Far East Journal of Mathematical Sciences (FJMS)
Volume 35, Number 1, 2009, Pages 71-76
Published Online: November 19, 2009
This paper is available online at http://www.pphmj.com

$\varphi - b$-GENERAL ORTHOGONALITY IN LINEAR $\varphi - 2$-NORMED SPACES

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Abstract

The purpose of this paper is to introduce the definition of φ-function and linear $\varphi - 2$-normed spaces given by Golet [5] and extend these to b-general orthogonality given by Kamali and Mazaheri [6] and obtain some results on $\varphi - b$-general orthogonality.

1. Introduction, Definitions, Notations and Preliminaries

The concept of linear 2-normed spaces has been investigated by S. Gahler and has been developed extensively in different subjects by many authors [1].

A real linear 2-normed space is a real linear space X equipped with a 2-norm $\|\cdot, \cdot\| : X \times X \to \mathbb{R}$ satisfying the four conditions:

1.1. $\|x, y\| \geq 0$, for every $x, y \in X$; $\|x, y\| = 0$ if and only if x and y are linearly dependent;

2000 Mathematics Subject Classification: Primary 44B20; Secondary 46C05.

Keywords and phrases: b-orthogonality, b-general orthogonality, φ-function, $\varphi - 2$-normed spaces, $\varphi - b$-general orthogonality.

Received June 26, 2009

1.2. $\| x, y \| = \| y, x \|$, for every $x, y \in X$;

1.3. $\| x, \alpha y \| = |\alpha| \| x, y \|$, for every $x, y \in X$ and $\alpha \in \mathbb{R}$;

1.4. $\| x, y + z \| \leq \| x, y \| + \| x, z \|$, for every $x, y, z \in X$.

Some of the basic properties of 2-norms that they are non-negative and $\| x, y + \alpha x \| = \| x, y \|$, for all $x, y \in X$ and $\alpha \in \mathbb{R}$.

Let $(X, \| \cdot, \cdot \|)$ be a 2-normed space and W_1 and W_2 be two subspaces of X. A map $f : W_1 \times W_2 \to \mathbb{R}$ is called a *bilinear* 2-*functional* on $W_1 \times W_2$ whenever for all $x_1, x_2 \in W_1$, $y_1, y_2 \in W_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.

(i) $f(x_1 + x_2, y_1 + y_2) = f(x_1, y_1) + f(x_1, y_2) + f(x_2, y_1) + f(x_2, y_2)$;

(ii) $f(\lambda_1 x_1, \lambda_2 y_1) = \lambda_1 \lambda_2 f(x_1, y_1)$.

A bilinear 2-functional $f : W_1 \times W_2 \to \mathbb{R}$ is called *bounded* if there exists a non-negative real number M (called a *lipschitz constant* for f) such that $| f(x, y) | \leq M \| x, y \|$, for all $x \in W_1$ and all $y \in W_2$. For a 2-normed space $(X, \| \cdot, \cdot \|)$ and $0 \neq b \in X$, we denote by X_b^*, the Banach space of all bounded bilinear 2-functionals on $X \times \langle b \rangle$, where $\langle b \rangle$ is the subspace of X generated by b.

Definition 1 [10]**.** Let $(X, \| \cdot, \cdot \|)$ be a linear 2-normed space, and $x, y \in X$. If there exists $b \in X$ such that $\| x, b \| \neq 0$ and $\| x, b \| \leq \| x + \alpha y, b \|$ for all scalars $\alpha \in \mathbb{R}$, then x is b-orthogonal to y and denoted by $x \perp^b y$.

If W_1 and W_2 are subsets of X, there exists $b \in X$ such that for all $y_1 \in W_1$, $y_2 \in W_2$, $y_1 \perp^b y_2$, then we say $W_1 \perp^b W_2$.

Definition 2 [6]**.** Let X be a linear 2-normed space and $x, y, b \in X$. x is called *b-general orthogonal* to y and write $x \perp_G^b y$, if and only if there exists a unique $\phi_x \in X_b^*$ such that

$$\phi_x(x, b) = \| x, b \|^2, \quad \| \phi_x \| = \| x, b \| \text{ and } \phi(y, b) = 0.$$

Definition 3 [5]**.** Let φ be a function defined on the real field $\mathbb{R}$ into itself with the following properties:

(a_1) $\varphi(-t) = \varphi(t)$, for every $t \in \mathbb{R}$;

(a_2) $\varphi(1) = 1$;

(a_3) φ is strict increasing and continuous on $(0, \infty)$;

(a_4) $\lim_{\alpha \to 0} \varphi(\alpha) = 0$ and $\lim_{\alpha \to \infty} \varphi(\alpha) = \infty$.

Based on this definition of φ function, we define linear $\varphi - 2$-normed spaces as follows:

Definition 4 [5]**.** Let X be a linear space over the field $\mathbb{R}$ of dimension greater than one and let $\| \cdot, \cdot \|$ be a mapping defined on $X \times X$ with real valued into the field $\mathbb{R}$ satisfying the following conditions:

(1) $\| x, y \| = 0$, if and only if x and y are linearly dependent;

(2) $\| x, y \| = \| y, x \|$, for all $x, y \in X$;

(3) $\| \alpha x, y \| = | \varphi(\alpha) | \| x, y \|$, for every $x, y \in X$ and all $\alpha \in \mathbb{R}$;

(4) $\| x + y, z \| \leq \| x, z \| + \| y, z \|$, for every $x, y, z \in X$.

If $\varphi(\alpha) = | \alpha |$, then one obtains the linear 2-normed spaces [4].

If $\varphi(\alpha) = | \alpha |^p$, $p \in (0, 1]$, then one obtains the $p - 2$-normed spaces as a generalization of 2-normed spaces. In this study, we assume $\varphi(\alpha) = | \alpha |$ and apply this to b-general orthogonality and its results.

2. Main Results

In this section, we state and prove some characterizations of the $\varphi - b$-general orthogonality in linear $\varphi - 2$-normed spaces.

Definition 5. Let $(X, \| \cdot, \cdot \|)$ be $\varphi - 2$-normed space and $x, y \in X$. If there exists $b \in X$ such that $\| x, b \| \neq 0$ and $\| x, b \| \leq \| x + \varphi(\alpha) y, b \|$ $(\alpha \in \mathbb{R})$. Then x is $\varphi - b$-orthogonal to y and denoted by $x \perp_{\varphi}^{b} y$.

Theorem 1. *Let X be a linear $\varphi - 2$-normed space, $b \in X$, $y \in X$ and $x \in$*

$X \setminus (\langle b \rangle)$. *Then the following statements are equivalent*:

(1) $x \perp_{\varphi}^{b} y$;

(2) *There exists* X_b^* *such that* $f(y, b) = 0$, $f(x, b) = \| x, b \|$ *and* $\| f \| = 1$.

Proof. (2) $\to$ (1) First suppose there exists $f \in X_b^*$ such that $f(y, b) = 0$. Then

$$\| x, b \| = f(x, b) = f(x + \varphi(\alpha) y, b) \leq \| f \| \cdot \| x + \varphi(\alpha) y, b \|$$
$$\leq \| x + \varphi(\alpha) y, b \|.$$

Therefore, $x \perp_{\varphi}^{b} y$.

(1) $\to$ (2) Suppose $x \perp_{\varphi}^{b} y$ and $W = \langle y \rangle$ is the subspace of X generated by b. Then

$$\inf\{\| x - \varphi(\alpha) y, b \| : \varphi(\alpha) y \in W\} \geq \| x, b \| > 0.$$

Then there exists $f \in X_b^*$ such that $g(y, b) = 0$, $g(x, b) = 1$ and $\| g \| = 1/\delta$. Put $f = \delta g$, then $f(y, b) = 0$, $f(x, b) = \| x, b \|$ and $\| f \| = 1$. □

Definition 6. Let X be a linear $\varphi - 2$-normed space and $x, y, b \in X$. x is called $\varphi - b$-*general orthogonal* to y and write $x \perp_{\varphi G}^{b} y$ if and only if there exists a unique $\phi_x \in X_b^*$ such that $\phi_x(x, b) = \| x, b \|^2$, $\| \phi_x \| = \| x, b \|$ and $\phi(y, b) = 0$.

Theorem 2. *Let X be a linear* $\varphi - 2$-*normed space. If* $x, y \in X$, *and* $x \perp_{G}^{b} y$, *then* $x \perp_{\varphi}^{b} y$.

Proof. Suppose $x, y \in X$ and $x \perp_{G}^{b} y$, then

$$\begin{aligned} \| x, b \|^2 &= \phi_x(x, b) \\ &= \phi_x(x + \varphi(\alpha) y, b) \\ &= \| \phi_x \| \cdot \| x + \varphi(\alpha) y, b \|. \end{aligned}$$

Therefore, $\| x, b \| \leq \| x + \varphi(x) y, b \|$. That is $x \perp_{\varphi}^{b} y$. □

Theorem 3. *Let X be a linear $\varphi - 2$-normed space. Then the following statements are true.*

(a) *For all $x \in X$ and all $\varphi(\alpha) > 0$, $\phi_{\varphi(\alpha)x} = \varphi(\alpha)\phi_x$.*

(b) *For all $x, y \in X$ and all $\alpha > 0$, if $x \perp_{\varphi G}^{b} y$, then $\varphi(\alpha)x \perp_{\varphi G}^{b} y$.*

(c) *For all $x \in X$, if $x \perp_{\varphi G}^{b} x$, then $x = 0$.*

(d) *For all $x, y \in X$, if $x \perp_{\varphi G}^{b} y$ and $x \neq 0$, then $\langle x \rangle \bigcap \langle y \rangle = \{0\}$.*

(e) *For all $x \in X$, $0 \perp_{\varphi G}^{b} y$ and $x \perp_{\varphi G}^{b} 0$.*

Proof. (a) Suppose $x \in X$ and $\varphi(x) > 0$. Then

$$\phi_{\varphi(x)x}(\varphi(\alpha)x, b) = \| \varphi(\alpha)x, b \|^2 = \varphi^2(\alpha) \cdot \| x, b \|^2$$

$$= \varphi^2(\alpha)\phi_x(x, b)$$

by using the linearity of ϕ_x, we have

$$\varphi(\alpha)\phi_{\varphi(\alpha)x}(x, b) = \varphi^2(\alpha) \cdot \phi_x(x, b)$$

and

$$\phi_{\varphi(\alpha)x}(x, b) = \varphi(\alpha) \cdot \phi_x(x, b).$$

Then we obtain $\phi_{\varphi(\alpha)x} = \varphi(\alpha)\phi_x$. □

(b) Suppose $x, y \in X$, $\beta \in \mathbb{R}$ and $\varphi(\alpha) > 0$. Then

$$\begin{aligned} \varphi^2(\alpha) \cdot \| x, b \|^2 &= \varphi^2(\alpha) \cdot \| x + \beta y, b \|^2 \\ &= \| \varphi(\alpha)(x + \beta y), b \|^2 \\ &= \| \varphi(\alpha)x, b \|^2 + \| \varphi(\alpha)\beta y, b \|^2 \\ &= \| \varphi(\alpha)x, b \|^2. \end{aligned}$$

Since $\phi(y, b) = 0$, $\varphi^2(\alpha) \cdot \beta^2 \| y, b \|^2 = 0$. Therefore, $\varphi(\alpha)x \perp_{\varphi G}^{b} y$. □

(c) For all $x \in X$, if $x \perp_{\varphi G}^{b} x$, then $\phi_{\varphi(\alpha)x}(\varphi(\alpha)x, b) = 0$ and $\phi_{\varphi(\alpha)x}(\varphi(\alpha)x, b)$ $= \| \varphi(\alpha)x, b \|^2$. Since $\varphi(\alpha) > 0$ and $b \in X$, then $x = 0$. □

(d) If $z \in \langle x \rangle \cap \langle y \rangle$ and since $\lim_{\alpha \to 0} \varphi(\alpha) = 0$, choose x_n and y_n such that $z = c_1 x_n = c_2 y_n$ for scalars c_1, c_2. Hence $\phi_{\varphi(\alpha)}(z, b) = 0$, it follows that

$$\phi_{\varphi(c_1)x}(\varphi(c_1)x_n, b) = \| \varphi(c_1)x_n, b \|^2 = \varphi^2(c_1) \| x_n, b \|^2$$

$\| x_n, b \| \not\to 0$, as $n \to \infty$. Therefore, $\varphi(c_1) \to 0$, thus $c_1 \to 0$.

(e) It is trivial.

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