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LOCALLY DEFINABLE $C^\infty G$ MANIFOLD STRUCTURES OF LOCALLY DEFINABLE $C^r G$ MANIFOLDS

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Abstract

Let G be a finite abelian group and $1 \leq r < \infty$. We prove that every locally definable $C^r G$ manifold admits a unique locally definable $C^\infty G$ manifold structure up to locally definable $C^\infty G$ diffeomorphism.

1. Introduction

Let G be a finite group and $1 \leq r < \infty$. Let $\mathcal{M}$ be an o-minimal exponential expansion $(\mathbb{R}, +, \cdot, <, e^x, \ldots)$ of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field $\mathbb{R}$ of real numbers admits the C^∞ cell decomposition and has piecewise controlled derivatives.

In this paper, we consider existence of locally definable $C^\infty G$ manifold structures of a locally definable $C^r G$ manifold and uniqueness of locally definable $C^\infty G$ manifold structure up to locally definable $C^\infty G$ diffeomorphism. If G is a

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finite abelian group and $1 \leq s < r < \infty$, then unique existence of locally definable C^rG manifold structure of a locally definable C^sG manifold is studied in [11].

Let $0 \leq r \leq \infty$. A *locally definable* C^r *manifold* is a C^r manifold admitting a countable system of charts whose gluing maps are of class definable C^r. If this system is finite, then it is called a *definable* C^r *manifold*. Definable C^rG manifolds are studied in [5], [6], [7], [8], [9]. A locally definable C^r manifold is *affine* if it can be imbedded into some $\mathbb{R}^n$ in a locally definable C^r way. We can define locally definable C^rG manifolds and affine locally definable C^rG manifolds in a similar way of equivariant definable cases. Locally definable C^rG manifolds are generalizations of definable C^rG manifolds and they are studied in [11] when *r* is a positive integer.

In this paper, everything is considered in $\mathcal{M}$, any map is continuous and every manifold does not have boundary unless otherwise stated.

Theorem 1.1. *Let G be a finite group and* $1 \leq r < \infty$. *Then every affine locally definable* C^rG *manifold is locally definably* C^rG *diffeomorphic to some locally definable* $C^\infty G$ *manifold*.

Theorem 1.2. *Let G be a finite group. Then for any two affine locally definable* $C^\infty G$ *manifolds, they are* C^1G *diffeomorphic if and only if they are locally definably* $C^\infty G$ *diffeomorphic*.

If $\mathcal{M}$ is polynomially bounded, then Theorem 1.2 is not always true. Even in the non-equivariant Nash category, there exist two affine Nash manifolds such that they are not Nash diffeomorphic but C^∞ diffeomorphic [14], and that for any two affine Nash manifolds, they are locally Nash diffeomorphic if and only if they are Nash diffeomorphic.

Existence of $C^\omega G$ manifold structures of proper $C^\infty G$ manifolds and uniqueness of them are studied in [3] and [4], respectively, when *G* is a C^ω Lie group. Moreover if *G* is a compact C^ω Lie group, then for any two $C^\omega G$ manifolds, they are $C^\infty G$ diffeomorphic if and only if they are $C^\omega G$ diffeomorphic [13].

Theorems 1.1 and 1.2 are locally definable C^∞ versions of [2] and [3], respectively, when G is a finite group.

The above theorems are locally definable C^∞ versions of results of [10].

In the non-equivariant setting, we have the following.

Theorem 1.3. *If* $1 \leq r \leq \infty$, *then every* n*-dimensional locally definable* C^r *manifold* X *is locally definably* C^r *imbeddable into* $\mathbb{R}^{2n+1}$.

The above theorem is the locally definable version of Whitney's imbedding theorem (e.g., 2.14 [2]). The definable C^r version of Theorem 1.1 is known in [8] when r is a non-negative integer.

If $\mathcal{M} = \mathcal{R}$ and $r = \infty$, then Theorem 1.3 is not true. The assumption that $\mathcal{M}$ is exponential is necessary.

As a corollary of Theorem 1.3, we have the following.

Theorem 1.4. *Let* G *be a finite abelian group and* $1 \leq r \leq \infty$. *Then every locally definable* $C^r G$ *manifold is affine*.

By Theorems 1.1-1.4, we have the following theorem.

Theorem 1.5. *Let* G *be a finite abelian group and* $1 \leq r < \infty$. *Then every locally definable* $C^r G$ *manifold admits a unique locally definable* $C^\infty G$ *manifold structure up to locally definable* $C^\infty G$ *diffeomorphism*.

2. Locally Definable $C^r G$ Manifolds

Let $f : U \to \mathbb{R}$ be a definable C^∞ function on a definable open subset $U \subset \mathbb{R}^n$. We say that f has *controlled derivatives* if there exist a definable continuous function $u : U \to \mathbb{R}$, real numbers $C_1, C_2, \ldots$ and positive integers $E_1, E_2, \ldots$ such that $| D^\alpha f(x) | \leq C_{|\alpha|} u(x)^{E_{|\alpha|}}$ for all $x \in U$ and $\alpha \in (\mathbb{N} \cup \{0\})^n$, where $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_1^{\alpha_n}}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We say that $\mathcal{M}$ has *piecewise controlled derivatives* if for every definable C^∞ function $f : U \to \mathbb{R}$ defined in a definable open subset U of $\mathbb{R}^n$, there exist definable open sets $U_1, \ldots, U_l \subset U$ such that $\dim(U - \bigcup_{i=1}^{l} U_i) < n$ and each $f|U_i$ has controlled derivatives.

A subset X of $\mathbb{R}^n$ is called *locally definable* if for every $x \in X$ there exists a definable open neighborhood U of x in $\mathbb{R}^n$ such that $X \cap U$ is definable in $\mathbb{R}^n$. Clearly every definable set is locally definable. Remark that any open subset of $\mathbb{R}^n$ is locally definable and that every compact locally definable set is definable. A more general setting of locally definable sets is studied in [1].

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be locally definable sets. We call a map $f : U \to V$ *locally definable* if for any $x \in U$ there exists a definable open neighborhood W_x of x in $\mathbb{R}^n$ such that $f|U \cap W_x$ is definable.

Note that for any locally definable map f between locally definable sets X and Y, if X is compact, then $f(X)$ is a definable set and $f : X \to f(X)\ (\subset Y)$ is a definable map.

Remark that the maps $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = \sin x$, $f_2(x) = \cos x$, respectively, are analytic but not definable in any o-minimal expansion of $\mathcal{R}$. However they are locally definable in $\mathbf{R}_{an}$. Remark further that the field $\mathbb{Q}\ (\subset \mathbb{R})$ of rational numbers is not a locally definable subset of $\mathbb{R}$.

Proposition 2.1 [11]**.** *Let X, Y and Z be locally definable sets and let $f : X \to Y$ and $g : Y \to Z$ be locally definable maps. Then $g \circ f : X \to Z$ is locally definable.*

We can define locally definable groups and affine locally definable groups in a similar way of definable cases. But we do not give their definitions here because we restrict our attention to finite groups.

A *representation map* of G is a group homomorphism from G to some $O(n)$. A *representation* of G means the representation space of a representation map of G. Recall the definition of locally definable $C^r G$ manifolds [11].

Definition 2.2 [11]**.** Let $1 \leq r \leq \omega$.

(1) A locally definable C^r submanifold of a representation Ω of G is called a

locally definable C^rG *submanifold of* Ω if it is G invariant.

(2) A *locally definable* C^rG *manifold* is a pair (X, θ) consisting of a locally definable C^r manifold X and a group action θ of G on X such that $\theta : G \times X \to X$ is a locally definable C^r map. For simplicity of notation, we write X instead of (X, θ). Clearly each definable C^rG manifold is a locally definable C^rG manifold.

(3) Let X and Y be locally definable C^rG manifolds. A locally definable C^r map is called a *locally definable* C^rG *map* if it is a G map. We say that X and Y are *locally definably* C^rG *diffeomorphic* if there exist locally definable C^rG maps $f : X \to Y$ and $h : Y \to X$ such that $f \circ h = id$ and $h \circ f = id$.

(4) A locally definable C^rG manifold is said to be *affine* if it is locally definably C^rG diffeomorphic to a locally definable C^rG submanifold of some representation of G.

Note that we can define locally definable G manifolds for a locally definable group G, but in this paper we do not use these notions.

Recall existence of definable C^rG tubular neighborhoods.

Theorem 2.3 ([9], [6]). *Let r be a non-negative integer*, ∞ *or* ω. *Then every definable* C^rG *submanifold X of a representation* Ω *of G has a definable* C^rG *tubular neighborhood* (U, θ_X) *of X in* Ω, *namely U is a G invariant definable open neighborhood of X in* Ω *and* $\theta_X : U \to X$ *is a definable* C^rG *map with* $\theta_X | X = id_X$.

Let $G = \{g_1, ..., g_m\}$ and let f be a C^rG map from a C^rG manifold M to a representation Ω of G. Then the averaging map $A : M \to \Omega$ is

$$A(f)(x) = \frac{1}{m}\sum_{i=1}^{m} g_i^{-1} f(g_i x).$$

By using [7], we have the following lemma.

Proposition 2.4 [7]. (1) $A(f)$ *is equivariant*, *and* $A(f) = f$ *iff f is equivariant*.

(2) *If f is a polynomial map, then so is* $A(f)$.

(3) *If* $0 \leq r \leq \infty$ *and f lies in the set* $C^r(M, \Omega)$ *of* C^r *maps from M to* Ω, *then* $A(f) \in C^r(M, \Omega)$.

(4) $A : C^r(M, \Omega) \to C^r(M, \Omega)$, $f \mapsto A(f)$ $(0 \leq r \leq \infty)$ *is continuous in the* C^r *Whitney topology*.

(5) *If M is a definable* C^rG *manifold, f is a definable* C^r *map and* $0 \leq r \leq \omega$, *then* $A(f)$ *is a definable* C^rG *map*.

(6) *If M is a locally definable* C^rG *manifold, f is a locally definable* C^r *map and* $0 \leq r \leq \omega$, *then* $A(f)$ *is a locally definable* C^rG *map*.

Let *K* be a subgroup of *G*. Suppose that *S* is an affine definable $C^\infty K$ manifold. Then we know that the twisted product $G \times_K S$ with the standard action $G \times (G \times_K S) \to G \times_K S$, $(g, [g', s]) \mapsto [gg', s]$ is a definable $C^\infty G$ manifold [9].

We need the following proposition to prove Theorem 1.1.

Proposition 2.5. *Let X be a locally definable* $C^\infty G$ *manifold. Suppose that K is a subgroup of G and N is an affine definable* $C^\infty K$ *manifold. If* $f : N \to X$ *is a locally definable* $C^\infty K$ *map, then*

$$\mu(f) : G \times_K N \to X, \quad \mu([g, n]) = gf(n)$$

is a locally definable $C^\infty G$ *map*.

Proof. By the property of quotient manifolds, $\mu(f)$ is a $C^\infty G$ map. Thus it suffices to prove that $\mu(f)$ is locally definable. Let π be the orbit map $G \times N \to G \times_K N$. Then π is a definable C^∞ map. Take $x \in G \times_K N$ and $y \in \pi^{-1}(x) \subset G \times N$. By the assumption and the definition of the G action on $G \times N$, $\overline{\mu}(f) : G \times N \to X$, $\overline{\mu}(f)(g, n) = gf(n)$ is a locally definable $C^\infty G$ map. Hence there exist definable open neighborhoods U of y and V of $\overline{\mu}(f)(y)$, respectively, such that $\overline{\mu}(f)(U) \subset V$ and $\overline{\mu}(f)|U : U \to V$ is a definable C^∞ map. In particular, $\overline{\mu}(f)|U : U \to V$ is definable. Hence $\pi(U)$ is open and definable. Since the graph of $\mu(f)|\pi(U) : \pi(U) \to V \subset X$ is the image of that of $\overline{\mu}(f)|U$ by $\pi \times id_V$, $\mu(f)|\pi(U)$ is definable. □

Definition 2.6. Let X be a definable $C^{\infty}G$ manifold.

(1) We say that a K invariant definable C^{∞} submanifold S of X is a *definable K slice* if GS is open in X, S is affine as a definable $C^{\infty}K$ manifold, and

$$\mu : G \times_K S \to GS\,(\subset X), \quad [g, x] \mapsto gx$$

is a definable $C^{\infty}G$ diffeomorphism.

(2) A definable $C^{\infty}K$ slice S is called *linear* if there exist a representation Ω of K and a definable $C^{r}K$ imbedding $j : \Omega \to X$ such that $j(\Omega) = S$.

(3) We say that a definable $C^{\infty}K$ slice (resp., a linear definable $C^{\infty}K$ slice) S is a *definable* C^{∞} *slice* (resp., a *linear definable* C^{∞} *slice*) *at x in X* if $K = G_x$ and $x \in S$ (resp., $K = G_x$, $x \in S$ and $j(0) = x$).

Recall existence of definable C^{∞} slices [9] to prove Theorem 1.2.

Theorem 2.7 [9]**.** *Let X be an affine definable* $C^{\infty}G$ *manifold*, $x \in X$. *Then there exists a linear definable* $C^{\infty}G$ *slice at x in X.*

3. Proof of Theorem 1.1

The following lemma is obtained by 2.2.8 [2] and Proposition 2.4.

Lemma 3.1. *Let K be a finite group. Suppose that f is a definable* $C^{\infty}K$ *map between definable* $C^{\infty}K$ *manifolds M and N. Suppose further that V is an open K invariant subset of M and that P is a K invariant definable* C^{∞} *submanifold of N with* $f(V) \subset P$. *Then there exist an open neighborhood* $\mathfrak{N}$ *of* $f|V$ *in the set* $Def_K^{\infty}(V, P)$ *of definable* $C^{\infty}K$ *maps from V to P such that for any* $h \in \mathfrak{N}$, *the map* $E(h) : M \to N$,

$$E(h)(x) = \begin{cases} h(x), & x \in V, \\ f(x), & x \in M - V \end{cases}$$

is a definable $C^\infty K$ *map and* $E : \mathfrak{N} \to Def_K^\infty(M, N)$, $h \mapsto E(h)$ *is continuous in the* C^∞ *Whitney topology*.

Proposition 3.2. *Let X be a locally definable* $C^\infty G$ *manifold and Y be an affine definable* $C^\infty G$ *manifold in a representation* Ω *of G. Then every* $C^\infty G$ *map* $f : X \to Y$ *is approximated by a locally definable* $C^\infty G$ *map* $h : X \to Y$ *in the* C^∞ *Whitney topology*.

In the Nash case, if $1 \leq r < \infty$, then locally C^r Nash diffeomorphisms are essentially different from C^r Nash diffeomorphisms because there exist two affine Nash manifolds such that they are C^∞ diffeomorphic but not Nash diffeomorphic [14], and that every C^r Nash diffeomorphism between affine Nash manifolds is approximated by a Nash diffeomorphism [15].

Proposition 3.3 [12]**.** *Every affine definable* $C^\infty G$ *manifold is definably* $C^\infty G$ *diffeomorphic to a definable* $C^\infty G$ *submanifold closed in some representation* Ω *of G*.

For the proof of Proposition 3.3, we need the condition that $\mathcal{M}$ is exponential, admits the C^∞ cell decomposition and has piecewise controlled derivatives.

Proof of Proposition 3.2. By Proposition 3.3, replacing Ω if necessary, we may assume that *Y* is a definable $C^\infty G$ submanifold closed in Ω. By a way similar to find a C^∞ partition of unity of C^∞ manifold, we have a locally definable C^∞ partition of unity $\{\phi_j\}_{j=1}^\infty$ subordinates to some locally finite definable open cover $\{X_j\}_{j=1}^\infty$ of *X* such that $X = \bigcup_{j=1}^\infty \operatorname{supp} \phi_j$ and $\overline{X_j}$ is compact. For any *j*, take an open neighborhood U_j of $\operatorname{supp} \phi_j$ in *X* such that $\overline{U_j}$ is compact. Applying the polynomial approximation theorem, we have a locally definable C^∞ map $h_j : U_j \to \Omega$ which approximates $f \,|U_j$. By Theorem 2.3, one can find a definable $C^\infty G$ tubular neighborhood (U, p) of *Y* in Ω. If our approximation is sufficiently close, then $p \circ \sum_{j=1}^\infty \phi_j h_j$ is a (non-equivariant) C^∞ approximation of *f*. Since *G* is a finite group, applying Proposition 2.4, we have the required locally definable $C^{\infty}G$ map *h* as a C^{∞} Whitney approximation of *f*. □

Proof of Theorem 1.1. Using Lemma 3.1 and Proposition 3.2, a similar proof of 1.1 [11] proves Theorem 1.1. □

4. Proof of Theorem 1.2

In this section, we prove the following theorem.

Theorem 4.1. *Let G be a finite group and let r be a positive integer. Suppose that Y and Z are affine locally definable* $C^{\infty}G$ *manifolds and there exists a* $C^{r}G$ *diffeomorphism* $f : Y \to Z$. *Then there exists a locally* $C^{\infty}G$ *diffeomorphism* $h : Y \to Z$ *which is G homotopic to f.*

Theorem 1.2 follows from Theorem 4.1.

Let *K* be a subgroup of *G* and let *X* be an affine definable $C^{\infty}G$ manifold. By Theorem 2.7, there exists a linear definable $C^{\infty}K$ slice *S*, namely there exists a definable $C^{\infty}K$ diffeomorphism *i* from some representation Ω of *K* to *S* such that *GS* is open in *X*, and that $\mu : G \times_K \Omega \to GS\,(\subset X)$, $\mu(i)([g, x]) = gi(x)$ is a definable $C^{\infty}G$ diffeomorphism.

For simplicity, we use the following notations. Set $B_s := \{x \in \Omega \,|\, \| x \| \leq s\}$, $B_s^{\circ} := \{x \in \Omega \,|\, \| x \| < s\}$, $s > 0$, $B := B_1$, and $B^{\circ} := B_1^{\circ}$, and let denote D_s, D_s°, D and D° by $i(B_s)$, $i(B_s^{\circ})$, $i(B)$, and $i(B^{\circ})$, respectively. Let GD (resp., GD°) denote the closed unit tube (resp., the open unit tube) and let GD_s (resp., GD_s°) stand for the closed tube (resp., the open tube) of radius *s*.

To prove Theorem 4.1, we prepare two preliminary results.

Lemma 4.2. *Let* Ω *and* Ξ *be representations of G and let M* (*resp.*, *N*) *be a definable* $C^{\infty}G$ *submanifold of* Ω (*resp.*, Ξ). *Suppose that F is a G invariant definable subset of M and that* $\alpha : M \to N$ *is a* $C^{\infty}G$ *map such that* $\alpha | F : F \to N$ *is definable. Let* $\mathfrak{N}$ *be a neighborhood of* α *in the set* $C_G^{\infty}(M, N)$ *of*

$C^\infty G$ maps from M to N and let V_1 and V_2 be compact G invariant definable subsets of M such that V_1 is properly contained in the interior $Int V_2$ of V_2. Then there exists $\kappa \in \mathfrak{N}$ such that:

(a) $\kappa | F \cup V_1 : F \cup V_1 \to N$ *is definable*.

(b) $\kappa = \alpha$ *on* $M - Int V_2$.

(c) κ *is G homotopic to* α *relative to* $M - Int V_2$.

Proof. Take a non-negative definable C^∞ function $f : M \to \mathbb{R}$ such that $f = 0$ on V_1 and $f = 1$ on $M - Int V_2$. Notice that if $\mathcal{M}$ is polynomially bounded, then such an *f* does not necessarily exist. Since *G* is a finite group and by Proposition 2.4, we may assume that *f* is *G* invariant.

We approximate α by a polynomial *G* map β on V_2 using the polynomial approximation theorem and Proposition 2.4. By Theorem 2.3, one can find a definable $C^\infty G$ tubular neighborhood (U, p) of *N* in Ξ. If the approximation is sufficiently close, then one can define $\kappa : M \to N$, $\kappa(x) = p(f(x)\alpha(x) + (1 - f(x))\beta(x))$. Then κ is a $C^\infty G$ map, and κ satisfies Properties (a) and (b). If this approximation is sufficiently close, then $\kappa \in \mathfrak{N}$ because κ and α coincide with outside of a compact set V_2.

The map $H : M \times [0, 1] \to N$ defined by $H(x, t) = p((1 - t)\alpha(x) + t\kappa(x))$ gives a *G* homotopy relative to $M - \mathrm{Int} V_2$ from α to κ. □

Proposition 4.3. *Let Ω and Ξ be representations of G. Let $M \subset \Omega$, $N \subset \Xi$ be affine locally definable $C^\infty G$ manifolds and A be a closed G invariant locally definable subset of M. Suppose that $f : M \to N$ is a $C^\infty G$ diffeomorphism such that $f | A : A \to N$ is locally definable, and that $x \in M$. Suppose further that $j : \Omega' \to S$ is a linear definable C^∞ slice at x in Ω. If $GD_{10} \cap M$ is compact, then there exists a $C^\infty G$ diffeomorphism $h : M \to N$ such that*:

(1) $h | A \cup (GD \cap M) : A \cup (GD \cap M) \to N$ *is locally definable*.

(2) $h = f$ *on* $M - GD_2^\circ \cap M$.

(3) *h is G homotopic to f relative to* $M - GD_2^{\circ} \cap M$.

The condition that $GD_{10} \cap M$ is compact is not essential. By Theorem 2.7, one can find a linear definable C^r slice S at $x \in M$ in Ω. Since S is a linear definable C^{∞} slice in Ω, there exists a definable $C^{\infty}K$ diffeomorphism j from some representation Ω' of G_x onto S such that $j(0) = x$, GS is open in Ω, and that

$$\mu(j) : G \times_{G_x} \Omega' \to GS(\subset \Xi),$$

$$\mu(j)([g, x]) = gj(x)$$

is a definable $C^{\infty}G$ diffeomorphism. Notice that M is locally compact. Thus replacing smaller S, if necessary, $GD_{10} \cap M$ is compact because M is locally compact.

Proof of Proposition 4.3. Since $GD_{10} \cap M$ is compact and A is closed in M, $A \cap GD_{10}$ $(= A \cap (GD_{10} \cap M))$ is a compact G invariant locally definable subset of $GS \cap M$. Thus $A \cap GD_{10}$ is a G invariant definable subset of Ω. Hence

$$E := \mu(j)^{-1}(A \cap GD_{10})$$

is a G invariant definable subset of $G \times_{G_x} \Omega'$. Let $L = j^{-1}(D_{10}^{\circ} \cap M)$. The map

$$\alpha := f \circ \mu(j) | G \times_{G_x} L : G \times_{G_x} L \to \Xi$$

is a $C^{\infty}G$ diffeomorphism onto an open G invariant subset $V := f(GD_{10}^{\circ} \cap M)$ of N. Since $A \cap GD_{10}$ is compact and $f | A$ is locally definable, $f | (A \cap GD_{10}) : A \cap GD_{10} \to f(A \cap GD_{10}) \subset N \subset \Xi$ is definable. The map $\alpha | (E \cap (G \times_{G_x} L)) : E \cap (G \times_{G_x} L) \to \Xi$ is definable because $\mu(j)$ and $f | (A \cap GD_{10}) : A \cap GD_{10} \to \Xi$ are definable. Since V is contained in a G invariant compact set $f(GD_{10} \cap M)$, and since N is a locally definable $C^{\infty}G$ submanifold of Ξ, there exists a G invariant definable set W of Ξ such that $V \subset W \subset N$ and that W is open in N. Notice that W is an affine definable $C^{\infty}G$ manifold. Since $G \times_{G_x} L$ is contained in a G invariant compact subset of $G \times_{G_x} j^{-1}(D_{20} \cap M)$, $G \times_{G_x} L$ is an affine definable $C^{\infty}G$

manifold. Applying Lemma 4.2 to $\alpha : G \times_{G_x} L \to W$, there exists a $C^\infty G$ map $\beta : G \times_{G_x} L \to W$ as a C^∞ Whitney approximation of α such that:

(a) $\beta | (G \times_{G_x} (j^{-1}(A \cap D_{10}^\circ) \cup (B \cap L))) : G \times_{G_x} (j^{-1}(A \cap D_{10}^\circ) \cup (B \cap L))) \to W (\subset N)$ is definable.

(b) $\beta = \alpha$ on $G \times_{G_x} (L - B_2^\circ \cap L)$.

(c) β is G homotopic to α relative to $G \times_{G_x} (L - B_2^\circ \cap L)$.

Then the map $h : M \to N$ defined by

$$h(x) = \begin{cases} \beta \circ \mu(j)^{-1}(x), & x \in GD_5 \cap M, \\ f(x), & x \in M - M \cap GD_5^\circ \end{cases}$$

is well-defined, and it is a $C^\infty G$ diffeomorphism if our approximation is sufficiently close. Since $h | (A \cap GD_5)$ and $h | (GD \cap M)$ are definable, and since $h | (A \cap (M - GD_5 \cap M)) (= f | (A \cap (M - GD_5 \cap M)))$ is locally definable, $h | A \cup (GD \cap M)$ is locally definable by Proposition 2.1. By the construction of *h*, *h* satisfies Properties (2) and (3). □

Proof of Theorem 4.1. Using Proposition 4.3, a similar proof of 4.1 [11] proves Theorem 4.1. □

5. Proofs of Theorems 1.3 and 1.4

Proof of Theorem 1.3. By Whitney's imbedding Theorem (e.g., 2.14 [2]), there exists a C^∞ imbedding $f : X \to \mathbb{R}^{2n+1}$. By Proposition 3.2 and since C^∞ imbeddings from *X* to $\mathbb{R}^{2n+1}$ are open in the set $C^\infty(X, \mathbb{R}^{2n+1})$ of C^∞ maps from *X* to $\mathbb{R}^{2n+1}$, we have the required locally definable C^∞ imbedding $h : X \to \mathbb{R}^{2n+1}$. □

Proof of Theorem 1.4. Let $G = \{g_1, ..., g_m\}$ and *X* be a locally definable $C^\infty G$ manifold of dimension *n*. By Theorem 1.3, there exists a locally definable C^∞ imbedding $f : X \to \mathbb{R}^{2n+1}$. Let Ω be the representation of *G* whose underlying space is $\mathbb{R}^{(2n+1)m} = \mathbb{R}^{2n+1} \times \cdots \times \mathbb{R}^{2n+1}$ and its action is defined by the permutation of coordinates $(x_1, ..., x_m) \mapsto (x_{\sigma(1)}, ..., x_{\sigma(m)})$ induced from $(gg_1, ..., gg_m) = (g_{\sigma(1)}, ..., g_{\sigma(m)})$. Then $F : X \to \Omega$, $F(x) = (f(g_1 x), ..., f(g_m x))$ is the required locally definable $C^\infty G$ imbedding. □

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