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BOUNDARY LAYER THEORY IN TWO-DIMENSIONAL LAMINAR MOTION WITH CIRCULAR HYDRAULIC JUMP

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Abstract

Watson [18] used boundary layer theory and a constant velocity profile to study the radial spread of a liquid jet over a horizontal plane. In this work, we also use boundary layer approach but with the incorporation of Lamb's velocity profile to study two-dimensional motion of laminar flow of a liquid with circular hydraulic jump. Based on this model, a new relation is obtained for the following parameters: displacement thickness, momentum thickness and the position of the jump. Comparison of the approximate values obtained based on Karman-Pohlhausen [14] method with the exact values due to Blasius [2] shows that the error in the shear rate relation on the plane is only about 1.5% while the error in the thickness ratio is about 2.6%. These percentages, which are smaller than those of Watson [18], show remarkable improvement of our work upon that of Watson. Besides, our values also compare rather favourably with the exact values [2]. Finally, our analysis reveals that the depth of the fluid on the plane is dependent on the velocity profile used.

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1. Introduction

When a vertical jet is directed upon a horizontal plate, it spreads out radially in a thin layer bounded by a circular hydraulic jump, on the inside where the depth is smaller and on the outside where the depth is greater (Watson [18]). Watson [18] studied the motion in the thin layer by means of boundary layer theory but with the inclusion of a constant velocity profile. He observed that the depth of the flow was much greater on the outside of the jump than on the inside, and hence the condition at the jump might be simplified. The formation of the thin layer and the circular jump was first noticed by Rayleigh [17] who derived the properties of bores and jumps.

Bohr et al. [3] extended the work on circular hydraulic jump using the shallow water approach. They showed that circular hydraulic jump could be qualitatively understood using simplified equations of the shallow water type which included viscosity. They also concluded that it was not possible to determine the position of the jump from ideal theory. Craik et al. [5] also contributed to the study of circular hydraulic jump using experimental approach. They described new observations of this phenomenon. In addition, they examined a previously unreported instability of the jump and showed this to arise when the local Reynolds number R_j just ahead of the jump exceeded a critical value of 147. Other contributors include, notably, Belanger [1], Bouhadef [4], Felice [6], Felice and Francesco [7], Glauert [8], Groves [9], Huguera [10], Kundu and Cohen [11], Lamb [12], Olsson and Turkdogan [13], Rainville and Pinkel [15], Rajput [16], etc.

In this paper, we discuss circular hydraulic jump by means of boundary layer theory but with the inclusion of a velocity profile due to Lamb [12]. Our results agree reasonably with the exact values. Our work shows an improvement upon that of Watson. For convenience, we consider the following regions of flow:

i. The region $0 < x < x_0$. Here the speed at the edge of the boundary layer is taken as the constant U_0. When $x < x_0$, we have $\delta < h$ and $U(x) = U_0$. Here also an approximation to the Blasius type of solution will be derived.

ii. The region $x > x_0$. Here, there is a similarity resolution with $\delta = h$ and $U(x) < U_0$.

iii. The region $x = x_0$. Here $\delta = h$ and the whole flow is of the boundary layer type.

Note that x_0 is given by the condition $\delta = h$, so that the whole flow passes through the boundary layer.

2. Ideal Theory

The treatment of the problem of two-dimensional flow here applies only to laminar flow in which viscosity is completely neglected. The flow here might be realized by a two-dimensional jet striking a horizontal plane, or by the flow of water under a sluice gate, Glauert [8]. If the flow was realized physically by one of the methods above, U_0 would be the speed of the impinging jet, or the speed attained by the flow under the sluice gate a short distance downstream of the gate.

The ideal or inviscid flow has the uniform depth, a, given by

$$a = \frac{Q}{u_0}. \tag{2.1}$$

The characteristic Reynolds number is

$$R = \frac{U_0 a}{\upsilon} = \frac{Q}{\upsilon}. \tag{2.2}$$

Here Q is the volume flux and υ is the kinematic viscosity. The condition to be applied at the jump (Belanger [1]) is that the thrust of the pressure is equal to the rate at which momentum is destroyed. If d is the depth outside the jump and h is the depth inside it, then the thrust of the pressure per unit length of wave (jump) is

$$\frac{1}{2}\rho g(d^2 - h^2),$$

where ρ is the density and g is the gravitational acceleration.

The speed of flow inside the jump is U_0 and outside it is U_1, where

$$U_1 = \frac{Q}{d}. \tag{2.3}$$

The rate of destruction of momentum per unit length of wave is therefore

$$\rho(U_0^2 h - U_1^2 d).$$

Thus

$$\frac{1}{2}\rho g(d^2 - h^2) = \rho(U_0^2 h - U_1^2 d). \tag{2.4}$$

Using (2.1) and (2.3) in (2.4) leads to

$$\frac{1}{2} g(d^2 - h^2) = Q^2\left[\frac{h}{a^2} - \frac{1}{d}\right]. \tag{2.5}$$

When $h << d$, (2.5) reduces to (by neglecting h^2 and $\frac{1}{d}$)

$$\frac{1}{2} gd^2 = \frac{Q^2 h}{a^2} \tag{2.6}$$

which becomes, when $h = a$,

$$\frac{Qd^2}{2} = \frac{Q^2}{a}. \tag{2.7}$$

A better approximation is to neglect only $\frac{h}{d}$ in (2.5), so that the pressure thrust inside the wave is ignored but the momentum outside it is included. Thus, from (2.5), we get

$$\frac{1}{2} g\left(1 - \frac{h^2}{d^2}\right) = \frac{Q^2}{a^2}\left(\frac{h}{d^2} - \frac{1}{d}\right). \tag{2.8}$$

Neglecting $\frac{h^2}{d^2}$ and setting $h = a$ in (2.8) and then using (2.7) yield after multiplying the resulting equation by $\frac{ad^2}{Q^2}$,

$$\frac{Qad^2}{2Q^2}\left(1 + \frac{a}{d}\right) = 1. \tag{2.9}$$

Since $h << d$, (2.5) becomes (by neglecting h^2)

$$\frac{g}{2} = \frac{Q^2 h}{a^2 d^2} - \frac{Q^2}{d^3}. \tag{2.10}$$

Putting $h = a$ in (2.10) and multiplying the result by $\frac{ad^2}{Q^2}$ leads to

$$\frac{gad^2}{2Q^2} + \frac{a}{d} = 1. \tag{2.11}$$

Thus, when the depth *h* is regarded as constant and equal to *a*, the ideal or inviscid theory, identical with the theory of Rayleigh [17], leaves the position of the jump indeterminate as in [3] but gives the result (2.9), or if the pressure thrust ahead of the wave is neglected, it leads to (2.11).

3. Blasius Solution of Two-dimensional Laminar Boundary Layer Equations

Let *x*, *y* be the rectangular coordinates with *y* vertically upwards and *u*, *v* the corresponding velocity components, then the equations for laminar flow are

$$\frac{\partial u}{\partial x} + \frac{dv}{dy} = 0, \tag{3.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \upsilon\frac{\partial^2 u}{\partial y^2}, \tag{3.2}$$

$$u = v = 0 \quad \text{at} \quad y = 0, \tag{3.3}$$

$$\frac{\partial u}{\partial y} = 0 \quad \text{at} \quad y = h(x), \tag{3.4}$$

$$\int_0^{h(x)} u dy = Q, \tag{3.5}$$

where *Q* is the volume flux in the positive *x*-direction. The total flow from the two-dimensional jet would be 2*Q*, and the flow under the sluice gate is *Q*.

A solution of these equations can be found based on Blasius type of velocity profile given by

$$u = U_0 f_1'(\eta), \qquad \eta = \left[\frac{U_0}{\upsilon x}\right]^{\frac{1}{2}} y, \tag{3.6}$$

and

$$f_1 f_1'' + 2 f_1''' = 0 \qquad (0 < \eta < \infty) \tag{3.7}$$

is the Blasius equation with boundary conditions

$$\left.\begin{aligned} f_1 = f_1' = 0 \quad & \text{at} \quad \eta = 0 \\ f_1' = 1 \quad & \text{at} \quad \eta = \infty \end{aligned}\right\}. \tag{3.8}$$

[Here $'$ denotes differentiation with respect to η.]

Thus the velocity distribution has the Blasius flat-plate profile, and the boundary layer thickness is $\left[\frac{\upsilon x}{U_0}\right]$. Before considering the approximate solution, we shall first find the exact solution of the boundary layer equations (3.1)-(3.5).

Thus following Blasius [2], it follows that

$$\text{(i)} \left(\frac{\upsilon x}{U_0}\right)^{\frac{1}{2}} \frac{1}{U_0} \left(\frac{\partial u}{\partial y}\right)_{y=0} = f''(0) = 0.332. \tag{3.9}$$

(ii) The displacement thickness δ_1 is

$$\delta_1 = 1.7208 \left(\frac{\upsilon x}{U_0}\right)^{\frac{1}{2}}. \tag{3.10}$$

(iii) The momentum thickness δ_2 is

$$\delta_2 = 0.664 \left(\frac{\upsilon x}{U_0}\right)^{\frac{1}{2}}. \tag{3.11}$$

From (3.10) and (3.11), we find

$$\delta_1 = 2.5915 \delta_2 \tag{3.12}$$

so that the thickness ratio H^* becomes

$$H^* = \frac{\delta_1}{\delta_2} = 2.5915. \tag{3.13}$$

Thus the value of $f_1''(0)$ in (3.9) and the value of H^* in (3.13) constitute the exact solution.

4. Similarity Solution of the Boundary Layer Equations (3.1)-(3.2)

In this section, a similarity solution will be derived by direct assumption of the velocity profile due to Lamb [12] given by

$$u = U(x)\, f(\eta), \qquad \eta = \frac{y}{h(x)}, \tag{4.la}$$

where

$$f(\eta) = \sin\left(\frac{\pi}{2}\eta\right) \tag{4.1b}$$

is the similarity-profile function.

When the boundary layer finally absorbs the whole flow, the velocity profile changes as x increases from the Blasius profile (3.6) to the similarity profile (Lamb's profile) (4.la)-(4.lb). Here $U(x)$ is the speed at the free surface $y = h(x)$ and $h(x)$ is the depth of the fluid on the plane. Using the boundary conditions (3.3) and (3.4), we find

$$\left.\begin{aligned} f(0) = \sin(0) = 0, &\qquad y = 0 \\ f(1) = \sin\frac{\pi}{2} = 1, &\qquad y = h(x) \\ f'(1) = \frac{\pi}{2}\cos\frac{\pi}{2} = 0, &\qquad y = h(x) \end{aligned}\right\}. \tag{4.2}$$

From (3.5),

$$Q = Uh\int_0^1 \sin\left(\frac{\pi}{2}\eta\right) d\eta. \tag{4.3}$$

Thus Uh is a constant, and (3.1) then leads to

$$\upsilon = Uh'\eta\, f(\eta) \tag{4.4}$$

or

$$\upsilon = Uh'\eta \sin\left(\frac{\pi}{2}\eta\right). \tag{4.5}$$

Using (4.la) and (4.4) equation of motion (3.2) reduces to

$$\upsilon f'''(\eta) = h^2 U' f'^2(\eta), \tag{4.6}$$

i.e.,

$$\upsilon\left[-\frac{\pi^2}{4}\sin\left(\frac{\pi}{2}\eta\right)\right] = h^2 U'^2 \sin^2\left(\frac{\pi}{2}\eta\right),$$

i.e.,

$$\upsilon\left[-\sin\left(\frac{\pi}{2}\eta\right)\right] = \frac{4}{\pi^2} h^2 U' \sin^2\left(\frac{\pi}{2}\eta\right) \tag{4.7}$$

from which it follows that $\frac{4h^2U'}{\pi^2}$ is a constant.

Also, $-\sin\left(\frac{\pi}{2}\eta\right) \le 0$, since the shearing stress $\tau = \mu\frac{\partial u}{\partial y}$ is greatest at the plate. Thus, it is convenient to write

$$\frac{4}{\pi^2} h^2 U' = \frac{-3}{2}\alpha^2\upsilon, \tag{4.8}$$

where α is a number. Using (4.8) in (4.7), we find

$$\upsilon\left[-\sin\left(\frac{\pi}{2}\eta\right)\right] = \frac{-3}{2}\alpha^2\upsilon\sin^2\left(\frac{\pi}{2}\eta\right)$$

or

$$2\left[-\sin\left(\frac{\pi}{2}\eta\right)\right] = -3\alpha^2\sin^2\left(\frac{\pi}{2}\eta\right). \tag{4.9}$$

Multiplying (4.9) by $f' = \frac{\pi}{2}\cos\left(\frac{\pi}{2}\eta\right)$, we find

$$2\left[-\sin\left(\frac{\pi}{2}\eta\right)\right]\cdot\frac{\pi}{2}\cos\left(\frac{\pi}{2}\eta\right) = -3\alpha^2\sin^2\left(\frac{\pi}{2}\eta\right)\cdot\frac{\pi}{2}\cos\left(\frac{\pi}{2}\eta\right)$$

or

$$\frac{\partial}{\partial\eta}\left[\cos^2\left(\frac{\pi}{2}\eta\right)\right] = -\alpha\frac{\partial}{\partial\eta}\left[\sin^3\left(\frac{\pi}{2}\eta\right)\right]. \tag{4.10}$$

Integrating (4.10), we have

$$\cos^2\left(\frac{\pi}{2}\eta\right) = -\alpha\sin^3\left(\frac{\pi}{2}\eta\right) + A \qquad (A = \text{const}). \tag{4.11}$$

Using the last two conditions of (4.2) in (4.11), we have

$$A = \alpha^2 \tag{4.12}$$

so that (4.11) becomes

$$\cos^2\left(\frac{\pi}{2}\eta\right) = \alpha^2\left[1 - \sin^3\left(\frac{\pi}{2}\eta\right)\right]. \tag{4.13}$$

Since $\cos\left(\frac{\pi}{2}\eta\right) \geq 0$, we have from (4.13),

$$\alpha = \frac{\cos\left(\frac{\pi}{2}\eta\right)}{\left[1 - \sin^3\left(\frac{\pi}{2}\eta\right)\right]^{\frac{1}{2}}}. \tag{4.14}$$

Let $t' = \sin\left(\frac{\pi}{2}\eta\right)$. Then $\frac{dt'}{d\eta} = \frac{\pi}{2}\cos\left(\frac{\pi}{2}\eta\right)$.

Substituting these in (4.14), we find

$$\alpha = \frac{2}{\pi}\frac{dt'}{d\eta}[1 - t'^3]^{-\frac{1}{2}}. \tag{4.14$'$}$$

Separating variables and integrating (4.14)′ becomes

$$\alpha\eta = \frac{2}{\pi}\int_0^{\sin\left(\frac{\pi}{2}\eta\right)}[1 - t'^3]^{-\frac{1}{2}}\,dt'. \tag{4.15}$$

Applying the condition $f(1) = 1$ of (4.2), we find

$$\alpha = \frac{2}{\pi}\int_0^1 [1 - t'^3]^{-\frac{1}{2}}\,dt'. \tag{4.16}$$

Using change of variables $t'^3 = s$, (4.16) becomes

$$\alpha = \frac{2}{\pi}\cdot\frac{1}{3}\int_0^1 s^{-\frac{2}{3}}(1 - s)^{-\frac{1}{2}}\,ds. \tag{4.17}$$

Equation (4.17) is a well known integral whose solution is written in the form

$$\alpha = \frac{2}{\pi} \cdot \frac{1}{3} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{5}{6}\right)}, \tag{4.18}$$

where $\Gamma(n)$ is the gamma function.

Now, (4.14) can also be written as

$$\alpha = \frac{\frac{2}{\pi} \cdot \frac{d}{d\eta}\left[\sin\left(\frac{\pi}{2}\eta\right)\right]}{\left[1 - \sin^3\left(\frac{\pi}{2}\eta\right)\right]^{\frac{1}{2}}}$$

or

$$\alpha = \frac{2}{\pi} \cdot \left[1 - \sin^3\left(\frac{\pi}{2}\eta\right)\right]^{-\frac{1}{2}} \cdot \frac{d}{d\eta}\left[\sin\left(\frac{\pi}{2}\eta\right)\right].$$

Multiplying both sides of this expression by $\sin\left(\frac{\pi}{2}\eta\right)$, we find

$$\alpha \sin\left(\frac{\pi}{2}\eta\right) = \frac{2}{\pi} \cdot \sin\left(\frac{\pi}{2}\eta\right) \cdot \left[1 - \sin^3\left(\frac{\pi}{2}\eta\right)\right]^{-\frac{1}{2}} \cdot \frac{d}{d\eta}\left[\sin\left(\frac{\pi}{2}\eta\right)\right]. \tag{4.18$'$}$$

Separating variables and integrating from $\eta = 0$ to 1, (4.18)′ becomes

$$\int_0^1 \sin\left(\frac{\pi}{2}\eta\right) d\eta = \frac{2}{\pi} \cdot \alpha^{-1} \int_0^1 \sin\left(\frac{\pi}{2}\eta\right) \cdot \left[1 - \sin^3\left(\frac{\pi}{2}\eta\right)\right]^{-\frac{1}{2}} \cdot d\left[\sin\left(\frac{\pi}{2}\eta\right)\right]. \tag{4.19}$$

Again, using change of variables $\sin^3\left(\frac{\pi}{2}\eta\right) = \lambda$ in the RHS of (4.19), we find

$$\int_0^1 \sin\left(\frac{\pi}{2}\eta\right) d\eta = \frac{2}{\pi} \cdot \alpha^{-1} \int_0^1 \lambda^{\frac{1}{3}} (1 - \lambda)^{-\frac{1}{2}} d(\lambda^{\frac{1}{3}}),$$

i.e.,

$$\int_0^1 \sin\left(\frac{\pi}{2}\eta\right) d\eta = \frac{2}{\pi} \cdot \frac{1}{3} \alpha^{-1} \int_0^1 \lambda^{-\frac{1}{3}} (1 - \lambda)^{-\frac{1}{2}} d\lambda. \tag{4.19$'$}$$

Simplifying the RHS of (4.19)′, we obtain

$$\int_0^1 \sin\left(\frac{\pi}{2}\eta\right)d\eta = \frac{2}{\pi}\cdot\frac{1}{3\alpha}\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{6}\right)} = \frac{2}{\pi}\cdot\frac{2\pi}{3\sqrt{3}\alpha^2} = \frac{4}{3\sqrt{3}\alpha^2}. \tag{4.20}$$

But

$$\int_0^1 \sin\left(\frac{\pi}{2}\eta\right)d\eta = \frac{2}{\pi}. \tag{4.21}$$

Therefore, from (4.20) and (4.21), we have

$$\frac{4}{3\sqrt{3}\alpha^2} = \frac{2}{\pi}$$

whence

$$\alpha = 1.09963. \tag{4.22}$$

Substituting (4.20) into (4.3), we find

$$Uh = \frac{3\sqrt{3}\alpha^2\cdot Q}{4}. \tag{4.23}$$

We now solve for $U(x)$ and $h(x)$ for the similarity solution using (4.8) and (4.23). From (4.8), we have

$$U' = \frac{-3}{2}\alpha^2\upsilon\cdot\frac{1}{h^2}\cdot\frac{\pi^2}{4}.$$

Using (4.23) in this last expression, we find after simplification

$$U' = \frac{dU}{dx} = \frac{-3}{2}\cdot\upsilon\cdot\frac{4\pi^2U^2}{27\alpha^2Q^2}. \tag{4.24}$$

Separating variables and integrating (4.24) gives

$$\frac{1}{U} = \frac{2\pi^2\upsilon x}{9\alpha^2Q^2} + \text{const.} \tag{4.25}$$

Putting the const. $= \dfrac{2\pi^2\upsilon\ell}{9\alpha^2Q^2}$ in (4.25) leads to

$$\frac{1}{U} = \frac{2\pi^2}{9\alpha^2}\frac{\upsilon(x+\ell)}{Q^2}$$

or

$$U(x) = \frac{9\alpha^2}{2\pi^2} \frac{Q^2}{\upsilon(x + \ell)}. \tag{4.26}$$

Here ℓ is an arbitrary constant whose presence merely indicates that a shift of origin is possible. Accordingly, substituting (4.26) into (4.23) and simplifying we find

$$h(x) = \frac{\pi^2}{2\sqrt{3}} \frac{\upsilon(x + \ell)}{Q}. \tag{4.27}$$

Equation (4.26) shows that $U(x)$ varies inversely as x while (4.27) shows that $h(x)$ grows in direct proportion to x. This is possible within the vicinity of the jump.

5. Approximate Solution using Karman-Pohlhausen Method with Lamb's Profile

Karman-Pohlhausen momentum integral equation for two-dimensional laminar flow (see [13]) is given by

$$\frac{\partial}{\partial x} \int_0^\delta (U_0 u - u^2)\, dy = \upsilon \left(\frac{\partial u}{\partial y} \right)_{y=0} \tag{5.1}$$

with Lamb's velocity profile (4.1a)-(4.1b) becoming

$$u = U_0 \sin\left(\frac{\pi}{2} \eta \right), \qquad \eta = \frac{y}{\delta(x)}. \tag{5.2}$$

Here δ is the boundary-layer thickness. From (5.2), we find

$$\left(\frac{\partial u}{\partial y} \right)_{y=0} = U_0 \frac{\pi}{2\delta(x)} \cos(0) = U_0 \frac{\pi}{2\delta(x)}. \tag{5.3}$$

From (5.1),

$$\int_0^\delta U_0 u dy = U_0^2 \delta(x) \int_0^1 \sin\left(\frac{\pi}{2} \eta \right) d\eta = U_0^2 \delta(x) \cdot \frac{4}{3\sqrt{3}\alpha^2} \quad \text{(using (4.20))} \tag{5.4}$$

and

$$\int_0^\delta u^2 dy = U_0^2 \delta(x) \int_0^1 \sin^2\left(\frac{\pi}{2} \eta \right) d\eta = \frac{U_0^2 \delta(x)}{2}. \tag{5.5}$$

Substituting (5.3), (5.4) and (5.5) into the momentum integral equation (5.1), we obtain

$$\left(\frac{8-3\sqrt{3}\alpha^2}{6\sqrt{3}\alpha^2}\right)U_0^2\,\frac{d}{dx}\,\delta(x) = \upsilon U_0\cdot\frac{\pi}{2\delta(x)}. \tag{5.6}$$

Integrating (5.6) and simplifying, we find

$$[\delta(x)]^2 = \frac{6\sqrt{3}\alpha^2\pi}{(8-3\sqrt{3}\alpha^2)}\cdot\frac{\upsilon x}{U_0} + C, \tag{5.7}$$

where C is a constant. If (5.7) were to remain valid as $x \to 0$, then $C = 0$; or when $x >> a$ (where a is the uniform depth of the ideal or inviscid flow), then C could be neglected.

Consequently, when $a << x < x_0$,

$$[\delta(x)]^2 = \frac{6\sqrt{3}\alpha^2\pi}{(8-3\sqrt{3}\alpha^2)}\cdot\frac{\upsilon x}{U_0} \tag{5.8}$$

or

$$[\delta(x)]^2 = \frac{6\sqrt{3}\alpha^2\pi}{(8-3\sqrt{3}\alpha^2)}\cdot\frac{\upsilon x a}{Q} \quad \text{(using (2.1))}. \tag{5.9}$$

From (5.8),

$$\delta(x) = \left[\frac{\upsilon x}{U_0}\right]^{\frac{1}{2}}\left[\frac{6\sqrt{3}\alpha^2\pi}{8-3\sqrt{3}\alpha^2}\right]^{\frac{1}{2}}. \tag{5.10}$$

Hence

$$\frac{d\delta}{dx} = \frac{1}{2}\left[\frac{\upsilon x}{U_0}\right]^{-\frac{1}{2}}\frac{\upsilon}{U_0}\left[\frac{6\sqrt{3}\alpha^2\pi}{8-3\sqrt{3}\alpha^2}\right]^{\frac{1}{2}}. \tag{5.11}$$

Comparing (5.11) and (5.6), we have

$$\upsilon U_0\cdot\frac{\pi}{2\delta(x)}\cdot\frac{1}{U_0^2}\left[\frac{6\sqrt{3}\alpha^2}{8-3\sqrt{3}\alpha^2}\right] = \frac{1}{2}\left[\frac{\upsilon x}{U_0}\right]^{-\frac{1}{2}}\frac{\upsilon}{U_0}\left[\frac{6\sqrt{3}\alpha^2\pi}{8-3\sqrt{3}\alpha^2}\right]^{\frac{1}{2}},$$

which on simplification yields

$$\left[\frac{\upsilon x}{U_0}\right]^{\frac{1}{2}} \frac{1}{U_0}\left[\frac{\partial u}{\partial y}\right]_{y=0} = \frac{\pi}{2}\left[\frac{8-3\sqrt{3}\alpha^2}{6\sqrt{3}\alpha^2\pi}\right]^{\frac{1}{2}} = f''(0) \tag{5.12}$$

(by virtue of (3.9)).

Substituting (4.22) into (5.12), we find

$$\left[\frac{\upsilon x}{U_0}\right]^{\frac{1}{2}} \frac{1}{U_0}\left[\frac{\partial u}{\partial y}\right]_{y=0} = f''(0) = 0.327. \tag{5.13}$$

Substituting Lamb’s velocity profile (5.2) into the displacement thickness

$$\delta_1 = \int_0^{\delta}\left[1-\frac{u}{U_0}\right]dy, \tag{5.14}$$

we find

$$\delta_1 = \delta\int_0^1\left[1-\sin\left(\frac{\pi}{2}\eta\right)\right]d\eta = \delta\left[1-\frac{4}{3\sqrt{3}\alpha^2}\right]$$

or

$$\delta_1 = \delta\left[\frac{3\sqrt{3}\alpha^2-4}{3\sqrt{3}\alpha^2}\right]. \tag{5.15}$$

Similarly, substituting (5.2) into the momentum thickness

$$\delta_2 = \int_0^{\delta}\frac{u}{U_0}\left[1-\frac{u}{U_0}\right]dy, \tag{5.16}$$

we have

$$\delta_2 = \delta\int_0^1\left[\sin\left(\frac{\pi}{2}\eta\right)-\sin^2\left(\frac{\pi}{2}\eta\right)\right]d\eta = \delta\left[\frac{4}{3\sqrt{3}\alpha^2}-\frac{1}{2}\right]$$

or

$$\delta_2 = \delta\left[\frac{8-3\sqrt{3}\alpha^2}{6\sqrt{3}\alpha^2}\right]. \tag{5.17}$$

Consequently, the thickness ratio H^* is

$$H^* = \frac{\delta_1}{\delta_2} = \frac{6\sqrt{3}\alpha^2 - 8}{8 - 3\sqrt{3}\alpha^2} = 2.6596. \tag{5.18}$$

Comparison of the approximate values (5.13) and (5.18) with the accurate values (3.9) and (3.13) shows that the error in the shear rate relation $f''(0)$ is about 1.5% while the error in the thickness ratio H^* is about 2.6%. The boundary layer just absorbs the whole flow when $x = x_0$.

When $x < x_0$, the total depth *h* is given by the volume flux condition

$$U_0\delta\int_0^1 \sin\left(\frac{\pi}{2}\eta\right)d\eta + U_0(h - \delta) = Q. \tag{5.19}$$

Substituting (4.21) and (2.1) into (5.19) and using the condition $\delta = h$, we have after simplification

$$h = \frac{\pi a}{2}, \tag{5.20}$$

where a is the jet radius. Since $\delta = h$ when $x = x_0$, then using this condition together with (5.20) and (2.2) in (5.10), we obtain

$$x_0 = \frac{[8 - 3\sqrt{3}\alpha^2]\pi a R}{24\sqrt{3}\alpha^2}, \tag{5.21}$$

where *R* given by (2.2) is the characteristic Reynolds number. Now the value of ℓ in (4.26) and (4.27) can be estimated as follows. Using $x = x_0$, $U(x) = U_0$ when $\delta = h$ in (4.26), we find

$$U_0 = \frac{9\alpha^2}{2\pi^2}\frac{Q^2}{\upsilon(x_0 + \ell)} \tag{5.22}$$

which gives, on solving for ℓ,

$$\ell = \frac{9\alpha^2}{U_0}\frac{Q^2}{2\pi^2\upsilon} - x_0. \tag{5.23}$$

Substituting (5.21) into (5.23) and using (2.1) and (2.2), we get after simplification

$$\ell = \frac{[216\sqrt{3}\alpha^4 - (16 - 6\sqrt{3}\alpha^2)\pi^3]aR}{48\sqrt{3}\alpha^2\pi^2}. \tag{5.24}$$

6. Jump Condition

The position $x = x_1$ of the hydraulic jump is determined by equating the rate of loss of momentum to the thrust of the pressure. The condition of the momentum is thus

$$\frac{1}{2}\rho g d^2 = \rho\int_0^h u^2 dy - \rho U_1^2 d \tag{6.1}$$

or using (2.3),

$$\frac{1}{2} g d^2 + \frac{Q^2}{d} = \int_0^h u^2 dy. \tag{6.2}$$

Equation (6.2) is the jump condition. We note that for the case $x_1 < x_0$ (see [18]),

$$\int_0^h u^2 dy = U_0^2\delta\int_0^1 \sin^2\left(\frac{\pi}{2}\eta\right)d\eta + U_0^2(h - \delta). \tag{6.3}$$

Substituting (6.3) into (6.2), the jump condition for the case $x_1 < x_0$ takes the form

$$\frac{1}{2} g d^2 + \frac{Q^2}{d} = U_0^2\delta\int_0^1 \sin^2\left(\frac{\pi}{2}\eta\right)d\eta + U_0^2(h - \delta), \tag{6.4}$$

i.e.,

$$\frac{1}{2} g d^2 + \frac{Q^2}{d} = U_0^2\delta \cdot \frac{1}{2} + U_0^2(h - \delta). \tag{6.5}$$

Using (5.20) in (6.5) gives

$$\frac{1}{2} g d^2 + \frac{Q^2}{d} = U_0^2\left[\frac{\pi a}{2} - \frac{\delta}{2}\right]. \tag{6.6}$$

Substituting (5.10) into (6.6), we find

$$\frac{1}{2}gd^2 + \frac{Q^2}{d} = U_0^2\left[\frac{\pi a}{2} - \frac{1}{2}\left[\frac{6\sqrt{3}\alpha^2\pi}{8-3\sqrt{3}\alpha^2}\frac{\upsilon x}{U_0}\right]^{\frac{1}{2}}\right]. \tag{6.7}$$

Applying (2.1) to (6.7) gives

$$\frac{gd^2a^2}{2Q^2} + \frac{a^2}{d} = \left[\frac{\pi a}{2} - \frac{1}{2}\left[\frac{6\sqrt{3}\alpha^2\pi}{8-3\sqrt{3}\alpha^2}\frac{\upsilon xa}{Q}\right]^{\frac{1}{2}}\right]. \tag{6.8}$$

Substituting (2.2) into the RHS of (6.8) and changing x to x_1, we have

$$\frac{gd^2a^2}{2Q^2} + \frac{a^2}{d} = \left[\frac{\pi a}{2} - \frac{1}{2}\left[\frac{6\sqrt{3}\alpha^2\pi}{8-3\sqrt{3}\alpha^2}\frac{x_1a}{R}\right]^{\frac{1}{2}}\right]. \tag{6.9}$$

Solving for x_1, we obtain

$$\frac{x_1}{aR} = \left[\frac{8-3\sqrt{3}\alpha^2}{6\sqrt{3}\alpha^2\pi}\right]\left[\pi - \frac{gd^2a}{Q^2} - \frac{2a}{d}\right]^2 \quad \text{for} \quad x_1 < x_0. \tag{6.10}$$

For the case $x_1 > x_0$, we have

$$\int_0^h u^2 dy = U^2 h\int_0^1 \sin^2\left(\frac{\pi}{2}\eta\right)d\eta. \tag{6.11}$$

Substituting (6.11) into the jump condition (6.2), we have

$$\frac{1}{2}gd^2 + \frac{Q^2}{d} = U^2 h\int_0^1 \sin^2\left(\frac{\pi}{2}\eta\right)d\eta = U^2 h\cdot\frac{1}{2}. \tag{6.12}$$

Using (4.26) and (4.27) in (6.12), we find after simplification

$$\frac{\upsilon(x+\ell)}{Q^3} = \left[\frac{27\sqrt{3}\alpha^4}{16\pi^2}\right]\left[\frac{1}{2}gd^2 + \frac{Q^2}{d}\right]^{-1}. \tag{6.13}$$

Using (2.2) in (6.13) leads to

$$x + \ell = RQ^2 \left[\frac{27\sqrt{3}\alpha^4}{16\pi^2}\right]\left[\frac{1}{2} gd^2 + \frac{Q^2}{d}\right]^{-1}, \tag{6.14}$$

i.e.,

$$\frac{x}{aR} = \left[\frac{27\sqrt{3}\alpha^4}{16\pi^2}\right]\left[\frac{gd^2 a}{2Q^2} + \frac{a}{d}\right]^{-1} - \ell. \tag{6.15}$$

Finally, substituting (5.24) into (6.15) and changing x to x_1, we obtain

$$\frac{x_1}{aR} = \left[\frac{27\sqrt{3}\alpha^4}{16\pi^2}\right]\left[\frac{gd^2 a}{2Q^2} + \frac{a}{d}\right]^{-1} - \frac{[216\sqrt{3}\alpha^4 - (16 - 6\sqrt{3}\alpha^2)\pi^3]aR}{48\sqrt{3}\alpha^2\pi^2}. \tag{6.16}$$

7. Discussion and Conclusion

The two-dimensional inviscid theory, which assumes uniform constant velocities U_0, U_1 before and after the jump respectively, leads to the result (2.11), which shows that it is not possible to determine the position of the jump from the inviscid theory [3]. We observe that incorporation of viscous effects (2.2) in (2.11) coupled with the use of the principles of momentum and continuity at the jump leads to the modified result (6.10) for $x_1 < x_0$ or (6.16) for $x_1 > x_0$. Thus, the position $x = x_1$ of the jump is given by (6.10) for $x_1 < x_0$ or (6.16) for $x_1 > x_0$. The difference between the inviscid result (2.11) and the result (6.10) or (6.16) due to viscous effects is that (6.10) or (6.16) shows that if the left hand side of (2.11) is less than 1, the flow loses total head by friction over the length x_1 until the jump can occur.

In the present work, we observe that comparison of the approximate value (5.13) with the exact value (3.9) shows that our percentage error in the shear rate relation $f''(0)$ on the plate is only about 1.5%. Similarly, the percentage error in the thickness ratio H^* obtained by comparing the approximate value (5.18) with the accurate value (3.13) is about 2.6%. These results are adequate for the present purpose since they closely tend to the accurate values [2]. Also, these percentages which are less than those of Watson show improvement of our work upon that of

Watson [18]. However, the position of the jump (6.10) or (6.16) based on viscous effects shows a good correspondence to that of Watson [18], provided the liquid flow remains laminar. Our analysis also shows that the total thickness of the layer h (5.20) is directly proportional to a, the radius of the impinging jet; that is, h depends chiefly on the model (velocity profile) used. Finally, (4.27) shows that $h(x)$ depends linearly on x, while (4.26) shows that $U(x)$ depends inversely on x. The relation (4.27) also means that x starts from the leading edge of the boundary layer, whereas the parameter ℓ in this relation is the distance from the centre of the impinging jet to the leading edge of the boundary layer. Here the boundary layer constitutes an obstacle over which the upstream (incident) flow jumps and falls off at the edge of the plate.

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