



SOME NOTES FOR SPLINE WAVELETS ON INTERVAL

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Abstract

Wavelets on the interval are important in many applications. In this paper, we make certain observations related to the spline wavelets on the interval $[0, 1]$. In particular, we are interested in knots of splines, smoothness and stability of spline wavelets.

1. Introduction

In many applications such as numerical solutions of differential equations and image processing, wavelets on the interval are required. Meyer constructed a family of orthogonal wavelets on the interval $[0, 1]$ in [7]. Cohen et al. gave a different construction which has certain advantages over Meyer's construction in [5]. But the dual scaling spaces have no polynomial exactness. In order to avoid that disadvantage and to have \tilde{d} vanishing moments of the wavelets, Dahmen et al. gave a general method to construct scaling function space and wavelets on the interval $[0, 1]$, where the method is based on the factorization of the refinable matrix in [6]. Following the approach of Chui and Quak [3], Jia constructed a class of wavelets on the interval $[0, 1]$ which is also a stable basis in the Sobolev space $H^\mu(0, 1)$ for a certain range of μ .

2000 Mathematics Subject Classification: 42C40.

Keywords and phrases: knots, smoothness, stability, spline wavelets.

Supported by the National Natural Science Foundation of China (Grant No. 10871012) and Natural Science Foundation of Beijing (No. 1082003).

Received May 14, 2009

By the investigation of the biorthogonal wavelets in [4], Bittner constructed a new class of wavelets with Schoenberg spline as the scaling function in [2]. As usual, he still uses the wavelets in [4] as the inner wavelets and constructs three types of boundary wavelets. Since the wavelets have zero boundary values, it will be useful in the numerical solution of PDE's with homogenous boundary conditions. Unfortunately, the Riesz basis property or stability of the wavelets is still needed to prove theoretically. In this paper, we make some notes of the wavelets on the interval $[0, 1]$.

2. Biorthogonal Wavelets on \mathbb{R}

The biorthogonal wavelets on \mathbb{R} were first introduced by Cohen et al. in [4]. They choose B -spline as the scaling function, i.e.,

$$\phi^d(x) := d[0, 1, \dots, d](\cdot - x)_+^{d-1},$$

with the refinable relation $\phi^d(x) = \sqrt{2} \sum_{k=0}^d h_k \phi^d(2x - k)$. Then they construct the dual scaling function $\tilde{\phi}^{d, \tilde{d}}$ of ϕ^d with the parameter $\tilde{d} \geq d$. And $\tilde{\phi}^{d, \tilde{d}}$ is also refinable, i.e.,

$$\tilde{\phi}^{d, \tilde{d}}(x) = \sqrt{2} \sum_{k=\tilde{\ell}_1}^{\tilde{\ell}_2} \tilde{h}_k \tilde{\phi}^{d, \tilde{d}}(2x - k),$$

where $\tilde{\ell}_1 = -\tilde{d} - \lfloor d/2 \rfloor + 1$, $\tilde{\ell}_2 = \tilde{d} + \lceil d/2 \rceil - 1$. Now the wavelets are defined by

$$\psi^{d, \tilde{d}}(x) := \sqrt{2} \sum_{k=1-\tilde{\ell}_2}^{1-\tilde{\ell}_1} b_k \phi(2x - k)$$

with $b_k := (-1)^k \tilde{h}_{1-k}$.

Bittner gave a new view on biorthogonal spline wavelets in [1]. Explicitly, he shows an expression of the spline wavelets in the following theorem.

Theorem 2.1 [1]. *Let ψ be a spline wavelet with support in $[0, r]$, $r \in \mathbb{N}$ which satisfies*

$$N_d(2x - \ell) = \sum_{k \in \mathbb{Z}} (c_{\ell-2k} N_d(x - k) + d_{\ell-2k} \psi(x - k)) \quad (2.1)$$

with finitely supported sequences \mathbf{c} and \mathbf{d} . Then ψ is a spline with knots

$$t_k = \begin{cases} k, & \text{if } k = 0, \dots, s-1; \\ k - \frac{1}{2}, & \text{if } k = s; \\ k-1, & \text{if } k = s+1, \dots, r+1, \end{cases}$$

for some $s \in \mathbb{Z}$, $1 \leq s \leq r$.

This theorem tells us that all the spline wavelets with (2.1) are splines with one and only one half integer knot. But we find a fault about this theorem. Since ψ is a spline wavelet and $\psi \in V_1$, we have

$$\psi(x) = \sum_{k=0}^{2r} \alpha_k \left(x - \frac{k}{2}\right)_+^{d-1}. \quad (2.2)$$

But $N_d(x-k)$ has only integer knots, so we can write (2.1) as

$$\begin{aligned} N_d(2x-\ell) &= \sum_k \beta_k (x-k)_+^{d-1} + \sum_{k \in \mathbb{Z}} d_{\ell-2k} \sum_{n=1}^r \alpha_{2n-1} \left(x-n-k+\frac{1}{2}\right)_+^{d-1} \\ &= \sum_k \beta_k (x-k)_+^{d-1} + \sum_{k \in \mathbb{Z}} \sum_{n=1}^r d_{\ell+2(n-k)} \alpha_{2n-1} \left(x-k+\frac{1}{2}\right)_+^{d-1} \end{aligned} \quad (2.3)$$

with some coefficients β_k which depend on \mathbf{c} and \mathbf{d} . On the other hand, we know

$$N_d(2x-\ell) = \frac{2^{d-1}}{(d-1)!} \sum_{k=\ell}^{d+\ell} (-1)^{k-\ell} \binom{d}{k-\ell} \left(x - \frac{k}{2}\right)_+^{d-1}. \quad (2.4)$$

By comparison of coefficients in (2.3) and (2.4) with $\ell = 0$, Bittner concludes that

$$\frac{2^{d-1}}{(d-1)!} \binom{d}{k} (-1)^k = \sum_{n=1}^r d_{2n-k-1} \alpha_{2n-1} \quad (2.5)$$

for all $k \in \mathbb{Z}$. In fact, (2.5) holds only for k that is odd.

Furthermore, we can take semi-orthogonal wavelets as an example. Let

$$\Psi(x) := \sum_{k=0}^{2m-2} (-1)^k N_{2m}(k+1) N_{2m}(2x-k).$$

Then the semi-orthogonal wavelet is given as $\psi(x) := \Psi^{(m)}(x)/2^{2m-1}$. Since

$$N_m(x) = \frac{1}{(m-1)!} \sum_{k=0}^m (-1)^k \binom{m}{k} (x-k)_+^{m-1},$$

we have

$$\begin{aligned} \Psi(x) &= \frac{1}{(2m-1)!} \sum_{k=0}^{2m-2} (-1)^k N_{2m}(k+1) \sum_{\ell=0}^{2m} (-1)^\ell \binom{2m}{\ell} (2x-k-\ell)_+^{2m-1} \\ &= \frac{1}{(2m-1)!} \sum_{k=0}^{2m-2} \sum_{\ell=k}^{2m+k} (-1)^\ell N_{2m}(k+1) \binom{2m}{\ell-k} (2x-\ell)_+^{2m-1} \\ &= \frac{1}{(2m-1)!} \sum_{\ell=0}^{4m-2} \sum_{k=0}^{2m-2} (-1)^\ell N_{2m}(k+1) \binom{2m}{\ell-k} (2x-\ell)_+^{2m-1} \\ &= \frac{2^{2m-1}}{(2m-1)!} \sum_{\ell=0}^{4m-2} (-1)^\ell \left(\sum_{k=0}^{2m-2} N_{2m}(k+1) \binom{2m}{\ell-k} \right) \left(x - \frac{\ell}{2} \right)_+^{2m-1}, \end{aligned}$$

where $\binom{m}{k}$ denotes the binomial coefficient when $0 \leq k \leq m$, zero otherwise.

Since $N_{2m}(k+1) \binom{2m}{\ell-k} \geq 0$, the coefficient $c_\ell := \sum_{k=0}^{2m-2} N_{2m}(k+1) \binom{2m}{\ell-k}$ a positive number for all $0 \leq \ell \leq 4m-2$. Thus, the semi-orthogonal wavelets ψ will not have the form as Theorem 2.1 except $m = 1$.

Specially, we take $m = 2$, then

$$\begin{aligned} \psi(x) &= \frac{4}{3} \left(x_+ - 8 \left(x - \frac{1}{2} \right)_+ + 23(x-1)_+ - 32 \left(x - \frac{3}{2} \right)_+ \right. \\ &\quad \left. + 23(x-2)_+ - 8 \left(x - \frac{5}{2} \right)_+ + (x-3)_+ \right). \end{aligned}$$

Obviously, one will not find an s which satisfies the condition of Theorem 2.1.

Although this is not right for all spline wavelets, but it is right for the CDF wavelets $\psi^{d, \tilde{d}}$. That is $\psi^{d, \tilde{d}} = C\psi(x)$ with some $C \neq 0$ and

$$\begin{aligned}\psi(x) &:= B_{(0,1,\dots,n-1,n-1/2,\dots,2n-1)}^{(\tilde{d})}(x+n-1) \\ &= \gamma_1 \left(\frac{1}{2} - x\right)_+^{d-1} + \sum_{k=0}^{2n-1} \gamma_k (k-n+1-x)_+^{d-1},\end{aligned}\quad (2.6)$$

where $\gamma_{\frac{1}{2}}, \gamma_k \neq 0, n = (d + \tilde{d})/2$. By this observation, Bittner gives three types

of boundary wavelets on the interval $[0, 1]$. We will just write out the first type while one can get the other details in [2]. Let left boundary wavelets $\Psi_k^L := B_{\tau^k}^{(\tilde{d})}$,

$0 \leq k \leq n-2$ as the \tilde{d} th derivative of a B -spline of order $2n$ with knots

$$\tau^k = \begin{cases} \underbrace{(0, \dots, 0, \dots, k, k + \frac{1}{2}, k + 1, \dots, 2n-1)}_d, & k = 0, \dots, n-d-1, \\ \underbrace{(0, \dots, 0, \dots, k, k + \frac{1}{2}, k + 1, \dots, n+k)}_{n-k}, & k = n-d, \dots, n-2. \end{cases}$$

One can find $\psi_{j,k}^L$ has \tilde{d} vanish moments. Take $\Psi_j^I := \{\psi_{j,k}^{d,\tilde{d}}, n-1 \leq k \leq 2^j - n\}$ as the inner wavelets, $\Psi^R := \text{span}\{\psi_{j,2^j-k}^R(x) := \psi_{j,k}^L(1-x), 1 \leq k \leq n-1\}$ as the right boundary wavelets, then the wavelets space is $W_j = \text{span}\{\Psi_j^L, \Psi_j^I, \Psi_j^R\}$. Furthermore, take Schoenberg spline as the scaling function space V_j , then $V_{j+1} = V_j \oplus W_j$. One can find that the wavelets have zero boundary values. This ensures that it will be useful in the numerical solution of PDE's with homogenous boundary conditions.

3. Smoothness of Boundary Wavelets

From the definition of the boundary wavelets, one can find that the multiple of the knots is important for the smoothness and vanishing moment of the boundary wavelets. In this section, we discuss the smoothness of the boundary wavelets.

Lemma 3.1. *Given knots $\tau = (t_0, \dots, t_m, \dots, t_{m+l}, \dots, t_n)$, where $t_m = t_{m+1} = \dots = t_{m+l}, t_i < t_{i+1}, i \in \{0, \dots, n\} \setminus \{m, \dots, m+l-1\}$ and $l \leq n$, the divided difference*

$$[t_0, \dots, t_n]f = \begin{cases} \frac{[t_1, \dots, t_n]f - [t_0, \dots, t_{n-1}]f}{t_n - t_0}, & \text{if } t_0 < t_n; \\ \frac{f^{(n)}(t_0)}{n!}, & \text{if } t_0 = t_n, \end{cases}$$

has the form

$$[t_0, \dots, t_n]f = \sum_{k=0}^{n-l} c_k [t_k, \dots, t_{k+l}]f, \quad (3.1)$$

where $c_k \in \mathbb{R}$.

Proof. We prove it by induction. It is trivial for $n = 1$. Suppose (3.1) holds for $n = r \geq 1$, we shall show for any $l \leq r + 1$, (3.1) holds for $n = r + 1$. If $l = r + 1$, then the divided difference $[t_0, \dots, t_{r+1}]f$ itself has the form as (3.1). If $l < r + 1$, i.e., $l \leq r$, then we have

$$[t_0, \dots, t_{r+1}]f = \frac{[t_1, \dots, t_{r+1}]f - [t_0, \dots, t_r]f}{t_{r+1} - t_0}.$$

By the assumption,

$$\begin{aligned} [t_0, \dots, t_{n+1}]f &= \frac{1}{t_{n+1} - t_0} \left(\sum_{k=0}^{n-l} c_k [t_{k+1}, \dots, t_{k+l+1}]f - \sum_{k=0}^{n-l} c_k [t_k, \dots, t_{k+l}]f \right) \\ &= \sum_{k=0}^{(n+1)-l} c'_k [t_k, \dots, t_{k+1}]f. \end{aligned}$$

Theorem 3.1. Given knots τ as in Lemma 3.1, let $g(x) := [t_0, \dots, t_n]f(\cdot - x)$. If $f \in C^r(\mathbb{R})$, then $g \in C^{r-l}(\mathbb{R})$. Additionally, if $f \in C^{r+l}(\mathbb{R}/t_p)$, $t_p \in \mathbb{R}$, then $g \in C^r(\mathbb{R}/(t_m - t_p))$.

Proof. We prove this also by induction. If $l = 0$, then

$$g(x) = \sum_k^n c_k [t_k]f(\cdot - x) = \sum_k^n c_k f(t_k - x) \in C^r.$$

Suppose it holds for $l \leq l'$, we show that it also holds for $l = l' + 1$. By Lemma 3.1, we have

$$\begin{aligned}
g(x) &= \sum_{k=0}^{n-(l'+1)} c_k [t_k, \dots, t_{k+l'+1}] f(\cdot - x) \\
&= \sum_{\substack{k=0 \\ k \neq m}}^{n-(l'+1)} c_k [t_k, \dots, t_{k+l'+1}] f(\cdot - x) + c_m [t_m, \dots, t_{m+l'+1}] f(\cdot - x) \\
&= \sum_{\substack{k=0 \\ k \neq m}}^{n-(l'+1)} c_k [t_k, \dots, t_{k+l'+1}] f(\cdot - x) + c_m \frac{f^{(l'+1)}(t_m - x)}{(l' + 1)!} \\
&= A + B,
\end{aligned}$$

where

$$A = \sum_{\substack{k=0 \\ k \neq m}}^{n-(l'+1)} c_k [t_k, \dots, t_{k+l'+1}] f(\cdot - x), \quad B = c_m \frac{f^{(l'+1)}(t_m - x)}{(l' + 1)!}.$$

If $f \in C^r(\mathbb{R})$, then $B \in C^{r-l'-1}(\mathbb{R})$ and by assumption, $A \in C^{r-l'}(\mathbb{R})$. So we get $g(x) = A + B \in C^{r-(l'+1)}(\mathbb{R})$. Additionally, if $f \in C^{r+l'+1}(\mathbb{R}/t_p)$, then $B \in C^r(\mathbb{R}/(t_m - t_p))$ and by assumption $A \in C^r(\mathbb{R}/(t_m - t_p))$. So we get $g(x) = A + B \in C^r(\mathbb{R}/(t_m - t_p))$.

Corollary 3.1. *Given knots $\tau = (t_0, t_1, \dots, t_{m-1}, t_m, t_{m+1}, \dots, t_n)$, where $t_i < t_{i+1}$, and $\tau' = (t_0, t_1, \dots, t_{m-1}, \underbrace{t_m, \dots, t_m}_{l+1}, t_{m+1}, \dots, t_n)$, then $B_\tau(x) \in C^{n-2}(\mathbb{R})$ and $B_{\tau'}(x) \in C^{n+l-2}(\mathbb{R}/t_m) \cap C^{n-2}(\mathbb{R})$.*

Proof. Since $B_\tau(x) = (t_n - t_0)[t_0, \dots, t_n] f(\cdot - x)$ with $f(x) = (x)_+^{n-1} \in C^{n-2}(\mathbb{R})$, using Theorem 3.1, we obtain $B_\tau(x) \in C^{n-2}(\mathbb{R})$. Analogously, $B_{\tau'}(x) = (t_n - t_0)[t_0, \dots, t_n] f'(\cdot - x)$ with $f'(x) = (x)_+^{n+l-1} \in C^{n+2l-2}(\mathbb{R}/0) \cap C^{n+l-2}(\mathbb{R})$. Again, using Theorem 3.1, we have $B_{\tau'} \in C^{n+l-2}(\mathbb{R}/t_m) \cap C^{n-2}(\mathbb{R})$.

The corollary shows that the boundary wavelets on the zero point have less smoothness than other points. We can add more knots which are different with the existing knots in the divided difference to get more smoothness boundary wavelets.

4. Stability of Wavelets on the Interval

If the MRA is constructed by the Schoenberg spline, Bittner proves that $V_{j+1} = V_j \oplus W_j$ in [2]. Thus, we have the wavelets $\Psi_j := M_{j,1}^T \Phi_{j+1}$. But the stability still needs to prove theoretically. According to Proposition 2.5 and Proposition 2.6 in [6], we only need to construct the new wavelets by taking linear components of the old wavelets. Suppose $V_j = \text{span}\{\Phi_j\}$, where Φ_j is the Schoenberg spline. Then there exists a matrix $M_{j,0}$ such that $\Phi_j^T = \Phi_{j+1}^T M_{j,0}$. Let $\tilde{\Phi}_j = \langle \Phi_j, \Phi_j \rangle^{-1} \Phi_j$. Then Φ_j and $\tilde{\Phi}_j$ are biorthogonal scaling functions, i.e., $\langle \Phi_j, \tilde{\Phi}_j \rangle = I$. Furthermore, we have

$$\begin{aligned} \tilde{\Phi}_j^T &= \Phi_{j+1}^T M_{j,0} \langle \Phi_j, \Phi_j \rangle^{-1} \\ &= \Phi_{j+1}^T \langle \Phi_{j+1}, \Phi_{j+1} \rangle^{-1} \langle \Phi_{j+1}, \Phi_{j+1} \rangle M_{j,0} \langle \Phi_j, \Phi_j \rangle^{-1} \\ &= \tilde{\Phi}_{j+1}^T \langle \Phi_{j+1}, \Phi_{j+1} \rangle M_{j,0} \langle \Phi_j, \Phi_j \rangle^{-1}. \end{aligned}$$

Let $\tilde{M}_{j,0} := \langle \Phi_{j+1}, \Phi_{j+1} \rangle M_{j,0} \langle \Phi_j, \Phi_j \rangle^{-1}$. Then we have the refinable relation in the dual side, i.e., $\tilde{\Phi}_j^T = \tilde{\Phi}_{j+1}^T \tilde{M}_{j,0}$. Now, define

$$\Psi_j^{\text{new}} := (I - M_{j,0} \tilde{M}_{j,0}^T) \Psi_j.$$

By using Proposition 2.6, we know that the new wavelets are uniformly stable. Since the new wavelets are the linear components of the old wavelets, they still have the zero boundary values and \tilde{d} vanishing moments.

In practice, one can find the new wavelets generated by the dual $\tilde{\Phi}_j$ are not good for applications. We need to find a suitable dual such that the support of the new wavelets is locally finite. Furthermore, we know the Schoenberg spline does not have zero boundary values. And this may produce a relatively bigger error when we project the space $\{f \in L_2([0, 1]) : f(0) = f(1) = 0\}$ into V_j . Thus, it leads us that we must also change the primal scaling function Φ_j . All the two problems will be discussed in our next paper.

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