psi\left(\frac{-\delta}{\varepsilon}\right) \leq 0$. But Ψ is univalent and $\Psi(0) = 0$. Therefore, $\Psi\left(\frac{-\delta}{\varepsilon}\right) < 0$ and hence the claim. □

Choose $u \in \sigma(D_1 \cap \mathbf{D})$ such that each of the following conditions is met.

- The line segment from 0 to *au* not including 0 is contained in $\sigma(D_1 \cap \mathbf{D})$.
- $p_0 = \sigma^{-1}(u)$ is in the first quadrant.
- The angle formed between $[-1, 0]$ and the line segment from 0 to u is acute.

Let $p_1 = \phi(p_0)$. Then $au = a\sigma(p_0) = \sigma(\phi(p_0)) = \sigma(p_1)$. Parameterize the line segment from 0 to au by tu for $0 \leq t \leq a$. Define $\gamma(t) = \sigma^{-1}(tu)$ for $0 \leq t \leq a$. Hence, γ is a curve in $(D_1 \cap \mathbf{D}) \cup \{1\}$ from 1 to p_1 through p_0.

For $0 \leq t \leq 1$, let $z = \sigma^{-1}(tu) = \gamma(t)$. Then $tu = \sigma(z)$ and $atu = a\sigma(z) = \sigma(\phi(z))$. Therefore, $\gamma(at) = \sigma^{-1}(atu) = \sigma^{-1}(\sigma(\phi(z))) = \phi(z) = \phi(\gamma(t))$. Further extend the definition of γ recursively as follows: For $s > 1$, define $\gamma(a^s) = \phi(\gamma(a^{s-1}))$. This gives a smooth continuous curve $\gamma : (0, \infty) \to \mathbf{D}$ guaranteed by analyticity of σ^{-1} and ϕ. From its definition γ is mapped onto itself by ϕ.

Using Lemma 2.4, we have $\arg(\phi(z)) > (k+1)\arg(z)$ for $z \in \mathbf{D}$ satisfying $0 < \arg(z) \leq \frac{\pi}{k+1}$. Thus, γ crosses $(-1, 0)$. Let s_0 be the smallest value such that $\gamma(a^{s_0}) \in (-1, 0)$. Define $z_0 = \gamma(a^{s_0})$ and for $m \in \mathbf{N}$ define $z_m = \gamma(a^{s_0 - m})$. Then $\phi(z_m) = \phi(\gamma(a^{s_0 - m})) = \gamma(a^{s_0 - m+1}) = z_{m-1}$ and $\lim_{m\to\infty} z_m = \lim_{m\to\infty} \gamma(a^{s_0 - m})$ $= \gamma(0) = 1$. Since from Claim 4.5, σ^{-1} maps the negative real axis into the interval conformally at 1, it maps the line segment from 0 to au onto a curve that makes an acute angle with $(0, 1)$ from the choice of u. Hence, $\{z_m\}$ approaches 1 nontangentially. The sequence $\{z_m\}$ is an interpolating sequence follows from Theorem 4.3. This completes the proof of Theorem 4.4. □

Corollary 4.6. *Let* $\{z_m\}_{m=0}^{\infty}$ *be the sequence obtained in Theorem* 4.4. *Then* $\{\overline{z_m}\}_{m=0}^{\infty}$ *is an interpolating sequence*.

Proof. Note that $\phi(\overline{z_m}) = \overline{z_m}$. □

Remark 4.7. In order to obtain an interpolating sequence approaching a boundary fixed point $b \neq 1$ we modify the proof of Theorem 4.4 by considering rays from 0 to b and 0 to $-b$. Another interpolating sequence can be obtained similar to Corollary 4.6 by reflecting the original interpolating sequence on the ray joining 0 and b.

5. Verification of Cowen-MacCluer's Conjecture

From the two interpolating sequences obtained in Theorem 4.4 and Corollary 4.6, we omit z_0 to obtain two disjoint sequences. The following theorem is needed to show that $\{z_m, \overline{z_m}\}_{m=1}^{\infty}$ is an interpolating sequence.

Theorem 5.1 [4, p. 314]. *If S and T are disjoint interpolating sequences, then $S \cup T$ is an interpolating sequence if and only if $\rho(S, T) = \inf\{\rho(z, w) : z \in S, w \in T\} > 0$, where $\rho(z, w) = \left|\frac{z - w}{1 - \overline{w}z}\right|$ is the pseudo-hyperbolic metric.*

In our case for $n, p > 1$ the Schwartz-Pick Theorem [2] gives

$$\rho(z_n, \overline{z_p}) = \left|\frac{z_n - \overline{z_p}}{1 - z_p z_n}\right| \geq \left|\frac{\phi(z_n) - \phi(\overline{z_p})}{1 - \overline{\phi(\overline{z_p})}\phi(z_n)}\right| = \left|\frac{z_{n-1} - \overline{z_{p-1}}}{1 - z_{p-1}z_{n-1}}\right| = \rho(z_{p-1}, \overline{z_{p-1}}).$$

We also have $\rho(z_n, \overline{z_p}) = \rho(z_p, \overline{z_n})$. Thus, $\rho(\{z_m\}_{m=1}^{\infty}, \{\overline{z_m}\}_{m=1}^{\infty}) = \inf\{\rho(z_n, \overline{z_p})\}$ $= \inf\{\rho(z_1, \overline{z_m})\}$. Since $\rho(z_1, \overline{z_j}) = \frac{|z_1 - \overline{z_j}|}{|1 - z_j z_1|} \geq \frac{\mathrm{Im}(z_1)}{1 + |z_1|} > 0$, we have proved the following proposition.

Proposition 5.2. *Let $\{z_m\}_{m=0}^{\infty}$ be the sequence from Theorem 4.4. Then $\{z_m, \overline{z_m}\}_{m=1}^{\infty}$ is an interpolating sequence.*

The following two theorems are needed for the development of an invariant subspace M for C_{ϕ}^{*}. A sequence $\{f_n\}$ in $H^2(\mathbf{D})$ is said to be a *basic sequence* if $\{f_n\}$ is a basis of the closed linear subspace of $H^2(\mathbf{D})$ that is spanned by $\{f_n\}$.

Theorem 5.3 [1, p. 88]. *If $\{z_m\} \subset \mathbf{D}$ is an interpolating sequence, then $\{\sqrt{1 - |z_m|^2}K_{z_m}\}$ is a basic sequence in $H^2(\mathbf{D})$ equivalent to an orthonormal set, that is, the series $\sum \alpha_m \sqrt{1 - |z_m|^2}K_{z_m}$ converges if and only if $\sum |\alpha_m|^2 < \infty$.*

Theorem 5.4 [9, p. 765]. *Suppose the vectors $\{e_m\}_{m=0}^{\infty}$ are equivalent to an orthonormal set. Then the vectors $\{e_{2m} - e_{2m+1}\}_{m=0}^{\infty}$ are equivalent to an orthonormal set.*

Since $\{z_m, \overline{z_m}\}_{m=1}^{\infty}$ is an interpolating sequence $\{\sqrt{1-|z_m|^2}K_{z_m}, \sqrt{1-|z_m|^2}K_{\overline{z_m}}\}_{m=1}^{\infty}$ in $H^2(\mathbf{D})$ equivalent to an orthonormal set by Theorem 5.3. Let $k_m = \sqrt{1-|z_m|^2}K_{z_m} - \sqrt{1-|z_m|^2}K_{\overline{z_m}}$. Then $\{k_m\}_{m=1}^{\infty}$ is equivalent to an orthonormal set using Theorem 5.4.

Let $M = \overline{\text{span}}\{k_m\}_{m=1}^{\infty}$. Consider $C_\phi^* k_m$, where $\{k_m\}_{m=1}^{\infty}$ is a basis for M,

$$\begin{aligned} C_\phi^* k_m &= C_\phi^*\left(\sqrt{1-|z_m|^2}K_{z_m} - \sqrt{1-|z_m|^2}K_{\overline{z_m}}\right) \\ &= \sqrt{1-|z_m|^2}K_{\phi(z_m)} - \sqrt{1-|z_m|^2}K_{\phi(\overline{z_m})} \\ &= \frac{\sqrt{1-|z_m|^2}}{\sqrt{1-|z_{m-1}|^2}}\left(\sqrt{1-|z_{m-1}|^2}K_{z_{m-1}} - \sqrt{1-|z_{m-1}|^2}K_{\overline{z_m}}\right) \\ &= \frac{\sqrt{1-|z_m|^2}}{\sqrt{1-|z_{m-1}|^2}}\, k_{m-1}. \end{aligned}$$

Thus, C_ϕ^* shifts the basis for M which gives the following theorem.

Theorem 5.5. *Let* $\phi(z) = \dfrac{(1-2c)z^{k+1}}{1-2cz^k}$ *with* $0 < c < \frac{1}{2}$ *and* $k \in \mathbf{N}$, *and* C_ϕ *be the associated composition operator on* $H^2(\mathbf{D})$. *Let* $k_m = \sqrt{1-|z_m|^2}K_{z_m} - \sqrt{1-|z_m|^2}K_{\overline{z_m}}$, *where* $\{z_m\}_{m=0}^{\infty}$ *is the interpolation sequence satisfying Theorem* 4.4. *Then* $M = \overline{\text{span}}\{k_m\}_{m=1}^{\infty}$ *is an invariant subspace for* C_ϕ^* *and* $C_\phi^*|_M$ *is similar to the backward weighted shift with weight sequence* $\left\{\dfrac{\sqrt{1-|z_m|^2}}{\sqrt{1-|z_{m-1}|^2}}\right\}_{m=1}^{\infty}$.

Theorem 5.6. $C_\phi^*|_M \sim e^{i\theta}C_\phi^*|_M$.

Proof. Since $C_\phi^*|_M$ is a weighted shift, theorem follows from the well-known result that a weighted shift is unitarily equivalent to rotates of itself [8, p. 52]. □

This verifies Cowen and MacCluer's conjecture for C_ϕ, where $\phi(z) = \frac{(1-2c)z^{k+1}}{1-2cz^k}$, $k \in \mathbf{N}$, and $0 < c < \frac{1}{2}$. We identify parts of the spectrum with the following result.

Theorem 5.7. *If* $|\lambda| < \frac{1}{\sqrt{a}}$, *then* λ *is an eigenvalue of* C_ϕ^*.

Proof. As before $a = \frac{k+1-2c}{1-2c}$. Suppose $|\lambda| < \frac{1}{\sqrt{a}}$ and $f_\lambda = \sum_{m=1}^\infty \lambda^m \left(\frac{1-|z_0|^2}{1-|z_m|^2}\right)^{\frac{1}{2}} k_m$, where $\{k_m\}_{m=1}^\infty$ is the basis for M.

From the Julia-Carathéodory Theorem, we have

$$a = \lim_{m\to\infty} \frac{1-|\phi(z_m)|}{1-|z_m|} = \lim_{m\to\infty} \frac{1-|z_{m-1}|}{1-|z_m|} = \lim_{m\to\infty} \frac{1-|z_{m-1}|^2}{1-|z_m|^2}.$$

Hence, there is a constant b such that $\frac{1-|z_0|^2}{1-|z_j|^2} = \prod_{i=1}^j \frac{1-|z_{i-1}|^2}{1-|z_i|^2} \le ba^j$. Now f_λ converges absolutely whenever $\sum_{m=1}^\infty |\lambda|^m a^{\frac{m}{2}}$ converges since $\{k_m\}_{m=1}^\infty$ is equivalent to an orthonormal set. Clearly, $f_\lambda \in H^2(\mathbf{D})$ whenever $|\lambda| < \frac{1}{\sqrt{a}}$. It is a routine calculation to show that $C_\phi^* f_\lambda = \lambda f_\lambda$ and $f_\lambda \neq 0$. □

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