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NEWTON'S METHOD OF ENTIRE FUNCTIONS WITH INFINITE ORDER

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Abstract

In this paper, Newton's method for a class of entire functions with infinite order is investigated. By using dynamical theory of functions meromorphic outside a small set, we find that there are infinitely many virtual immediate basins and supper-attracting immediate basins in the Fatou set of Newton's method. We also show each supper-attracting immediate basin has finite area while each is unbounded.

1. Introduction

Newton's method is a classical way to approximate roots of differentiable functions f by an iterative procedure. We can investigate the procedure in view of complex dynamical systems (see [3] for general references on this subject).

Newton's method for a complex polynomial p is the iteration of a rational function $N_p = z - \frac{p(z)}{p'(z)}$ on the Riemann sphere. If f is a transcendental entire function, then the associated Newton map $N_f(z) = z - f/f'(z)$ will generally be

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transcendental meromorphic, except in the special case $f = pe^q$ with polynomials p and q which was studied by Haruta ([6]). Mayer and Schleicher ([8]) have shown that immediate basins for Newton maps of entire functions are simply connected and unbounded, extending a result of Przytycki ([9]) in the polynomial case. They have also shown that Newton maps of transcendental functions may exhibit a type of Fatou component called *virtual immediate basin* from which the iteration converges to infinity, while it does not appear for Newton maps of polynomials.

Definition 1.1. Let ξ be an attracting fixed point of $N_f(z)$. The basin of attraction of ξ is a open set of all points z such that $\{N_f^m(z)\}$ converges to ξ as $m \to \infty$. The connected component containing ξ of the basin is called the *immediate basin of* ξ.

Definition 1.2. An unbounded domain $U \subset \mathbb{C}$ is called *virtual immediate basin of* $N_f(z)$ if it is maximal (among domains in $\mathbb{C}$) with respect to the following properties:

(i) $\lim_{n\to\infty} N_f^n(z) = \infty$ for all $z \in U$;

(ii) there is a connected and simply connected subdomain $S_0 \subset U$ such that $N_f(\overline{S}_0) \subset S_0$ and for all $z \in U$ there is an $m \in N$ such that $N_f^m(z) \in S_0$. We call the domain S_0 an absorbing set of U.

In this paper, we investigate the Newton's map $N_f(z)$ for a class of entire functions in the form $f(z) = q(e^z)e^{p(e^z)}$, where $p(z)$ and $q(z)$ are polynomial with $\deg(p) \geq 1$, $\deg(q) \geq 1$ and $q(0) \neq 0$.

To investigate the dynamics of the meromorphic function $N_f(z)$, we need to analyse the dynamics of function in following class M.

$M = \{f$: there is a compact totally disconnected set $E = E(f)$ such that f is meromorphic in $E^c = \hat{\mathbb{C}} \setminus E$ and $C(f, E^c, z_0) = \hat{\mathbb{C}}$ for all $z_0 \in E$. If $E = \varnothing$, we make the further assumption that f is neither constant nor univalent in $\hat{\mathbb{C}}\}$, where the cluster set $C(f, E^c, z_0) = \{w : w = \lim_{n\to\infty} f(z_n)$ for some $z_n \in E^c$ with $z_n \to z_0\}$.

The class M was studied in [1, 2, 4, 5, 7]. In [1] and [2], where the basic concepts such as Fatou and Julia sets and the basic properties of dynamics of functions in M were established. It was proved in [1] that the class M is closed under composition and if $f, g \in M$, then $E(f \circ g) = E(g) \bigcup g^{-1}(E(f))$. For $f \in M$, we define f^0 to be the identity function with $E_0 = \varnothing$, $f^n = f \circ f^{n-1}$, then $f^n \in M$, $n \in N$, and $E_n = E(f^n) = \bigcup_{j=0}^{n-1} f^{-j}(E) = \{\text{singularities of } f^{-n}\}$. Let $J_1(f) = \overline{\bigcup_{n=0}^{+\infty} E_n}$ and $F(f) = \hat{\mathbb{C}} \setminus J_1(f)$. Then $F_1(f)$ is the largest open set in which all f^n are defined and $f(F_1(f)) \subset F_1(f)$. As in [1], for $f \in M$, we define the Fatou set of f, denoted by $F(f)$, to be the largest open set in which (i) all composition f^n are meromorphic and (ii) the family $\{f^n\}$ is a normal family; and the Julia set of f, denoted by $J(f)$, to be the complement of $F(f)$. If the set $J_1(f)$ is either empty or contains one point or two points, then f is conjugate to a rational map or entire function or an analytic map of the punctured plane $\mathbb{C}^*$, respectively. In these cases the condition (i) is trivial and the Fatou sets are determined by (ii). In all other cases, by Montels theorem, $F(f) = F_1(f)$ and $J(f) = J_1(f)$. It is clear that for $f \in M$, $F(f)$ is open and completely invariant. Let U be a connected component of $F(f)$. Then $f^n(U)$ is contained in a component U_n of $F(f)$. If for some pair of $m \neq n$, $U_m \neq U_n$, then U is called a *wandering domain of f*, otherwise U is said to be *preperiodic*. If for some $n \in N$, $U_n = U$, namely $f^n(U) \subset U$, then U is said to be *periodic*. For a periodic component of $F(f)$, we have the following classification theorem:

Theorem A [1]**.** *Let U be a periodic component of the Fatou set of period p. Then precisely one of the following is true*:

(i) *U is a* (*super*) *attracting domain of a* (*super*) *attracting periodic point a of f of period p such that* $f^{np}|_U \to a$ *as* $n \to +\infty$ *and* $a \in U$.

(ii) *U is a parabolic domain of a rational neutral periodic point a of f of period p such that* $f^{np}|_U \to a$ *as* $n \to +\infty$ *and* $a \in \partial U$.

(iii) *U is a Siegel disk of period p such that there exists an analytic homeomorphism* $\varphi : U \to \Delta$, *where* $\Delta = \{z : |z| < 1\}$, *satisfying* $\varphi(f^p(\varphi^{-1}(z))) = e^{2\pi\alpha i}z$ *for some irrational number* α *and* $\varphi^{-1}(0) \in U$ *is an irrational neutral periodic point of f of period p.*

(iv) *U is a Herman ring of period p such that there exists an analytic homeomorphism* $\varphi : U \to A$, *where* $A = \{z : 1 < |z| < r\}$, *satisfying* $\varphi(f^p(\varphi^{-1}(z))) = e^{2\pi\alpha i}z$ *for some irrational number* α.

(v) *U is a Baker domain of period p such that* $f^{np}|_U \to a \in J(f)$ *as* $n \to +\infty$ *but* f^p *is not meromorphic at a. If* $p = 1$, *then* $a \in E(f)$.

With similar discussion as that of Subsection 6.5 in [3], or refer to Subsection 3.1.6 in [10], we have the following Theorems B and C:

Theorem B. *Suppose that the map* $f \in M$ *has a Taylor expansion* $f(z) = z - z^{p+1} + O(z^{2p+1})$ *at the origin. Then for sufficiently small t, f has p petals lying in distinct parabolic domains at the origin, such that*:

(i) *f maps each petal* $\Pi_k(t)$ *into itself, and* $f : \Pi_k(t) \mapsto \Pi_k(t)$ *is conjugate to* $T(z) = z + 1$;

(ii) $f^n(z) \mapsto 0$ *uniformly on each petal as* $n \mapsto \infty$;

(iii) $\arg(f^n(z)) \mapsto \frac{2k\pi}{p}$ *locally uniformly on* Π_k *as* $n \mapsto \infty$;

(iv) $|f(z)| < |z|$ *on a neighborhood of the axis of each petal, where*

$$\Pi_k(t) = \left\{re^{i\theta} : r^p < t(1 + \cos(p\theta)); \left|\frac{2k\pi}{p} - \theta\right| < \frac{\pi}{p}\right\}, \ (k = 0, 1, \ldots, p - 1).$$

Theorem C. *Suppose that the map* $f \in M$ *has a Taylor expansion* $f(z) = z + az^{p+1} + O(z^{p+2})$ *at the origin with* $a \neq 0$. *Then* $f(z)$ *is conjugate near* 0 *to a function* $F(z) = z - z^{p+1} + O(z^{2p+1})$, *via a polynomial* $\varphi(z) = \lambda z + \beta z^2 + \cdots + \gamma z^{p!}$, *where* $\lambda = |a|^{-p} e^{\frac{\arg(a)}{n} i}$.

Theorem D [10, Theorem 3.1.17]**.** *Let* $f, g \in M$, *and* $\exp(f(z)) = g(e^z)$. *If* $\infty \in E(f)$ *or* $f(\infty) \neq \infty$, *then* $\exp(J(f)) = J(g)\setminus\{0\}$ *and* $\exp(F(f)) = F(g)\setminus\{0\}$.

Let $f(z) = q(e^z)e^{p(e^z)}$, $R(z) = -\frac{q(z)}{z(q'(z) + q(z)p'(z))}$, $g(z) = ze^{R(z)}$. Then $N_f(z) = z + R(e^z)$, $e^{N_f(z)} = g(e^z)$. According to the nature of logarithmic function and $e^{N_f(z)} = g(e^z)$, Theorem 3.1.17 in [10] implies that the dynamics of N_f in horizontal strip regions $\{z : (2m-1)\pi < Imz < (2m+1)\pi\}$ are same for different $m \in \mathbb{Z}$. So, we just need to consider dynamics of N_f in the horizontal strip region

$$\Xi = \{z : -\pi < Imz < \pi\}$$

and obtain the following results:

Theorem 1. *In* Ξ, *there are m simple connected supper-attracting immediate basins, and n invariant Baker domains. Moreover, each Baker domain is virtual immediate basin.*

Theorem 2. *Each supper-attracting immediate basin of* $N_f(z)$ *has finite area.*

2. Immediate Basins and Virtual Immediate Basins

To prove Theorem 1, we need the following Lemma 2.1:

Lemma 2.1. *In Fatou set of* $g(z)$, *there are n invariant parabolic domains* V_∞^k $(k = 0, 1, ..., n-1)$ *such that* $g^n(z) \mapsto \infty$ *for* $z \in V_\infty^k$ *as* $n \mapsto \infty$; *and for any root a of* $q(z)$ *there is an invariant supper-attracting component* V_a *containing a.*

Proof. It is easy to see that the essential singularities of $g(z)$ are poles of $R(z)$, i.e., $g(z) \in M$. Note that $\deg(p) = n \geq 1$, $R(z) \to 0$ as $z \to \infty$, $g(z)$ has only one pole at infinity. Obviously any zero point *a* of $q(z)$ is fixed point of $g(z)$ and $g'(a) = 0$. So by Theorem A, there is an invariant supper-attracting component V_a which contains *a* in Fatou set $F(g)$ of $g(z)$.

With no loss of generality, let

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

and

$$q(z) = b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0.$$

Then

$$N_f(z) = -\frac{b_m w^m + b_{m-1} w^{m-1} + \cdots + b_1 w + b_0}{n a_n b_m w^{n+m} + \cdots + (a_1 b_0 + b_1) w} \circ (e^z) + z$$

and

$$g(z) = z e^{-\frac{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0}{n a_n b_m z^{n+m} + \cdots + (a_1 b_0 + b_1) z}}.$$

Let $\sigma(z) = \frac{1}{z}$ and $h(z) = z e^{\frac{z^n (b_m + b_{m-1} z + \cdots + b_1 z^{m-1} + b_0 z^m)}{n a_n b_m + \cdots + (a_1 b_0 + b_1) z^{n+m-1}}}$. Then $\sigma \circ g(z) = h \circ \sigma(z)$.

Because $h(z)$ has a Taylor expansion $h(z) = z + \frac{1}{n a_n} z^{n+1} + O(z^{n+2})$ at the origin, by Theorem A there are n invariant parabolic domains B^k $(k = 0, 1, \ldots, n-1)$, in which $h^n(z) \to 0$ as $n \to \infty$. So, there are n invariant parabolic domains $V_\infty^k = \sigma(B^k)$ $(k = 0, 1, \ldots, n-1)$, in which $g^n(z) \to \infty$ as $n \to \infty$. □

Proof of Theorem 1. Since $e^{N_f(z)} = g(e^z)$ and $\infty \in E(N_f)$, based on Lemma 2.1 and Theorem D, there is a supper-attracting immediate basin $U_a = ln(V_a)$ in Fatou set $F(N_f)$ of $N_f(z)$, and the corresponding supper-attracting fixed point is a. According to Theorem 2.7 in [8], U_a are simply connected.

On the other hand, $h(z)$ has a Taylor expansion $h(z) = z + \frac{1}{n a_n} z^{n+1} + O(z^{n+2})$ at the origin, Theorem C implies that $h(z)$ is conjugate to a function $H(z) = z$

$-z^{n+1} + O(z^{2n+1})$ via a certain polynomial $\psi(z) = \lambda z + \beta z^2 + \cdots + \gamma z^{n!}$, where $\lambda = (n|a_n|)^{-\frac{1}{n}} e^{-\frac{\arg(a_n)}{n}i}$. Using Theorem B, for sufficiently small positive numbers t_0, s_0, each parabolic domains in Fatou set $F(H)$ has an absorbing sets

$$\Pi_k(t_0) = \left\{re^{i\theta} : r^n < t_0(1 + \cos n\theta); \left|\frac{2k}{n}\pi - \theta\right| < \frac{1}{n}\pi\right\},$$

at the origin, $L^k = \left\{re^{i\theta} : 0 < r < s_0,\ \theta = \frac{2k+1}{n}\pi\right\}$ and $l^k = \left\{re^{i\theta} : 0 < r < s_0,\ \theta = \frac{2k}{n}\pi\right\}$ is the corresponding repelling and absorbing axis, respectively, where $k = 0, 1, \ldots, n-1$ and same hereinafter. Note the conjugate among $H(z)$, $h(z)$ and $g(z)$, each parabolic domains V_∞^k in Fatou set $F(g)$ has an absorbing sets $\sigma \circ \psi^{-1}(\Pi_k(t_0))$ with absorbing axis $\sigma \circ \psi^{-1}(l^k)$.

Note the semiconjugate between $g(z)$ and $N_f(z)$, Theorem D implies $N_f(z)$ has a component $U_\infty^k = \ln(V_\infty^k)$ such that $N_f^n(z) \mapsto \infty$ for $z \in U_\infty^k$ as $n \mapsto \infty$. So, each U_∞^k is not wandering domain but Baker domain with absorbing sets $\ln \circ \sigma \circ \psi^{-1}(\Pi_k(t_0))$, then each U_∞^k is virtual immediate basin. □

In order to prove Theorem 2, we need the following Lemma 2.2:

Lemma 2.2. *In* Ξ, *complement of the union of all virtual immediate basins of* $N_f(z)$ *has finite area*.

Proof. From the proof of Theorem 1, for sufficiently small positive numbers t_0, each virtual immediate basin U_∞^k of $N_f(z)$ has an absorbing set $\ln \circ \sigma \circ \psi^{-1}$ $(\Pi_k(t_0))$. Note that complement of the union of all virtual immediate basins of $N_f(z)$ is subset of the complement of union of absorbing sets. To complete this proof, we need only to show that the complement of union of absorbing sets has finite area in Ξ.

Now, we analyse those absorbing sets in parabolic domains in $F(g)$.

Let $0 < t < t_0$, $\frac{3}{4n}\pi < \theta_0 < \frac{1}{n}\pi$, and

$$\gamma_1^k = \left\{re^{i\theta} : r^n = t(1+\cos n\theta),\ \frac{2k}{n}\pi + \theta_0 < \theta < \frac{2k}{n}\pi + \frac{1}{n}\pi\right\},$$

$$\gamma_2^k = \left\{re^{i\theta} : r^n = t(1+\cos n\theta),\ \frac{2k\pi}{n} - \theta_0 > \theta > \frac{2k\pi}{n} - \frac{\pi}{n}\right\}.$$

Choosing the branch of ψ^{-1} which is in the form $\psi^{-1}(z) = \frac{1}{\lambda}z + \alpha_1 z^2 + \alpha_2 z^3 + \cdots$, then $\Gamma_1^k = \psi^{-1}(\gamma_1^k)$ and $\Gamma_2^k = \psi^{-1}(\gamma_2^k)$ are two simple curves in the parabolic domain in Fatou set $F(h)$.

Since $e^{N_f(z)} = g(e^z)$ and $\sigma \circ g(z) = h \circ \sigma(z)$, $\tilde{\Gamma}_1^k = \ln \circ \sigma \circ \psi^{-1}(\gamma_1^k)$ and $\tilde{\Gamma}_2^k = \ln \circ \sigma \circ \psi^{-1}(\gamma_2^k)$ are simple curves in Baker domains in Fatou set $F(N_f)$.

Let $\psi^{-1}(z) = r_v e^{i\theta_v} z$. Then

$$\tilde{\Gamma}_1^k = \left\{X(\theta) + iY(\theta) : \begin{array}{l} X(\theta) = -\ln(r_v(t + t\cos n\theta)^{1/n}), \\ Y(\theta) = -\theta - \theta_v, \end{array} \quad \frac{2k}{n}\pi + \theta_0 < \theta < \frac{2k+1}{n}\pi\right\},$$

$$\tilde{\Gamma}_2^k = \left\{X(\theta) + iY(\theta) : \begin{array}{l} X(\theta) = -\ln(r_v(t + t\cos n\theta)^{1/n}), \\ Y(\theta) = -\theta - \theta_v, \end{array} \quad \frac{2k-1}{n}\pi < \theta < \frac{2k}{n}\pi - \theta_0\right\},$$

where $r_v = \left|\frac{1}{\lambda} + \alpha_1 z + \alpha_2 z^2 + \cdots\right|$ and $\theta_v = \arg\left(\frac{1}{\lambda} + \alpha_1 z + \alpha_2 z^2 + \cdots\right)$ are continuous functions. Furthermore, $r_v(z) \to \left|\frac{1}{\lambda}\right|$, $\theta_v(z) \to \arg\frac{1}{\lambda}$, as $z \to 0$. Hence, the curve $\tilde{\Gamma}_1^k$ is monotonously decreasing, under which there is an asymptote $Y = -\frac{2k+1}{n}\pi - \arg\frac{1}{\lambda}$ as $\theta \to \frac{2k+1}{n}\pi$, the curve $\tilde{\Gamma}_2^k$ is monotonously increasing, above which there is an asymptote $Y = -\frac{2k-1}{n}\pi - \arg\frac{1}{\lambda}$ as $\theta \to \frac{2k-1}{n}\pi$.

Now, we analyse those absorbing set $\ln \circ \sigma \circ \psi^{-1}(\Pi_k(t))$ in Baker domain U_∞^k of $F(N_f)$.

The repelling axis $L^k = \left\{re^{i\theta} : 0 < r < s_0,\ \theta = \frac{2k+1}{n}\pi\right\}$ of $g(z)$ produces repelling axis

$$\ln \circ \sigma \circ \psi^{-1}(L^k) = \left\{ X(r) + iY(r) : \begin{array}{l} X(r) = -\ln(r_v r), \\ Y(r) = -\frac{2k+1}{n-1}\pi - \arg\frac{1}{\lambda}, \end{array} \quad 0 < r < s_0 \right\}$$

of $N_f(z)$ at infinity.

It is easy to see that the asymptote of $\tilde{\Gamma}_1^k$ or $\tilde{\Gamma}_2^k$ is the horizontal line in which $\ln \circ \sigma \circ \psi^{-1}(L^k)$ lies.

Next we show the area of unbounded wedge sharp region W_1^k between $\tilde{\Gamma}_1^k$ and the corresponding asymptote $Y = -\frac{2k+1}{n-1}\pi - \arg\frac{1}{\lambda}$ is finite. For positive numbers δ_1 and δ_2, we construct another unbounded wedge sharp region $\tilde{W}_1^k$, which is between curve

$$\overline{\Gamma}_1^k = \ln \circ \sigma \circ \overline{\psi}(\gamma_1^k) = \left\{ X(\theta) + iY(\theta) : \begin{array}{l} X(\theta) = -\ln\left(\frac{\delta_1}{|\lambda|}(t(1+\cos n\theta))^{1/n}\right), \\ Y(\theta) = -\theta - \left(\arg\frac{1}{\lambda} - \delta_2\left(\frac{2k+1}{n}\pi - \theta\right)\right), \end{array} \right.$$

$$\left. \frac{2k}{n}\pi + \theta_0 < \theta < \frac{2k+1}{n}\pi \right\}$$

and the corresponding asymptote $Y = -\frac{2k+1}{n}\pi - \arg\frac{1}{\lambda}$, where $\overline{\psi}(z) = z\frac{\delta_1}{|\lambda|} e^{i\left(\arg\frac{1}{\lambda} - \delta_2\left(\frac{2k+1}{n}\pi - \theta\right)\right)}$. It is easy to see that for some appropriate small positive numbers δ_1 and δ_2, if $z = re^{i\theta} \in \gamma_1^k$, the Euclidian distance of point $\ln \circ \sigma \circ \overline{\psi}(z)$ to line $Y = -\frac{2k+1}{n}\pi - \arg\frac{1}{\lambda}$ is greater than that of point $\ln(\sigma \circ \psi^{-1}(z))$ to the same line, i.e., $\overline{\Gamma}_1^k$ lies above $\tilde{\Gamma}_1^k$. While the area of unbounded wedge sharp region $\tilde{W}_1^k$ is the following integration:

$$\int_{\frac{2k}{n}\pi+\theta_0}^{\frac{2k+1}{n}\pi}\left(Y(\theta)-\left(-\frac{2k+1}{n}\pi-\arg\frac{1}{\lambda}\right)\right)dX(\theta)$$

$$=\int_{\frac{2k\pi}{n}+\theta_0}^{\frac{2k+1}{n}\pi}\frac{(\delta_2+1)\left(\frac{2k+1}{n}\pi-\theta\right)\sin n\theta}{1+\cos n\theta}\,d\theta$$

$$=\int_0^{\frac{\pi}{n}-\theta_0}\frac{(\delta_2+1)\theta\sin n\theta}{1-\cos n\theta}\,d\theta$$

$$=\frac{\delta_2+1}{n^2}\int_0^{\pi-n\theta_0}\frac{\theta\sin\theta}{(1-\cos\theta)}\,d\theta$$

$$<\frac{2(\delta_2+1)(\pi-\theta_0)}{n^2},$$

where

$$X(\theta)=-\ln\left(\frac{\delta_1}{|\lambda|}(t(1+\cos n\theta))^{1/n}\right)\text{ and }Y(\theta)=-\theta-\left(\arg\frac{1}{\lambda}-\delta_2\left(\frac{2k+1}{n}\pi-\theta\right)\right).$$

It is clear that the difference between areas of $\tilde{W}_1^k$ and W_1^k is finite, so W_1^k has finite area. The symmetry implies that the area of unbounded wedge sharp region W_2^k between $\tilde{\Gamma}_2^k$ and the corresponding asymptote $Y=-\frac{2k+1}{n}\pi-\arg\frac{1}{\lambda}$ takes the same value as the area of W_1^k.

Denote the union of these wedge sharp regions by W and $\bigcup_{k=0}^{n-1}\ln\circ\sigma\circ\psi^{-1}(\Pi_k(t))$ by $\prod$. Note that $\ln\circ\sigma\circ\psi^{-1}(\Pi_k(t))$ are also absorbing sets of those Baker domains, respectively, and that those asymptotes exist alternately with alternation angle as $\frac{2\pi}{n-1}$, $\Xi\setminus(W\cup\Pi)$ is a bounded domain, and the area of $\Xi\setminus\Pi=W\cup(\Xi\setminus W\cup\Pi)$ is finite. So, the complement of the union of all virtual immediate basins of $N_f(z)$, a subset of $\Xi\setminus\Pi$, is with finite area. □

Proof of Theorem 2. Theorem 2 follows from Lemma 2.2 directly. □

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