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SHIFT-INVARIANT COMPLEX CQF WAVELETS

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Abstract

This work introduces a framework for the shift-invariant complex conjugate quadrature filter (CQF) wavelet transform. The shift-invariance of the CQF wavelets is attained by a half-sample delay operator. The real and imaginary parts of the CQF wavelet sequences form Hilbert pairs, which yields analytic transform coefficients. The present method can be adapted to all existing orthogonal real-valued CQF wavelet filter banks.

1. Introduction

In many areas in research and industry, the discrete wavelet transform (DWT) [1, 7, 12, 14] has a well assisted position in processing of signals and images. One of the main difficulties in DWT analysis is the dependence of the total energy of the wavelet coefficients in different scales on the fractional shifts of the analysed signal. For example, in the time-shifted signal $x[n-\tau]$, where $\tau \in [0, 1]$, there may appear significant differences in the energy of the wavelet coefficients as a function of the

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time-shift. Kingsbury [4] suggested that a nearly shift-invariant method was the real and imaginary parts of the complex wavelet coefficients are approximately a Hilbert transform pair. The energy (absolute value) of the wavelet coefficients equals the envelope, which provides smoothness and approximate shift-invariance. Selesnick [13] observed that using two parallel DWT filter banks, which are constructed so that the impulse responses of the scaling filters have half-sample delayed versions of each other: $h_0[n]$ and $h_0[n-0.5]$, the corresponding wavelets are a Hilbert transform pair. Selesnick [13] proposed a spectral factorization method based on the half delay all-pass Thiran filters. As a disadvantage the constructed scaling filters do not have coefficient symmetry and the nonlinearity interferes with the spatial timing in different scales and prevents accurate statistical correlations. Gopinath [2] generalized the idea for N parallel filter banks, which are phase shifted versions of each other. Gopinath showed that increasing N, the shift-invariance of the wavelet transform improves. However, the greatest advantage comes from the change $N = 1$ to 2.

In our previous works, we have used computationally expensive FFT based methods for construction of the shift-invariant analytic wavelet transform [8], [9]. In this work, we show that the shift-invariance of the conjugate quadrature filter (CQF) wavelets is obtained by a half-sample delay operator. The real and imaginary parts of the CQF wavelets form a Hilbert pair, which yields analytic transform coefficients.

2. Theoretical Considerations

A. CQF wavelet filter bank

The CQF DWT filter bank consists of the $H_0(z)$ and $H_1(z)$ analysis filters and $G_0(z)$ and $G_1(z)$ synthesis filters for N odd [12],

$$\begin{aligned} H_0(z) &= (1+z^{-1})^K P(z), \\ H_1(z) &= z^{-N} H_0(-z^{-1}), \\ G_0(z) &= H_1(-z), \\ G_1(z) &= -H_0(-z), \end{aligned} \tag{1}$$

where $P(z)$ is a polynomial in z^{-1}. The filters are related via the perfect reconstruction (PR) condition

$$H_0(z)G_0(z) + H_1(z)G_1(z) = 2z^{-k},$$
$$H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0, \tag{2}$$

where $k \in N$. The tree structured implementation of the real-valued CQF filter bank is described in Figure 1. Let us denote the frequency response of the z-transform filter as

$$H(z) = \sum_n h_n z^{-n} \Rightarrow H(\omega) = \sum_n h_n e^{-j\omega n}. \tag{3}$$

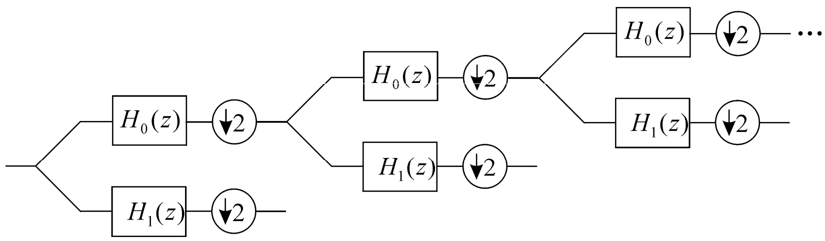

Figure 1. The real-valued CQF wavelet transform.

Correspondingly, we have the relations

$$H(-z) \Rightarrow H(\omega - \pi),$$
$$H(-z^{-1}) \Rightarrow H^*(\omega - \pi), \tag{4}$$

where * denotes complex conjugation. In M-stage CQF tree, the frequency response of the wavelet is

$$\psi_M(\omega) = H_1(\omega/2) \prod_{k=2}^{M} H_0(\omega/2^k). \tag{5}$$

B. Complex CQF filter bank

We consider the design of a phase shifted CQF filter bank consisting of the scaling filter $\overline{H}_0(z)$ and the wavelet filter $\overline{H}_1(z)$. Let us suppose that the scaling filters in parallel CQF trees are related as

$$\overline{H}_0(\omega) = e^{-j\phi(\omega)} H_0(\omega), \tag{6}$$

where $\phi(\omega)$ is a 2π periodic phase function. Then the corresponding CQF wavelet filters are related as

$$H_1(\omega) = e^{-j\omega N} H_0^*(\omega - \pi)$$

and

$$\begin{aligned}\overline{H}_1(\omega) &= e^{-j\omega N}\overline{H}_0^*(\omega - \pi)\\ &= e^{-j\omega N} e^{j\phi(\omega-\pi)} H_0^*(\omega - \pi)\\ &= e^{j\phi(\omega-\pi)} H_1(\omega).\end{aligned} \tag{7}$$

We may easily verify that the phase shifted CQF bank (6, 7) obeys the PR condition (2). Correspondingly, the frequency response of the M-stage CQF wavelet sequence is

$$\begin{aligned}\overline{\psi}_M(\omega) &= \overline{H}_1(\omega/2)\prod_{k=2}^{M}\overline{H}_0(\omega/2^k)\\ &= e^{j\phi(\omega/2-\pi)} H_1(\omega/2)\prod_{k=2}^{M} e^{-j\phi(\omega/2^k)} H_0(\omega/2^k)\\ &= e^{j\phi(\omega/2-\pi)} H_1(\omega/2)\prod_{k=2}^{M} H_0(\omega/2^k)\, e^{-j\sum_{k=2}^{M}\phi(\omega/2^k)}\\ &= e^{j\theta}\psi_M(\omega),\end{aligned} \tag{8}$$

where the phase function is

$$\theta = \phi(\omega/2 - \pi) - \sum_{k=2}^{M}\phi(\omega/2^k). \tag{9}$$

The wavelet sequences yielded by the CQF bank (1) and the phase shifted CQF bank (6) and (7) can be interpreted as real and imaginary parts of the complex wavelet

$$\psi_{MC}(\omega) = \psi_M(\omega) + j\overline{\psi}_M(\omega). \tag{10}$$

The requirement for the shift-invariance comes from

$$\overline{\psi}_M(\omega) = \mathcal{H}\{\psi_M(\omega)\}, \tag{11}$$

where $\mathcal{H}$ denotes the Hilbert transform. The frequency response of the Hilbert transform operator is defined as

$$\mathcal{H}(\omega) = -j\,\mathrm{sgn}(\omega), \tag{12}$$

where the sign function is defined as

$$\mathrm{sgn}(\omega) = \begin{cases} 1 & \text{for } \omega \geq 0, \\ -1 & \text{for } \omega < 0. \end{cases} \tag{13}$$

In this work, we apply the Hilbert transform operator in the form

$$\mathcal{H}(\omega) = e^{-j\pi/2}\,\mathrm{sgn}(\omega). \tag{14}$$

If we select the phase function $\phi(\omega)$ in (9) as

$$\phi(\omega) = \omega/2, \tag{15}$$

then the scaling filters (6) are half-sample delayed versions of each other. By inserting (15) in (9), we have

$$\theta = \frac{\omega/2 - \pi}{2} - \omega\sum_{k=2}^{M}\frac{1}{2^{k+1}} = -\frac{\pi}{2} + \frac{\omega}{2^{M+1}}. \tag{16}$$

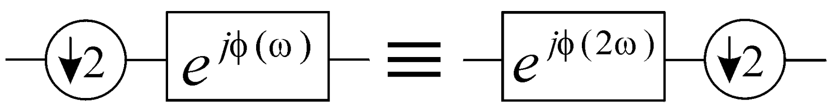

Figure 2. Two equivalents for realization of the phase function.

To evaluate the frequency response of the phase shifted CQF tree we move the phase functions in front of the CQF tree using the equivalence described in Figure 2. All the phase functions are reduced to a single phase function

$$\Phi(\omega) = \phi(2^{M-1}\omega - \pi) - \sum_{k=1}^{M-1}\phi(2^{k-1}\omega) \tag{17}$$

and the parallel M-stage wavelet sequences are related as

$$\overline{\psi}_M(\omega) = e^{j\Phi(\omega)}\psi_M(\omega). \tag{18}$$

If we suppose that the scaling filters (6) of the parallel trees are related via the half-sample delay relation (15), then the phase function (17) can be written as

$$\Phi(\omega) = \omega/2 - \pi/2. \tag{19}$$

We may note that the result does not depend on the number of stages M in the CQF tree. The elimination of the first term $\omega/2$ in the phase function can be made by prefiltering the analyzed signal by the half-sample delay operator $D(z) = z^{-1/2}$, which has the frequency response $D(\omega) = e^{-j\omega/2}$. The total phase function is then for $-\pi \le \omega \le \pi$,

$$\Phi(\omega) + \angle D(\omega) = -\pi/2 \tag{20}$$

which warrants that M-stage wavelets are Hilbert transform pairs.

C. Design of the half-delay filter

Our approach is to construct a half-delaying prefilter $D(z) = z^{-1/2}$, which has a precisely linear phase. We have previously described the fractional delay (FD) filter based on the *B*-spline transform interpolation and decimation procedure for implementation of the fractional delays $\tau = N/M \ \ (N, M \in \mathbb{Z})$ [6]. The FD filter has the following representation

$$D(N, M, z) = \frac{1}{\beta_p(z)}[z^{-N}\beta_p(z)F(z)]_{\downarrow M}, \tag{21}$$

where $\beta_p(z)$ is the discrete *B*-spline and polynomial

$$F(z) = \frac{1}{M^{p-1}}\frac{1-z^{-M}}{1-z^{-1}} = \frac{1}{M^{p-1}}\sum_{k=0}^{M-1} z^{-k}. \tag{22}$$

The half-sample delay operator $D(z) = z^{-1/2}$ is yielded by inserting $N = 1$ and $M = 2$ in (21). We have

$$D(z) = D(1, 2, z) = \frac{R_p(z)}{\beta_p(z)}, \tag{23}$$

where $R_p(z)$ is a polynomial in z^{-1}. For the discrete *B*-spline order $p = 4$, we obtain

$$\beta_4(z) = \frac{1 + 4z^{-1} + z^{-2}}{6} \tag{24}$$

and

$$R_4(z) = \frac{1 + 23z^{-1} + 23z^{-2} + z^{-3}}{48}. \tag{25}$$

The phase of the half-delay operator is exactly linear in the frequency range $-\pi \leq \omega \leq \pi$.

D. Complex CQF wavelets

In many real-time applications, the IIR-type prefilter (24) is difficult to implement, since one of the roots of the denominator is outside the unit circle and the computation needs the time-reversed convolution procedure [6]. The key idea in this work is based on the fact that only the relative time-shift of the wavelet coefficients is essential for shift-invariance. Hence, the half-sample shift operator can be divided between the real and imaginary parts of the wavelet sequences via the prefilters $\beta_p(z)$ and $R_p(z)$,

$$\psi_{MC}(z) = [\beta_p(z) + jR_p(z)]\psi_M(z). \tag{26}$$

This warrants that the real and imaginary parts of the M-stage wavelets are Hilbert transform pairs.

The present result implies a considerable simplification in the computation of the shift-invariant CQF wavelet transform. Instead of the design of the two parallel CQF trees, we only need to design one real-valued CQF wavelet tree. The real and imaginary parts of the complex shift-invariant CQF wavelets are then obtained by the prefilters (26) in front of the real-valued CQF tree.

The theoretical considerations in previous section are valid only if the phase function $\phi(\omega)$ obeys the half-sample delay relation (15) in the frequency range $-\pi \leq \omega \leq \pi$. It appears that for any number of vanishing moments K in (1), the scaling filters $H_0(z)$ and $\overline{H}_0(z)$ yielded by the spectral factorization method [13] do not have an exactly linear phase relationship. On the other hand, the equivalence in Figure 2 cannot be used for deduction of the half-delay prefilter relation (19) for such filters.

3. Discussion

In this work, we analysed the shift-invariance property of the M-stage CQF wavelets. The key observation is that if the scaling filters are half-sample delayed versions of each other, the corresponding wavelets are not precisely Hilbert transform pairs, but there occurs a phase error (16). The error term depends on the frequency and the number of stages *M*. In previous works, the half-sample relation has been deduced from the infinite product formula for wavelet bases [13]. However, in many applications such as image processing and multi-scale analysis, the number of stages in the CQF tree is limited and we should consider the shift-invariance of the *M*-stage wavelet sequences.

We introduced a new way to analyse the phase dependence of the CQF tree by using the equivalence described in Figure 2. By moving the phase functions in front of the tree structure a surprisingly simple phase condition (19) was observed. The result suggests that to obtain the Hilbert transform relation the phase error can be compensated by prefiltering the signal by a single half-sample delay operator $D(z) = z^{-1/2}$. Many types of fractional delay filters are suitable for that purpose [3, 5, 10, 11, 15]. In data acquisition devices, the half-sample delayed signal can be obtained by increasing the sampling rate by two.

In this work, the half-sample delaying prefilter $D(z)$ was constructed by the *B*-spline transform interpolation and decimation procedure [6]. The method yields a half-sample delay filter, which has a precisely linear phase. Some competing FD design methods, such as Thiran filters and Taylor series expansions produce phase distortion [5]. To avoid the implementation of the IIR-type prefilter, we divided the half-sample delay operator into two FIR-type prefilters (26). This idea yields shift-invariant complex wavelets, whose real and imaginary parts are Hilbert pairs. In VLSI applications, the need of the design of only one real-valued CQF tree means a significant advantage.

In this work, we applied a single scaling function $H_0(z)$ for the construction of the complex CQF tree. However, the present results can be generalized to include the CQF trees consisting of a sequence of different scaling filters and the corresponding wavelet filters obeying the PR condition (2). For example, we may apply the time-reversed CQF filter bank having the impulse response

$$\tilde{h}_0[n] = h_0[N - n]. \tag{27}$$

In *z*-transform domain, we have

$$\begin{aligned} \tilde{H}_0(z) &= H_1(-z), \\ \tilde{H}_1(z) &= H_0(-z) \end{aligned} \tag{28}$$

which are the quadrature mirror filters (QMFs) in respect to the original CQF bank. A drawback in complex CQF filter banks comes from the asymmetric energy distribution of the wavelets (skewed envelope), which may arise unwanted blurring in image processing and destroys the precise statistical processing of signals in multi-scale analysis [9]. By altering the scaling functions in the CQF tree the nonlinear phase effects can be highly reduced.

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