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SOME PROPERTIES OF GRADED MULTIPLICATION MODULES

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Abstract

Let G be a group with identity e, R be a G-graded commutative ring, and M be a graded R-module. This paper is devoted to study some properties of graded multiplication modules. First, we characterize graded multiplication modules by using the graded localization of graded R-module M. Next, we give a condition which allows us to determine whether graded submodules of a graded module have graded prime radical.

1. Introduction

Let G be a group. A ring (R, G) is called a *G-graded ring* if there exists a family $\{R_g : g \in G\}$ of additive subgroups of R such that $R = \oplus_{g \in G} R_g$ such that $1 \in R_e$ and $R_g R_h \subseteq R_{gh}$ for each g and h in G. For simplicity, we will denote the graded ring (R, G) by R. A G-graded ring is graded domain, if $ab = 0$, where $a, b \in h(R)$, then $a = 0$ or $b = 0$. A G-graded ring R is said to be *graded principal ideal domain*, if R is a graded domain, and for each graded ideal of R is graded principal. If R is G-graded, then an R-module M is said to be *G-graded* if it has a

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direct sum decomposition $M = \oplus_{g \in G} M_g$ such that for all $g, h \in G$; $R_g M_h \subseteq M_{gh}$. An element of some R_g or M_g is said to be *homogeneous element*. A submodule $N \subseteq M$, where M is G-graded, is called *G-graded* if $N = \oplus_{g \in G}(N \cap M_g)$ or if, equivalently, N is generated by homogeneous elements. Moreover, M/N becomes a G-graded module with g-component $(M/N)_g = (M_g + N)/N$ for $g \in G$. We write $h(R) = \bigcup_{g \in G} R_g$ and $h(M) = \bigcup_{g \in G} M_g$. The graded radical of I (in the abbreviation, $Gr(I)$) is the set of all $x \in R$ such that for each $g \in G$, there exists $n_g > 0$ with $x_g^{n_g} \in I$. Note that if r is homogeneous element of (R, G), then $r \in Gr(I)$ iff $r^n \in I$ for some $n \in N$. A graded ideal I of R is said to be *graded prime* (resp. *graded primary*) *ideal* if $I \neq R$; and whenever $ab \in I$, we have $a \in I$ or $b \in I$ (resp. $a \in I$ or $b \in Gr(I)$), where $a, b \in h(R)$. A graded ideal I of R is said to be *graded maximal* if $I \neq R$ and there is no graded ideal J of R such that $I \subsetneq J \subsetneq R$. The intersection of all graded maximal ideals of R is called *graded Jacobson radical* of R and denoted by gr-Jac(R). A graded ring R is called *graded local* if it has a unique graded maximal ideal. A graded submodule N of R-module M is called *graded prime* (resp. *graded primary*) if $rm \in N$, then $m \in N$ or $r \in (N : M)$ (resp. $m \in N$ or $r \in Gr(N : M)$), where $r \in h(R)$, $m \in h(M)$. Let N be a graded submodule of graded R-module M, graded radical N is the intersection of all graded prime submodules of M containing N, and denoted by gr-rad(N). A graded R-module M is called *graded finitely generated* if $M = \sum_{i=1}^{n} Rx_{g_i}$, where $x_{g_i} \in h(M)$. A graded R-module M is called *graded cyclic* if $M = Rx_g$, where $x_g \in h(M)$. It is clear that every graded finitely generated and graded cyclic modules are finitely generated and cyclic, respectively. A graded submodule of a graded R-module M is called *graded maximal* if it is maximal in the lattice of graded R-modules. An R-module M is called a *multiplication module* provided for each submodule N of M there exists an ideal I of R such that $N = IM$. Multiplication modules have been studied in details in [5] and [10]. An ideal I of a ring R is multiplication if it is multiplication as an R-module.

Let R be a G-graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then the ring of fraction $S^{-1}R$ is a graded ring which is called the *graded ring* of fractions. Indeed, $S^{-1}R = \oplus_{g \in G}(S^{-1}R)_g$, where $(S^{-1}R)_g = \{r/s : r \in R, s \in S$ and $g = (degs)^{-1}(degr)\}$.

Let M be a graded module over a ring R and $S \subseteq h(R)$ be a multiplicatively closed subset of R. The module of fraction $S^{-1}M$ over a graded ring $S^{-1}R$ is a graded module which is called the *module of fractions*, if $S^{-1}M = \oplus_{g \in G}(S^{-1}M)_g$, where $(S^{-1}M)_g = \{m/s : m \in M, s \in S$ and $g = (degs)^{-1}(degm)\}$. Consider the graded homomorphism $\eta : M \to S^{-1}M$ defined by $\eta(m) = m/1$. For any graded submodule N of M, the submodule of $S^{-1}M$ generated by $\eta(N)$ is denoted by $S^{-1}N$. Similar to non-graded case, one can prove that $S^{-1}N = \{\beta \in S^{-1}M : \beta = m/s$ for $m \in N$ and $s \in S\}$ and that $S^{-1}N \neq S^{-1}M$ if and only if $S \cap (N : M) = \varnothing$. Let P be any graded prime ideal of a graded ring R and consider the multiplicatively closed subset of $S = h(R) - P$. We denote the graded ring of fraction $S^{-1}R$ of R by R_P^g and we call it the *graded localization* of R. This ring is graded local with the unique graded maximal $S^{-1}P$ which will be denoted by PR_P^g. Moreover, R_P^g-module $S^{-1}M$ is denoted by M_P^g. For graded submodules N and K of M, if $N_P^g = K_P^g$ for every graded prime (graded maximal) ideal of R, then $N = K$.

Moreover, similar to non-graded case, we have the following properties for graded submodules N and K of M:

(1) $S^{-1}(N \cap K) = S^{-1}N \cap S^{-1}K$.

(2) $S^{-1}(N : K) = (S^{-1}N : S^{-1}K)$ if K is finitely generated.

If K is a graded submodule of $S^{-1}R$-module $S^{-1}M$, then $K \cap M$ will denote the graded submodule $\eta^{-1}(K)$ of M. Moreover, similar to the non-graded case, one can prove that $S^{-1}(K \cap M) = K$. In this paper, we study properties of graded multiplication modules and give a characterization for graded multiplication modules by using the graded localization of R-module M. Also, we study the product of graded submodules of graded multiplication modules and determine when the graded radical of graded submodules is graded prime.

2. Properties for Graded Multiplication Modules

A graded module M over a G-graded ring R is called to be *graded multiplication* if for each graded submodule N of M; $N = IM$ for some graded ideal I of R. One can easily show that if N is graded submodule of a graded multiplication module M, then $N = (N : M)M$. We can take $I = (N : M)$.

A graded ideal I of a G-graded ring R is called *graded multiplication* if it is graded multiplication as graded R-modules.

The following lemma is known, but we write it here for the sake of references:

Lemma 2.1. *Let M be a graded module over a graded ring R. Then the following hold*:

(i) *If I and J are graded ideals of R, then $I + J$ and $I \bigcap J$ are graded ideals.*

(ii) *If N is graded submodule, $r \in h(R)$ and $x \in h(M)$, then Rx, IN and rN are graded submodules of M.*

(iii) *If N and K are graded submodules of M, then $N + K$ and $N \bigcap K$ are also graded submodules of M and $(N : M)$ is a graded ideal of R.*

(iv) *Let N_λ be a collection of graded submodules of M. Then $\sum_\lambda N_\lambda$ and $\bigcap_\lambda N_\lambda$ are graded submodues of M.*

Lemma 2.2. *Let R be a G-graded ring and M be a graded R-module.*

(i) *Let $S \subseteq h(R)$ be a multiplicatively closed subset of R. If M is a graded multiplication R-module, then $S^{-1}M$ is a graded multiplication $S^{-1}R$-module.*

(ii) *A graded finitely generated R-module M is a graded multiplication module if and only if the R_P^g-module M_P^g is a graded multiplication module for all graded prime* (*graded maximal*) *ideal P of R.*

Proof. (i) Let L be a graded submodule of $S^{-1}M$. Then $L = S^{-1}N$ for some graded submodule N of M. So, there exists a graded ideal I of R such that $N = IM$. Hence $S^{-1}N = S^{-1}(IM) = S^{-1}IS^{-1}M$, as required.

(ii) Let N be a graded submodule of M, and let P be a graded ideal of R. So, N_P^g is a graded submodule of M_p^g. Hence $N_P^g = (N_P^g :_{R_p g} M_p^g)M_P^g = (N : M)_P^g M_p^g$ $= ((N : M)M)_P^g$ since M_p^g is graded finitely generated. Thus $N = (N : M)M$, as required. □

Definition 2.3. A graded module M over a G-graded ring R is called *locally graded cyclic* if M_P^g is graded cyclic R_p^g-module for all graded maximal ideal P of R.

Proposition 2.4. *A graded finitely generated module is a graded multiplication module if and only if it is locally graded cyclic.*

Proof. Let M be a graded multiplication module and P be a graded maximal ideal of R. So, M_P^g is a graded multiplication R_p^g-module by Lemma 2.2. Therefore, M_P^g is graded cyclic by [9, Theorem 2.14]. Conversely, let M be a graded locally cyclic. M_p^g is graded cyclic for any graded maximal ideal P of R, by definition. Hence M_p^g is a graded multiplication module for any graded maximal ideal P of R, so by Lemma 2.2, M is a graded multiplication module. □

If M is a graded module over a G-graded ring R, then we define the subset $\theta^g(M)$ of M as $\theta^g(M) = \sum_{x \in h(M)}(Rx : M)$. Since $(Rx : M)$ is graded ideal of R, so $\theta^g(M) = \sum_{x \in h(M)}(Rx : M)$ is graded submodule of M by Lemma 2.1.

Lemma 2.5. *Let M be a graded multiplication module over a G-graded ring R. Then the following hold*:

(i) $M = \theta^g(M)M$.

(ii) $N = \theta^g(M)N$ *for any graded submodule N of M.*

Proof. (i) For $x \in h(R)$, $Rx \subseteq M$ and so $Rx = (Rx : M)M$ since M is graded multiplication. Therefore,

$$M = \sum_{x \in h(M)} Rx = \sum_{x \in h(M)} (Rx : M)M$$

$$= \left(\sum_{x \in h(M)} (Rx : M)\right) M = \theta^g(M)M.$$

(ii) Suppose that N is a graded submodule of M. Then $N = IM$ for some graded ideal I of R. Hence $N = IM = I(\theta^g(M)M) = \theta^g(M)(IM) = \theta^g(M)N$, as needed. □

Theorem 2.6. *Let M be a graded module over a G-graded ring R. Then M is graded finitely generated and locally graded cyclic if and only if $\theta^g(M) = R$.*

Proof. Let P be a graded maximal ideal of R. Then $M_P^g = R_P^g x$ for some $x \in h(M_P^g)$. Hence $R_P^g = (R_P^g x : M_P^g) = (Rx :_R M)R_P^g$ since M is graded finitely generated. Therefore, $R_P^g = \theta^g(M)R_P^g$, and so, $R = \theta^g(M)$. Conversely, suppose that $R = \theta^g(M)$. Then there exist $x_i \in h(M)$ $(1 \le i \le n)$ such that $R = \sum_{i=1}^n (Rx_i : M)$. Therefore, $M = \theta^g(M)M = \left(\sum_{i=1}^n (Rx_i : M)\right)M \subseteq \sum_{i=1}^n Rx_i \subseteq M$, so, $M = \sum_{i=1}^n Rx_i$ is a graded finitely generated. Now, let P be a graded maximal ideal of R. Since $\theta^g(M) = R$, there exists $x \in h(M)$ with $(Rx : M) \not\subseteq P$. Therefore, there exists $r \in R \backslash P$ with $rM \subseteq Rx$ and then $(rM)_p^g \subseteq (Rx)_P^g$, so, $(r/1)R_P^g M_P^g \subseteq (Rx)_P^g$. Thus $M_P^g = (Rx)_P^g$ for any graded maximal ideal P of R and so M is locally graded cyclic. □

3. The Product of Multiplication Graded Submodules

Let M be a graded multiplication module over a G-graded ring R. Let N and K be graded submodules of M with $N = IM$ and $K = JM$ for some graded ideals of R. The product of N and K denoted by $NK = IJM$. Moreover, for $a, b \in h(M)$, by ab, we mean the product of Ra and Rb. Clearly, NK is a graded submodule of M by Lemma 2.1 and $NK \subseteq N \cap K$.

Lemma 3.1. *Let N and K be graded submodules of a graded multiplication R-module M and $S \subseteq h(M)$ be a multiplicatively closed subset of R. Then*

(i) $\theta^g(N)\theta^g(K) \subseteq \theta^g(NK)$.

(ii) $S^{-1}(\theta^g(M)) \subseteq \theta^g(S^{-1}M)$.

Proof. (i) Let $a \in N \bigcap M_g$ and $b \in K \bigcap M_h$ for $g, h \in G$. It is enough to prove that $(Ra : N)(Rb : K) \subseteq (Rab : NK)$. Assume that $\sum_{i=1}^n x_i y_i \in (Ra : N)(Rb : K)$, where $x_i \in (Ra : N)$ and $y_i \in (Rb : K)$ for $i = 1, 2, ..., n$. Hence $x_i N \subseteq Ra$ and $y_i K \subseteq Rb$ for $i = 1, 2, ..., n$. Thus $x_i y_i NK \subseteq Rab$ and then $x_i y_i \in (Rab : NK)$. So, $\sum_{i=1}^n x_i y_i \in (Rab : NK)$.

(ii)

$$S^{-1}(\theta^g(M)) = S^{-1}\left(\sum_{x \in h(M)} (Rx : M)\right)$$

$$= \sum_{x \in h(M)} S^{-1}(Rx : M) \subseteq \sum_{x \in h(M)} (\langle x/1 \rangle : S^{-1}M) \subseteq \theta(S^{-1}M). \square$$

Theorem 3.2. *Let R be a G-graded ring, N be a proper graded submodule of a graded multiplication R-module M and* $I = (N : M)$. *Then* $gr\text{-}rad(N) = Gr(I)M$.

Proof. Without loss of generality M is a faithful graded R-module. Let Λ be the collection of all graded prime ideals P of R such that $I \subseteq P$. If $J = Gr(I)$, then $J = \bigcap_{P \in \Lambda} P$ and hence, by [9, Theorem 2.11], $JM = \bigcap_{P \in \Lambda}(PM)$. Let $P \in \Lambda$. If $M = PM$, then $gr\text{-}rad(N) \subseteq PM$. If $M \neq PM$, then $N = IM \subseteq PM$ implies that $gr\text{-}rad(N) \subseteq PM$ by [9, Theorem 3.6]. It follows that $gr\text{-}rad(N) \subseteq JM$. Conversely, suppose that K is a graded submodule of M containing N. By [9, Theorem 3.6], there exists a graded prime ideal Q of R such that $I \subseteq Q$ and $K = QM$. Since $IM = N \subseteq K = QM \neq M$ it follows that $I \subseteq Q$, by [9, Proposition 3.3], and hence $J \subseteq Q$. Thus $JM \subseteq K$. It follows that $JM \subseteq gr\text{-}rad(N)$. Therefore, $gr\text{-}rad(N) = JM$. $\square$

Proposition 3.3. *Let N be a graded submodule of a faithful graded multiplication module over a graded PID. Then N is graded multiplication module.*

Proof. There exists a graded ideal I of R such that $N = IM$, so I is graded principal ideal of R, and hence it is graded ideal by [12, Theorem 2.3]. Now the assertion follows from [9, Corollary 2.9]. □

Lemma 3.4. *Let N, K be graded submodules of a graded multiplication R-module M. Then*:

(i) *If $N \subseteq K$, then $(K/N)^n = (K^n + N)/N$ for each positive integer n.*

(ii) *If $N \subseteq K$, then $gr\text{-}rad(K/N) = (gr\text{-}rad(K))/N$.*

(iii) *If M is finitely generated and N is a graded prime submodule of M, then $gr\text{-}rad(N^n) = N$ for each positive integer n.*

Proof. (i) Since a quotient of any graded multiplication R-module is graded multiplication by [9, Proposition 2.10], it follows from [1, Lemma 2.6], that

$$(K/N)^n = ((K/N : M/N))^n = (K : M)^n M/N$$

$$= ((K : M)^n M + N)/N = (K^n + N)/N.$$

(ii) We have

$$gr\text{-}rad(K/N) = (gr\text{-}rad(K/N : M/N)) \cdot (M/N)$$

$$= (gr\text{-}rad(K : M)) \cdot (M/N)$$

$$= (gr\text{-}radK + N)/N$$

$$= (gr\text{-}rad(K))/N,$$

by Theorem 3.2.

(iii) Since N is graded prime it follows that $(N : M) = P$ is a graded prime of R. By Theorem 3.2 and [1, Lemma 2.6], we have

$$gr\text{-}rad(N^n) = (gr\text{-}rad(N^n : M))M = (gr\text{-}rad(N : M))^n M$$

$$= gr\text{-}rad(P^n)M = PM = N.$$ □

Proposition 3.5. *Let N be a graded primary submodule of a graded finitely generated multiplication module over a G-graded ring R. Then whenever $a, b \in h(M)$ with $(Ra)(Rb) \subseteq N$ but $Ra \nsubseteq N$, then $Rb \subseteq gr\text{-}rad(N)$.*

Proof. Assume that N is graded primary and let $(Ra)(Rb) \subseteq N$ with $Ra \not\subseteq N$. There exist graded ideals I, J and A of R such that $Ra = IM$, $Rb = JM$ and $N = AM$, respectively (see [9, Proposition 2.3]). Since $Ra = IM \not\subseteq N$, there exist $r \in I$ and $m \in M - N$ such that $rm \notin N$. We can write $r = \sum_{i=1}^{m} r_{h_i}$ with $0 \neq r_{h_i}$ and $m = \sum_{i=1}^{n} m_{g_i}$ with $0 \neq m_{g_i}$. Therefore, there are $1 \leq j \leq m$ and $1 \leq t \leq n$ such that $r_{h_j} m_{g_t} \notin N$. If $s = \sum_{i=1}^{t} s_{g_i} \in J$, then for each $1 \leq i \leq t$; $r_{h_j} m_{g_t} s_{g_i} = s_{g_i}(r_{h_j} m_{g_t}) \in IJM \subseteq N$, so $s_{g_i} \in Gr(A)$ for any $1 \leq i \leq t$, and hence $J \subseteq Gr(A)$. Thus, $Rb = JM \subseteq Gr(A)M = gr\text{-}rad(N)$ by Theorem 3.2, as needed. □

Lemma 3.6. *Let N, K be graded submodules of a graded multiplication R-module M with K is graded finitely generated and $K \subseteq gr\text{-}rad(N)$. Then $K^t \subseteq N$ for some t.*

Proof. Let $K = Ra_{g_1} + \cdots + Rag_n$ for $a_{g_i} \in K \cap M_{g_i}$ $(1 \leq i \leq n)$. So, there exist graded ideals $I_1, \ldots, I_n$ such that $Ra_{g_i} = I_i M$ $(1 \leq i \leq n)$. There exist positive integers $m_1, \ldots, m_n$ such that $a_{g_i}^{m_i} = I_i^{m_i} M \subseteq N$ by [1, Theorem 3.13] $(1 \leq i \leq n)$. Let $t = \mathrm{Max}\{m_1, \ldots, m_n\}$. It follows that $K^t = (I_1 M + \cdots + I_n M)^t = (I_1 + \cdots + I_n)^t M \subseteq N$. □

Theorem 3.7. *Let M be a graded finitely generated faithful graded multiplication over a graded ring R. Then R is graded integral domain if and only if, whenever $N \cdot K = 0$, then either $N = 0$ or $K = 0$ for all graded submodules N and K of M.*

Proof. Assume that R is a graded integral domain and let N and K be graded submodules of M. There exist graded ideals I and J of R such that $N \cdot K = (IM)(JM) = IJM = 0$, so $IJ = 0$ since M is faithful, and hence $I = 0$ or $J = 0$. Thus, either $N = IM = 0$ or $K = JM = 0$. Conversely, suppose that $ab = 0$ with $a \neq 0$ for some $a, b \in h(R)$. Then $A = (Ra)M$ and $B = (Rb)M$ are graded submodules of M with $AB = 0$. By hypothesis, $B = 0$, and hence $b = 0$, as required. □

4. The Graded Radical of a Graded Submodule

Lemma 4.1. *Let R be a G-graded ring. If every proper graded ideal of R is graded primary, then R is a graded local ring.*

Proof. First, we show that if P and I are graded ideals of R, then either $I \subseteq P$ or $P \subseteq I$. Assume that $I \not\subseteq P$ and choose $a \in I \cap h(R) - P$. Let $b \in P$. So, $b = \sum b_{g_i}$, where $0 \neq b_{g_i} \in h(R)$. Then $ab_{g_i} \in PI$ for each $1 \leq i \leq n$ and PI is graded primary ideal, so, we have $b_{g_i} \in PI$ or $a^m \in PI$ for some m. But if $a^m \in PI \subseteq P$, then $a \in P$, a contradiction. Thus we must have $b_{g_i} \in PI$ for each $1 \leq i \leq n$. Hence $b \in PI$, and so $P \subseteq PI$. Therefore, $P = PI \subseteq I$. Now let M be any graded maximal ideal of R. Then M is comparable to any proper graded ideal. If M' is any graded maximal, then M and M' are comparable, so, $M = M'$, as needed. □

Theorem 4.2. *Let R be a graded ring and* $M \neq 0$ *be a graded R-module. If every proper graded submodule of M is a graded primary submodule of M and* $M \neq T(M)$, *where* $T(M) = \{m \in M : rm = 0$ *for some* $0 \neq r \in h(R)\}$, *then R is graded local.*

Proof. Let $m \in h(M) - T(M)$, so, $(0 : m) = 0$, and hence $Rm \cong R$ as R-modules. Clearly, every proper graded submodule of graded R-module $Rm \cong R$ is graded primary, hence R is graded local by Lemma 4.1. □

Proposition 4.3. *Let* $M \neq 0$ *be a graded multiplication module over a graded ring R. If every proper graded submodule of M is a graded primary submodule of M, then M is graded cyclic.*

Proof. This follows from Theorem 4.2 and [9, Theorem 2.15]. □

Proposition 4.4. *Let M be a graded multiplication R-module. Then if N is a graded primary submodule of M, then* $gr\text{-}rad(N)$ *is graded prime submodule of M.*

Proof. Assume that N is a graded primary submodule of M. Then $I = (N : M)$ is a graded primary ideal of R by [8, Proposition 2.7]. Since $Gr(I)$ is a graded prime ideal of R, it follows from Theorem 3.2 that $gr\text{-}rad(N) = Gr(I)M$ is graded prime, as required. □

Theorem 4.5. *Let R be a graded domain with* $G\dim(G) = 1$ *and M be a graded P-secondary R-module. Then for any graded submodule N of M,* $gr\text{-}rad(N)$ *is a graded prime submodule.*

Proof. Consider the ideal $(K : M)$ for any graded prime submodule K containing N. These graded ideals are graded prime and $N \subseteq K$ implies that $(N : M) \subseteq (K : M)$ which in turn implies that $P = Gr(N : M) \subseteq Gr(K : M) = (K : M)$ for all such K (note that M/N is graded P-secondary by [8, Theorem 2.7]). For one of these graded prime submodules K, we obtain the chain of graded ideals $0 \subsetneq gr\text{-}rad(N : M) \subseteq (K : M)$. Then $gr\text{-}rad(K : M) = (K : M) = P$ for any graded prime submodule K containing N since $G\dim n(R) = 1$. Moreover, $(gr\text{-}rad(N) : M) = (\bigcap_{N \subseteq K} (K : M)) = \bigcap_{N \subseteq K} (K : M) = P$. Let $ra \in gr\text{-}rad(N)$ for some $r \in h(R)$ and $a \in h(M) - (gr\text{-}rad(N))$. Then there exists a graded prime submodule N of M such that $a \in K$, so $r \in P$, as needed. □

Corollary 4.6. *Let R be a graded domain with* $G\dim(R) = 1$ *and M be a graded torsion R-module such that* 0 *is a graded prime submodule. Then for any graded submodule N of M,* $gr\text{-}rad(N)$ *is a graded prime submodule.*

Proof. By Theorem 4.5, it is enough to show that M is a graded secondary module. As M is graded torsion, $(0 : M) = P \neq 0$ and since 0 is graded prime, P is a graded prime ideal of R. Let $a \in h(R)$. If $a \in P$, then $aM = 0$. If $a \notin P$, then we have $aM = M$, as required. □

References

[1] R. Ameri, On the prime submodules of multiplication modules, Int. J. Math. Math. Sci. 27 (2003), 1715-1724.

[2] D. D. Anderson, Multiplication ideals, multiplication rings, and the ring $R(x)$, Canad. J. Math. 28(4) (1976), 760-768.

[3] D. D. Anderson, Some remarks on multiplication ideals, Math. Japon. 25(4) (1980), 463-469.

[4] F. W. Anderson and K. R. Fuller, Rings and Categories of Modules, Vol. 13, Springer-Verlag, New York, Heidelberg, 1974.

[5] A. Barnard, Multiplication modules, J. Algebra 71(1) (1981), 174-178.

[6] S. Ebrahimi Atani and F. Farzalipour, On prime and primary submodules, Chiang Mai. J. Sci. 32(1) (2005), 5-9.

[7] S. Ebrahimi Atani and F. Farzalipour, Notes on the graded prime submodules, Int. Math. Forum 1(37-40) (2006), 1871-1880.

[8] S. Ebrahimi Atani and F. Farzalipour, On graded secondary modules, Turkish J. Math. 31(4) (2007), 371-378.

[9] S. Ebrahimi Atani and F. Farzalipour, On graded multiplication modules, Chiang Mai. J. Sci., to appear.

[10] Z. Abd El-Bast and P. F. Smith, Multiplication modules, Comm. Algebra 16(4) (1988), 755-779.

[11] J. Escoriza and B. Torrecillas, Multiplication graded rings, Algebra and Number Theory (Fez), Lecture Notes in Pure and Appl. Math., 208, Dekker, New York, 2000, pp. 127-136.

[12] Hani A. Khashan, Some properties of gr-multiplication ideals, Turkish J. Math. 32 (2008), 1-12.